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Research Article

Rational Divide-and-Conquer Relations

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A rational divide-and-conquer relation, which is a natural generalization of the classical divide-and-conquer relation, is a recursive equation of the form $f(bn) = R(f(n), f(n), \ldots, f(b-1)n) + g(n)$, where b is a positive integer ≥ 2; R a rational function in $b-1$ variables and g a given function. Closed-form solutions of certain rational divide-and-conquer relations which can be used to characterize the trigonometric cotangent-tangent and the hyperbolic cotangent-tangent function solutions are derived and their global behaviors are investigated.

1. Introduction

The classical divide-and-conquer relation is a recursive relation of the form ([1–3])

$$F(n) = aF\left(\frac{n}{b}\right) + G(n), \tag{1.1}$$

where $a, b (\geq 2)$ are positive integers and $G(n)$ is a given function. This class of recurrence relations arises frequently in the analysis of recursive computer algorithms. Such algorithms split a problem of size n into a subproblems each of size $[n/b]$, with $G(n)$ extra operations being required when this split of a problem of size n into smaller problems is made. Although, there are certain cases, see for example the table on page 273 of [3], where the relation (1.1) can be solved explicitly, it is generally impossible to solve (1.1) for all values of n. However, when a starting value $\mathrm{F}(b^{\lambda})$ is given, a solution for $n = b^k (k > \lambda)$ can be found by making a change of variables $F(b^k) = \phi(k)$ which turns (1.1) into a first order difference equation of the form ([1, page 137])

$$\phi(k) = a\phi(k-1) + G\left(b^k\right), \tag{1.2}$$

and this last recursive equation can be easily solved. Another aspect of importance in the study of divide-and-conquer relations deals with the size of $F(n)$ which is used in analyzing the complexity of corresponding divide-and-conquer algorithms ([2, Section 5.3]).

Generalizing the above notion, by a *rational divide-and-conquer (RDAC) relation*, we refer to a recursive relation of the form

$$f(bn) = R(f(n), f(2n), \ldots, f(b-1)n) + g(n), \tag{1.3}$$

where $b \in \mathbb{N}$, $b \geq 2$, $R(x_1, \ldots, x_{b-1})$ a rational function in $x_1, \ldots, x_{b-1}$, and $g(n)$ a given function. Here we aim to find explicit closed form solutions of certain nonlinear divide-and-conquer relations which is closely related to identities of the trigonometric and hyperbolic cotangent identities. Our investigation arises from an observation that the trigonometric cotangent function satisfies, among a number of other identities, the following identity:

$$\cot(3A) = \frac{\cot 2A \cot A - 1}{\cot 2A + \cot A}, \tag{1.4}$$

which leads to an RDAC relation of the form

$$x_{3n} = \frac{x_{2n}x_n - 1}{x_{2n} + x_n}. \tag{1.5}$$

This relation can be rewritten as

$$\frac{x_{3n} - i}{x_{3n} + i} = \left(\frac{x_n - i}{x_n + i}\right)\left(\frac{x_{2n} - i}{x_{2n} + i}\right) \quad \left(i = \sqrt{-1}\right), \tag{1.6}$$

which is a simpler looking RDAC relation of the form

$$U_{3n} = U_n U_{2n} \quad \left(U_n := \frac{x_n - i}{x_n + i}\right) \tag{1.7}$$

that can be immediately solved. Let us mention in passing that similar substitution techniques have been employed earlier in [4, 5].

Our first objective here is to find, in the next section, a closed form solution of

$$U_{bn} = U_n^{\alpha_1} U_{2n}^{\alpha_2} \cdots U_{(b-1)n}^{\alpha_{b-1}}, \tag{1.8}$$

an RDAC relation generalizing (1.7). Experiences from (1.7) with the cotangent function lead us to apply the results from our first objective to use such RDAC relations to characterize the trigonometric and hyperbolic tangent and cotangent functions, and this will be carried out in the following section as applications.

2. Closed Form Solutions

Before stating our main result, it is convenient to introduce a new notation. For $k \in \mathbb{N}$, let us write

$$\left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right)^{*k} = \sum_{i_1+i_2+\cdots+i_{b-1}=k} \binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}} V_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell}, \tag{2.1}$$

where $\binom{k}{i_1, i_2, \ldots, i_{b-1}} := k!/i_1! i_2! \cdots i_{b-1}!$ denote the customary multinomial coefficients. Our main result is:

Theorem 2.1. *Let $b \in \mathbb{N}$, $b \geq 2$, and $\alpha_1, \ldots, \alpha_{b-1} \in \mathbb{R}$. If the sequence $\{U_n\}_{n\geq 0}$ satisfies the RDAC relation*

$$U_{bn} = U_n^{\alpha_1} U_{2n}^{\alpha_2} \cdots U_{(b-1)n}^{\alpha_{b-1}} \quad (n \geq 1), \tag{2.2}$$

then for $\ell \not\equiv 0 \pmod{b}$, one has

$$U_{b^k \ell} \prod_{i_1+i_2+\cdots+i_{b-1}=\ell} U_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell}^{\binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}}} \quad (k \in \mathbb{N}). \tag{2.3}$$

Proof. Taking principal logarithms of (2.2), the relation becomes

$$V_{bn} = \alpha_1 V_n + \alpha_2 V_{2n} + \cdots + \alpha_{b-1} V_{(b-1)n} \quad (n \geq 1), \tag{2.4}$$

where $V_i = \log U_i$. For $\ell \not\equiv 0 \pmod{b}$, evaluating (2.4) at $n = b\ell$, we get

$$\begin{aligned}
V_{b^2\ell} &= \alpha_1 V_{b\ell} + \alpha_2 V_{b2\ell} + \cdots + \alpha_{b-1} V_{b(b-1)\ell} \\
&= \alpha_1 \left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right) \\
&\quad + \alpha_2 \left(\alpha_1 V_{2\ell} + \alpha_2 V_{2^2\ell} + \cdots + \alpha_{b-1} V_{2(b-1)\ell}\right) \\
&\quad + \cdots + \alpha_{b-1} \left(\alpha_1 V_{(b-1)\ell} + \alpha_2 V_{2(b-1)\ell} + \cdots + \alpha_{b-1} V_{(b-1)^2\ell}\right) \\
&= \alpha_1^2 V_\ell + \alpha_2^2 V_{2^2\ell} + \cdots + \alpha_{b-1}^2 V_{(b-1)^2\ell} \\
&\quad + 2\alpha_1\alpha_2 V_{2\ell} + \cdots + 2\alpha_i\alpha_j V_{ij\ell} + \cdots + 2\alpha_{b-2}\alpha_{b-1} V_{(b-2)(b-1)\ell}.
\end{aligned} \tag{2.5}$$

Using the notation introduced above, we see at once that

$$\begin{aligned}
V_{b\ell} &= \left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right)^{*1} \\
V_{b^2\ell} &= \left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right)^{*2}.
\end{aligned} \tag{2.6}$$

To finish the proof, we need only show that for all $k \in \mathbb{N}$

$$V_{b^k \ell} = \left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right)^{*k}. \tag{2.7}$$

We proceed by induction. For any $\ell \not\equiv 0 \pmod b$, assume that (2.7) holds up to k. Thus, by (2.4) and the induction hypothesis one has

$$\begin{aligned}
V_{b^{k+1}\ell} &= \alpha_1 V_{b^k \ell} + \alpha_2 V_{b^k 2\ell} + \cdots + \alpha_{b-1} V_{b^k (b-1)\ell} \\
&= \alpha_1 \left(\alpha_1 V_\ell + \alpha_2 V_{2\ell} + \cdots + \alpha_{b-1} V_{(b-1)\ell}\right)^{*k} + \alpha_2 \left(\alpha_1 V_{2\ell} + \alpha_2 V_{2^2\ell} + \cdots + \alpha_{b-1} V_{2(b-1)\ell}\right)^{*k} \\
&\quad + \cdots + \alpha_{b-1} \left(\alpha_1 V_{(b-1)\ell} + \alpha_2 V_{2(b-1)\ell} + \cdots + \alpha_{b-1} V_{(b-1)^2\ell}\right)^{*k} \\
&= \alpha_1 \sum_{i_1+i_2+\cdots+i_{b-1}=k} \binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}} V_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell} \\
&\quad + \alpha_2 \sum_{i_1+i_2+\cdots+i_{b-1}=k} \binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}} V_{(1^{i_1} 2)(2^{i_2} 2) \cdots ((b-1)^{i_{b-1}} 2)\ell} \\
&\quad + \cdots + \alpha_{b-1} \sum_{i_1+i_2+\cdots+i_{b-1}=k} \binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}} V_{(1^{i_1}(b-1))(2^{i_2}(b-1)) \cdots (b-1)^{i_{b-1}+1}\ell} \\
&= \sum_{i_1+i_2+\cdots+i_{b-1}=k+1} \binom{k}{i_1, i_2, \ldots, i_{b-1}} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_{b-1}^{i_{b-1}} V_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell}.
\end{aligned} \tag{2.8}$$

□

The cases $b = 2$ and 3 are of particular interest and we record them here for future reference.

Corollary 2.2. *(I) Let $\alpha \in \mathbb{R}$. If the sequence $\{U_n\}_{n \ge 0}$ satisfies the RDAC relation*

$$U_{2n} = U_n^{\alpha} \quad (n \ge 1), \tag{2.9}$$

then for $\ell \not\equiv 0 \pmod 2$, one has

$$U_{2^k \ell} = U_\ell^{\alpha^k} \quad (k \in \mathbb{N}). \tag{2.10}$$

(II) Let $\alpha_1, \alpha_2 \in \mathbb{R}$. If the sequence $\{U_n\}_{n \ge 0}$ satisfies the RDAC relation

$$U_{3n} = U_n^{\alpha_1} U_{2n}^{\alpha_2} \quad (n \ge 1), \tag{2.11}$$

then for $\ell \not\equiv 0 \pmod 3$, one has

$$U_{3^k \ell} = U_\ell^{\binom{k}{0}\alpha_1^k \alpha_2^0} U_{2\ell}^{\binom{k}{1}\alpha_1^{k-1}\alpha_2} \cdots U_{2^k \ell}^{\binom{k}{k}\alpha_1^0 \alpha_2^k} \quad (k \in \mathbb{N}). \tag{2.12}$$

3. Applications

We now apply the result of Theorem 2.1 and Corollary 2.2 to several RDAC relations including those that can be used to characterize the trigonometric and hyperbolic tangent and cotangent functions.

Proposition 3.1. *(I) Suppose that the sequence* $\{x_n\}_{n\geq 0}$ *satisfies an RDAC relation of the form*

$$x_{2n} = \frac{x_n^2 - 1}{2x_n} \quad (n \geq 1). \tag{3.1}$$

For $\ell \not\equiv 0 \pmod 2$ *and* $k \in \mathbb{N}$, *if the condition* $2^k\theta_\ell \not\equiv 0 \pmod{2\pi}$ *is fulfilled, then*

$$x_{2^k\ell} = \cot\left(\frac{-2^k\theta_\ell}{2}\right) = \cot\left(2^k \operatorname{arccot} x_\ell\right), \tag{3.2}$$

where

$$\theta_\ell = -2 \operatorname{arccot} x_\ell. \tag{3.3}$$

(II) Assume that the sequence $\{x_n\}_{n\geq 0}$ *satisfies an RDAC relation of the form*

$$x_{2n} = \frac{2x_n}{1 - x_n^2} \quad (n \geq 0). \tag{3.4}$$

For $\ell \not\equiv 0 \pmod 2$ *and* $k \in \mathbb{N}$, *if* $2^k\theta_\ell$ *is not an odd multiple of* π, *then*

$$x_{2^k\ell} = \tan\left(\frac{-2^k\theta_\ell}{2}\right) = \tan\left(2^k \arctan x_\ell\right), \tag{3.5}$$

where

$$\theta_\ell = -2 \arctan x_\ell. \tag{3.6}$$

(III) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{2n} = \frac{x_n^2 + 1}{2x_n} \quad (n \geq 0), \tag{3.7}$$

then, for $\ell \not\equiv 0 \pmod 2$, $k \in \mathbb{N}$, *one has*

$$x_{2^k\ell} = \coth\left(\frac{-2^k\theta_\ell}{2}\right) = \coth\left(2^k \operatorname{arccoth} x_\ell\right), \tag{3.8}$$

where

$$\theta_\ell = -2 \operatorname{arccoth} x_\ell. \tag{3.9}$$

(IV) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{2n} = \frac{2x_n}{1+x_n^2} \quad (n \geq 0), \tag{3.10}$$

then, for $\ell \not\equiv 0 \pmod 2$, $k \in \mathbb{N}$, *one has*

$$x_{2^k\ell} = \tanh\left(\frac{-2^k\theta_\ell}{2}\right) = \tanh\left(2^k \operatorname{arctanh} x_\ell\right), \tag{3.11}$$

where

$$\theta_\ell = -2 \operatorname{arctanh} x_\ell. \tag{3.12}$$

Proof. (I) As seen in Section 1, the RDAC relation (3.1) is equivalent to

$$U_{2n} = U_n^2 \quad \left(U_n = \frac{x_n - i}{x_n + i}\right), \tag{3.13}$$

whose solution is, by virtue of Corollary 2.2, $U_{2^k\ell} = U_\ell^{2^k}$. Thus,

$$\frac{x_{2^k\ell} - i}{x_{2^k\ell} + i} = \left(\frac{x_\ell - i}{x_\ell + i}\right)^{2^k}. \tag{3.14}$$

Setting $e^{i\theta_\ell} = (x_\ell - i)/(x_\ell + i)$, one has

$$x_{2^k\ell} = i\left(\frac{1+e^{i2^k\theta_\ell}}{1-e^{i2^k\theta_\ell}}\right) = \cot\left(\frac{-2^k\theta_\ell}{2}\right) = \cot\left(2^k \operatorname{arccot} x_\ell\right), \tag{3.15}$$

provided $2^k\theta_\ell \not\equiv 0 \pmod{2\pi}$.

(II) Substituting x_n by $1/x_n$ turns (3.4) into (3.1) and so the result follows at once from part (I).

(III) Substituting x_n by ix_n in (3.7) turns it into a rational recursive equation of the form (3.1) and so part (I) yields the desired result.

(IV) Replacing x_n by ix_n in (3.10), we get a rational recursive equation of the form (3.4) and part (II) yields the result. □

Remark 3.2. Although the substitution x_n by $1/x_n$ employed in part (II) of Proposition 3.1 allows us to obtain a closed form solution of the RDAC relation (3.4), there remains a difficulty should there exist integer N such that $x_N = 0$. To overcome this shortcoming, we may either interpret the infinite value of the two expressions on both sides of the solution as equal or repeat the technique used in the proof of Proposition 3.1 to solve (3.4).

Proposition 3.3. *(I) Suppose that the sequence $\{x_n\}_{n\geq 0}$ satisfies an RDAC relation of the form*

$$x_{3n} = \frac{x_{2n}x_n - 1}{x_{2n} + x_n} \quad (n \geq 1). \tag{3.16}$$

For $\ell \not\equiv 0 \pmod 3$ and $k \in \mathbb{N}$, if the condition

$$\binom{k}{0}\theta_{\ell} + \binom{k}{1}\theta_{2\ell} + \cdots + \binom{k}{k}\theta_{2^k\ell} \not\equiv 0 \pmod{2\pi} \tag{3.17}$$

is fulfilled, then

$$\begin{aligned} x_{3^k\ell} &= \cot\left(\frac{-\binom{k}{0}\theta_{\ell} - \binom{k}{1}\theta_{2\ell} - \cdots - \binom{k}{k}\theta_{2^k\ell}}{2}\right) \\ &= \cot\left(\binom{k}{0}\operatorname{arccot} x_{\ell} + \binom{k}{1}\operatorname{arccot} x_{2\ell} + \cdots + \binom{k}{k}\operatorname{arccot} x_{2^k\ell}\right), \end{aligned} \tag{3.18}$$

where

$$\theta_j = -2 \operatorname{arccot} x_j \quad \left(j \in \left\{\ell, 2\ell, \ldots, 2^k\ell\right\}\right). \tag{3.19}$$

(II) Assume that the sequence $\{x_n\}_{n\geq 0}$ satisfies an RDAC relation of the form

$$x_{3n} = \frac{x_n + x_{2n}}{1 - x_n x_{2n}} \quad (n \geq 0). \tag{3.20}$$

For $\ell \not\equiv 0 \pmod 3$ and $k \in \mathbb{N}$, if $\theta_{\ell} + \binom{k}{1}\theta_{2\ell} + \cdots + \binom{k}{k}\theta_{2^k\ell}$ is not an odd multiple of π, then

$$\begin{aligned} x_{3^k\ell} &= \tan\left(\frac{-\binom{k}{0}\theta_{\ell} - \binom{k}{1}\theta_{2\ell} - \cdots - \binom{k}{k}\theta_{2^k\ell}}{2}\right) \\ &= \tan\left(\binom{k}{0}\arctan x_{\ell} + \binom{k}{1}\arctan x_{2\ell} + \cdots + \binom{k}{k}\arctan x_{2^k\ell}\right), \end{aligned} \tag{3.21}$$

where

$$\theta_j = -2\arctan x_j \quad \left(j \in \left\{\ell, 2\ell, \ldots, 2^k\ell\right\}\right). \tag{3.22}$$

(III) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{3n} = \frac{x_n x_{2n} + 1}{x_n + x_{2n}} \quad (n \geq 0), \tag{3.23}$$

then, for $\ell \not\equiv 0 \pmod{3}, k \in \mathbb{N}$, *one has*

$$\begin{aligned} x_{3^k\ell} &= \coth\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{2\ell} - \cdots - \binom{k}{k}\theta_{2^k\ell}}{2}\right) \\ &= \coth\left(\binom{k}{0}\operatorname{arccoth} x_\ell + \binom{k}{1}\operatorname{arccoth} x_{2\ell} + \cdots + \binom{k}{k}\operatorname{arccoth} x_{2^k\ell}\right), \end{aligned} \tag{3.24}$$

where

$$\theta_j = -2\operatorname{arccoth} x_j \quad \left(j \in \left\{\ell, 2\ell, \ldots, 2^k\ell\right\}\right). \tag{3.25}$$

(IV) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{3n} = \frac{x_n + x_{2n}}{1 + x_n x_{2n}} \quad (n \geq 0), \tag{3.26}$$

then, for $\ell \not\equiv 0 \mod 3$, $k \in \mathbb{N}$, *one has*

$$\begin{aligned} x_{3^k\ell} &= \tanh\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{2\ell} - \cdots - \binom{k}{k}\theta_{2^k\ell}}{2}\right) \\ &= \tanh\left(\binom{k}{0}\operatorname{arctanh} x_\ell + \binom{k}{1}\operatorname{arctanh} x_{2\ell} + \cdots + \binom{k}{k}\operatorname{arctanh} x_{2^k\ell}\right), \end{aligned} \tag{3.27}$$

where

$$\theta_j = -2\operatorname{arctanh} x_j \quad \left(j \in \left\{\ell, 2\ell, \ldots, 2^k\ell\right\}\right). \tag{3.28}$$

Proof. (I) As seen in Section 1, the RDAC relation (3.16) is equivalent to

$$U_{3n} = U_n U_{2n} \quad \left(U_n = \frac{x_n - i}{x_n + i}\right), \tag{3.29}$$

whose solution is, by virtue of Corollary 2.2, $U_{3^k\ell} = U_\ell^{\binom{k}{0}} U_{2\ell}^{\binom{k}{1}} \cdots U_{2^k\ell}^{\binom{k}{k}}$. Thus,

$$\frac{x_{3^k\ell} - i}{x_{3^k\ell} + i} = \left(\frac{x_\ell - i}{x_\ell + i}\right)^{\binom{k}{0}} \left(\frac{x_{2\ell} - i}{x_{2\ell} + i}\right)^{\binom{k}{1}} \cdots \left(\frac{x_{2^k\ell} - i}{x_{2^k\ell} + i}\right)^{\binom{k}{k}}. \tag{3.30}$$

Setting $e^{i\theta_j} = (x_j - i)/(x_j + i)$ $(j \in \{\ell, 2\ell, \ldots, 2^k\ell\})$, one has

$$\begin{aligned} x_{3^k\ell} &= i\left(\frac{1 + e^{i\left(\binom{k}{0}\theta_\ell + \cdots + \binom{k}{k}\theta_{2^k\ell}\right)}}{1 - e^{i\left(\binom{k}{0}\theta_\ell + \cdots + \binom{k}{k}\theta_{2^k\ell}\right)}}\right) = \cot\left(\frac{-\binom{k}{0}\theta_\ell - \cdots - \binom{k}{k}\theta_{2^k\ell}}{2}\right) \\ &= \cot\left(\binom{k}{0}\operatorname{arccot} x_\ell + \binom{k}{1}\operatorname{arccot} x_{2\ell} + \cdots + \binom{k}{k}\operatorname{arccot} x_{2^k\ell}\right), \end{aligned} \tag{3.31}$$

provided $\binom{k}{0}\theta_\ell + \cdots + \binom{k}{k}\theta_{2^k\ell} \not\equiv 0 \pmod{2\pi}$.

(II) Substituting x_n by $1/x_n$ turns (3.20) into (3.16) and so the result follows at once from part (I).

(III) Substituting x_n by ix_n in (3.23) turns it into a rational recursive equation of the form (3.16), and so part (I) yields the desired result.

(IV) Replacing x_n by ix_n in (3.26), we get a rational recursive equation of the form (3.20), and part (II) yields the result. □

Remark 3.4. As in the remark following Proposition 3.1, the substitution x_n by $1/x_n$ in part (II) causes no harm should there exists integer N such that $x_N = 0$ by either interpreting the infinite value of the expressions on both sides of the solution as equal. Alternately, we may repeat the technique used in the proof of Proposition 3.3 to solve (3.20).

As for the case of general b, an entirely analogous proof as that in Proposition 3.3, which we omit here, leads to Proposition 3.5.

Proposition 3.5. *Let $b \in \mathbb{N}$, $b \geq 2$.*

(I) Suppose that the sequence $\{x_n\}_{n\geq 0}$ satisfies an RDAC relation of the form

$$x_{bn} = \frac{x_{(b-1)n}x_n - 1}{x_{(b-1)n} + x_n} \quad (n \geq 1). \tag{3.32}$$

For $\ell \not\equiv 0 \pmod{b}$ and $k \in \mathbb{N}$, if the condition

$$\binom{k}{0}\theta_\ell + \binom{k}{1}\theta_{(b-1)\ell} + \cdots + \binom{k}{k}\theta_{(b-1)^k\ell} \not\equiv 0 \mod 2\pi \tag{3.33}$$

is fulfilled, then

$$\begin{aligned} x_{b^k\ell} &= \cot\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{(b-1)\ell} - \cdots - \binom{k}{k}\theta_{(b-1)^k\ell}}{2}\right) \\ &= \cot\left(\binom{k}{0}\operatorname{arccot} x_\ell + \binom{k}{1}\operatorname{arccot} x_{(b-1)\ell} + \cdots + \binom{k}{k}\operatorname{arccot} x_{(b-1)^k\ell}\right), \end{aligned} \tag{3.34}$$

where

$$\theta_j = -2 \operatorname{arccot} x_j \quad \left(j \in \left\{\ell, (b-1)\ell, \ldots, (b-1)^k \ell\right\}\right). \tag{3.35}$$

(II) Assume that the sequence $\{x_n\}_{n\geq 0}$ *satisfies an RDAC relation of the form*

$$x_{bn} = \frac{x_n + x_{(b-1)n}}{1 - x_n x_{(b-1)n}} \quad (n \geq 0). \tag{3.36}$$

For $\ell \not\equiv 0 \pmod{b}$ *and* $k \in \mathbb{N}$, *if* $\theta_\ell + \binom{k}{1}\theta_{(b-1)\ell} + \cdots + \binom{k}{k}\theta_{(b-1)^k\ell}$ *is not an odd multiple of* π, *then*

$$\begin{aligned} x_{b^k\ell} &= \tan\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{(b-1)\ell} - \cdots - \binom{k}{k}\theta_{(b-1)^k\ell}}{2}\right) \\ &= \tan\left(\binom{k}{0}\arctan x_\ell + \binom{k}{1}\arctan x_{(b-1)\ell} + \cdots + \binom{k}{k}\arctan x_{(b-1)^k\ell}\right), \end{aligned} \tag{3.37}$$

where

$$\theta_j = -2 \arctan x_j \quad \left(j \in \left\{\ell, (b-1)\ell, \ldots, (b-1)^k \ell\right\}\right). \tag{3.38}$$

(III) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{bn} = \frac{x_n x_{(b-1)n} + 1}{x_n + x_{(b-1)n}} \quad (n \geq 0), \tag{3.39}$$

then, for $\ell \not\equiv 0 \pmod{b}$, $k \in \mathbb{N}$, *one has*

$$\begin{aligned} x_{b^k\ell} &= \coth\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{(b-1)\ell} - \cdots - \binom{k}{k}\theta_{(b-1)^k\ell}}{2}\right) \\ &= \coth\left(\binom{k}{0}\operatorname{arccoth} x_\ell + \binom{k}{1}\operatorname{arccoth} x_{(b-1)\ell} + \cdots + \binom{k}{k}\operatorname{arccoth} x_{(b-1)^k\ell}\right), \end{aligned} \tag{3.40}$$

where

$$\theta_j = -2 \operatorname{arccoth} x_j \quad \left(j \in \left\{\ell, (b-1)\ell, \ldots, (b-1)^k \ell\right\}\right). \tag{3.41}$$

(IV) If the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{bn} = \frac{x_n + x_{(b-1)n}}{1 + x_n x_{(b-1)n}} \quad (n \geq 0), \tag{3.42}$$

then, for $\ell \not\equiv 0 \pmod{b}$, $k \in \mathbb{N}$, *one has*

$$\begin{aligned} x_{b^k \ell} &= \tanh\left(\frac{-\binom{k}{0}\theta_{\ell} - \binom{k}{1}\theta_{(b-1)\ell} - \cdots - \binom{k}{k}\theta_{(b-1)^k \ell}}{2} \right) \\ &= \tanh\left(\binom{k}{0} \operatorname{arctanh} x_{\ell} + \binom{k}{1} \operatorname{arctanh} x_{(b-1)\ell} + \cdots + \binom{k}{k} \operatorname{arctanh} x_{(b-1)^k \ell} \right), \end{aligned} \tag{3.43}$$

where

$$\theta_j = -2 \operatorname{arctanh} x_j \quad \left(j \in \left\{ \ell, (b-1)\ell, \ldots, (b-1)^k \ell \right\} \right). \tag{3.44}$$

When all the exponents α_j in (2.2) are equal to 1, RDAC relations, even more general than those in Proposition 3.5 which can be explicitly solved by our device, are given in the next proposition.

Proposition 3.6. *Let* $b \in \mathbb{N}$, $b \geq 2$, $w \in \mathbb{C} \setminus \{0\}$. *If the sequence* $\{x_n\}_{n \geq 0}$ *satisfies*

$$\frac{x_{bn}}{w} = \frac{(x_n + w)(x_{2n} + w) \cdots \left(x_{(b-1)n} + w\right) + (x_n - w)(x_{2n} - w) \cdots \left(x_{(b-1)n} - w\right)}{(x_n + w)(x_{2n} + w) \cdots \left(x_{(b-1)n} + w\right) - (x_n - w)(x_{2n} - w) \cdots \left(x_{(b-1)n} - w\right)}, \tag{3.45}$$

then for $\ell \not\equiv 0 \pmod{b}$ *and* $k \in \mathbb{N}$ *one has*

$$x_{b^k \ell} = w \left(\frac{A_+ + A_-}{A_+ - A_-} \right), \tag{3.46}$$

provided the values exist, where

$$\begin{aligned} A_+ &:= \prod_{i_1 + i_2 + \cdots + i_{b-1} = \ell} \left(x_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell} + w \right)^{\binom{k}{i_1, i_2, \ldots, i_{b-1}}} \\ A_- &:= \prod_{i_1 + i_2 + \cdots + i_{b-1} = \ell} \left(x_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell} - w \right)^{\binom{k}{i_1, i_2, \ldots, i_{b-1}}}. \end{aligned} \tag{3.47}$$

Proof. Rewriting (3.45), we get

$$\frac{x_{bn} - w}{x_{bn} + w} = \left(\frac{x_n - w}{x_n + w} \right) \left(\frac{x_{2n} - w}{x_{2n} + w} \right) \cdots \left(\frac{x_{(b-1)n} - w}{x_{(b-1)n} + w} \right) \tag{3.48}$$

or

$$U_{bn} = U_n U_{2n} \cdots U_{(b-1)n} \qquad \left(U_j = \frac{x_j - w}{x_j + w}, \; j \in \{n, 2n, \ldots, (b-1)n\} \right). \tag{3.49}$$

Theorem 2.1 thus yields for $\ell \not\equiv 0 \pmod{b}$ and $k \in \mathbb{N}$

$$U_{b^k \ell} = \prod_{i_1+i_2+\cdots+i_{b-1}=\ell} U_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell}^{\binom{k}{i_1, i_2, \ldots, i_{b-1}}}, \tag{3.50}$$

that is,

$$\frac{x_{b^k \ell} - w}{x_{b^k \ell} + w} = \prod_{i_1+i_2+\cdots+i_{b-1}=\ell} \left(\frac{x_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell} - w}{x_{1^{i_1} 2^{i_2} \cdots (b-1)^{i_{b-1}} \ell} + w} \right)^{\binom{k}{i_1, i_2, \ldots, i_{b-1}}}, \tag{3.51}$$

and the result follows. □

4. Global Behaviors

It is often desirable to know about global behaviors of the solutions of recursive equations, such as those in [6]. Using the explicit forms found above, this question is easily solved for RDAC relations in Proposition 3.5 with $b = 2$.

Proposition 4.1. *Let the notation be as in Proposition 3.1.*

(I) Suppose that the sequence $\{x_n\}_{n \geq 0}$ satisfies an RDAC relation of the form

$$x_{2n} = \frac{x_n^2 - 1}{2x_n} \quad (n \geq 1). \tag{4.1}$$

For each fixed $\ell \not\equiv 0 \pmod{2}$ and $k \in \mathbb{N}$,

(a) *if θ_ℓ is a rational multiple of π, then either $\{x_{2^k \ell}\}_{k \in \mathbb{N}}$ diverges in finitely many steps or $\{x_{2^k \ell}\}$ is periodic;*

(b) *if θ_ℓ is not a rational multiple of π, then $x_{2^k \ell}$ exists for all $k \in \mathbb{N}$ and the sequence $\{x_{2^k \ell}\}_{k \in \mathbb{N}}$ is never periodic.*

(II) Suppose that the sequence $\{x_n\}_{n \geq 0}$ satisfies an RDAC relation of the form

$$x_{2n} = \frac{2x_n}{1 - x_n^2} \quad (n \geq 0). \tag{4.2}$$

For each fixed $\ell \not\equiv 0 \pmod{2}$,

(a) *if θ_ℓ is a rational multiple of π, then either $\{x_{2^k \ell}\}_{k \in \mathbb{N}}$ diverges in finitely many steps or $\{x_{2^k \ell}\}$ is periodic;*

(b) *if θ_ℓ is not a rational multiple of π, then $x_{2^k \ell}$ exists for all $k \in \mathbb{N}$ and the sequence $\{x_{2^k \ell}\}_{k \in \mathbb{N}}$ is never periodic.*

(III) Suppose that the sequence $\{x_n\}_{n \geq 0}$ satisfies

$$x_{2n} = \frac{x_n^2 + 1}{2x_n} \quad (n \geq 0). \tag{4.3}$$

For each fixed $\ell \not\equiv 0 \pmod 2$,

(a) *if* $\theta_\ell = 0$, *then the sequence* $\{x_{2^k\ell}\}_{k\geq 1}$ *does not exist;*

(b) *if* $\theta_\ell > 0$, *the sequence* $\{x_{2^k\ell}\}$ *is strictly decreasing in the interval* $[\coth(\theta_\ell), 1)$;

(c) *if* $\theta_\ell < 0$, *then* $\{x_{2^k\ell}\}_{k\geq 1}$ *is strictly increasing in the interval* $[\coth(\theta_\ell), -1)$.

(IV) Suppose that the sequence $\{x_n\}_{n\geq 0}$ *satisfies*

$$x_{2n} = \frac{2x_n}{1 + x_n^2} \quad (n \geq 0). \tag{4.4}$$

For each fixed $\ell \not\equiv 0 \pmod 2$,

(z) *if* $\theta_\ell = 0$, *then* $\{x_{2^k\ell}\}_{k\in\mathbb{N}}$ *is the zero sequence;*

(b) *if* $\theta_\ell > 0$, *then the sequence* $\{x_{2^k\ell}\}$ *is strictly increasing in* $[\tanh(\theta_\ell), 1)$;

(c) *if* $\theta_\ell < 0$, *the sequence is strictly decreasing in* $[\tanh(\theta_\ell), -1)$.

Proof. (I) From part (I) of Proposition 3.1, we know that

$$x_{2^k\ell} = \cot\left(-2^{k-1}\theta_\ell\right) \tag{4.5}$$

provided $2^k\theta_\ell \not\equiv 0 \pmod{2\pi}$. Consider the case where θ_ℓ is a rational multiple of π, say,

$$\theta_\ell = \frac{m\pi}{t} \quad \text{with } m, t(>0) \in \mathbb{Z}, \quad \gcd(m, t) = 1. \tag{4.6}$$

If t is a multiple of 2, then it is easily checked that $\{x_{2^k\ell}\}_{k\in\mathbb{N}}$ diverges in finitely many steps. If $t \geq 2$ is not a multiple of 2, let $t = 2^v T$, where $2^v \parallel t$, $T \geq 3$. Observe that for all large $n \in \mathbb{N}$, when evaluating the values of cotangent, we need only look at

$$2^n\theta_\ell = \frac{2^n m\pi}{t} = \frac{2^{n-v} m\pi}{T} \pmod{2\pi}, \tag{4.7}$$

which is equivalent to looking at

$$G_n := 2^{n-v} m \pmod{2T}. \tag{4.8}$$

Since each G_n takes at most $2T$ values and the sequence $\{x_{2^k\ell}\}_{k\in\mathbb{N}}$ is infinite, there are positive integers $N_1 < N_2$ such that $G_{N_1} = G_{N_2}$, which in turn implies that $\{x_{2^k\ell}\}_{k\in\mathbb{N}}$ is periodic.

Finally, if θ_ℓ is not a rational multiple of π, then $2^{k-1}\theta_\ell$ is not a multiple of π showing that the sequence $\{x_{2^k\ell}\}_{k\in\mathbb{N}}$ is well defined and never periodic.

The proof of part (II) is similar to that of part (I).

(III) If $\theta_\ell = 0$, then the values $x_{2^k\ell} = \coth(2^k\theta_\ell)$ become infinite for all $k \in \mathbb{N}$ and part (a) follows. Since $x_{2^k\ell} = \coth(2^k\theta_\ell)$ is a strictly decreasing (resp. increasing) function of k according as $\theta_\ell > 0$ (resp. $\theta_\ell < 0$), the results in (b) and (c) are immediate.

(IV) If $\theta_\ell = 0$, then $x_{2^k\ell} = \tanh(\theta_\ell) = 0$. Arguments for the other two cases $\theta_\ell > 0$ and $\theta_\ell < 0$ are similar to those in part (III). □

Note from Proposition 4.1 that global behaviors of solutions in the case $b = 2$ depend solely on the single value θ_ℓ. The situation when $b \geq 3$ is more complex since their global behaviors depend heavily on the variable k as we see in the following illustration. Keeping the notation of Proposition 3.5, suppose that the sequence $\{x_n\}_{n\geq 0}$ satisfies an RDAC relation of the form

$$x_{bn} = \frac{x_{(b-1)n}x_n - 1}{x_{(b-1)n} + x_n} \quad (n \geq 1). \tag{4.9}$$

From part (I) of Proposition 3.5, we know that

$$x_{b^k\ell} = \cot\left(\frac{-\binom{k}{0}\theta_\ell - \binom{k}{1}\theta_{(b-1)\ell} - \cdots - \binom{k}{k}\theta_{(b-1)^k\ell}}{2}\right). \tag{4.10}$$

This explicit form shows that, for each fixed $\ell \not\equiv 0 \pmod{b}$, the behavior of $x_{b^k\ell}$ considered as a function of $k \in \mathbb{N}$ depends on all $\theta_\ell, \ldots, \theta_{(b-1)^k\ell}$, and we can merely infer that the values $x_{b^k\ell}$ are well defined (i.e., finite) if and only if

$$\binom{k}{0}\theta_\ell + \binom{k}{1}\theta_{(b-1)\ell} + \cdots + \binom{k}{k}\theta_{(b-1)^k\ell} \not\equiv (\bmod\ 2\pi). \tag{4.11}$$

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