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Research Article

The Stability Cone for a Difference Matrix Equation with Two Delays

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We provide geometric algorithms for checking the stability of matrix difference equations $x_n = Ax_{n-m} + Bx_{n-k}$ with two delays m, k such that the matrix $AB - BA$ is nilpotent. We give examples of how our results can be applied to the study of the stability of neural networks.

1. Introduction

The problem of the stability of the equation

$$x_n = ax_{n-m} + bx_{n-k} \tag{1.1}$$

with real coefficients a, b is basically solved [1–4].

The stability of matrix (1.1) with special 2×2 matrices a, b and $m = 1$ was studied in [5, 6]. The case $m = 1$, $a = I$, where I is the identity matrix, was studied in [7] without dimension restriction. In the paper [8] the dimension is also not bounded, and the results of [7] are generalized: it assumes that $a = \alpha I$, $\alpha \in \mathbb{R}$, $0 \leqslant \alpha \leqslant 1$. The representation of the solutions of (1.1) with commuting matrices a, b is given in [9] without considering a stability problem.

To the best of the authors' knowledge, the stability of (1.1) with complex coefficients a, b has not been studied yet.

In this paper we provide geometric algorithms for checking the stability of (1.1) with two delays $m, k \in \mathbb{Z}_+$, $k > m \geqslant 1$, for two cases: (1) a, b are complex numbers, (2) a, b

are simultaneously triangularizable matrices. The results of this paper are based on the D-decomposition method (parameter plane method) [10, 11].

Matrices a, b commute in all the above articles, which implies the possibility of simultaneous triangularization [12]. Therefore, our method can be applied to all of the above-mentioned cases. In the present paper the case $m = 1$ is studied along with other values $m \in \mathbb{Z}_+$. The case $m = 1$ is very important, so we separately examined it in detail in the paper [13].

The paper is organized as follows. In Section 2, we introduce the curve of D-decomposition and point out its key property of symmetry. In Section 3, we define the basic ovals and formulate their properties. In Section 4, we define a property of ρ-stability, which coincides with usual stability if $\rho = 1$. Later in that section we solve a problem of geometric checking ρ-stability of (1.1) with positive real a and complex b. In Sections 5 and 6, we give a method of geometric checking the stability of (1.1) with complex coefficients and simultaneously triangularizable matrices, correspondingly. Finally, in Section 7, we employ our results to derive the stability conditions for neural nets.

2. D-Decomposition Curve for Given k, m, a, ρ

Consider the scalar variant of (1.1). The characteristic polynomial for (1.1) is

$$f(\lambda) = \lambda^k - a\lambda^{k-m} - b. \tag{2.1}$$

If $k = dk_1, m = dm_1$, $d > 1$, then the trajectory of (1.1) splits into d independent trajectories, and degree of polynomial (2.1) gets smaller after the substitution $\lambda^d = \mu$:

$$f_1(\lambda) = \mu^{k_1} - a\mu^{k_1-m_1} - b. \tag{2.2}$$

Therefore, often (but not always) we will assume that the delays m, k are relatively prime.

Definition 2.1. D-decomposition curve for given $k, m \in \mathbb{Z}_+$, $a \in \mathbb{C}$, $\rho \in \mathbb{R}_+$ is a curve on the complex plane of the variable b defined by the equation

$$b(\omega) = \rho^k \exp(ik\omega) - |a|\rho^{k-m} \exp(i(k-m)\omega), \quad \omega \in \mathbb{R}. \tag{2.3}$$

Parameter ω moves along the interval of length 2π, the starting point of which is not fixed. We also call the curve (2.3) hodograph.

In this and the next sections we will consider only real positive values of a. Starting from Section 5 we will get rid of this restriction. Obviously, if we assume in (2.1) that $b = b(\omega)$ and $a \in \mathbb{R}$, $a \geq 0$, then (2.1) will have a root $\lambda = \rho \exp(i\omega)$. Hodograph (2.3) splits the complex plane into the connected components. This decomposition is called the D-decomposition [10]. If we put $a \in \mathbb{R}$, $a \geq 0$, and substitute any two internal points b_1, b_2 from one of the connected components of D-decomposition for coefficient b in polynomial (2.1), then the polynomials obtained will have equal number of roots inside the circle of radius ρ centred at the origin of the complex plane. In particular, if $a \in \mathbb{R}$, $a \geqslant 0$, $\rho = 1$ and the substitution of some inner point of a component of D-decomposition into (1.1) gives a stable equation, then the substitution of any other internal point from that component also gives a stable equation.

Let us point out a key property of symmetry of hodograph (2.3).

Lemma 2.2 (symmetry). *If k, m are coprime, then hodograph* (2.3) *is invariant under the rotation by $2\pi/m$.*

Proof. For coprime k, m there exist $s, t \in \mathbb{Z}_+$ such that

$$ks - mt = 1. \tag{2.4}$$

From (2.3), (2.4) it follows that

$$\exp\left(\frac{i2\pi}{m}\right) b(\omega) = b\left(\omega + \frac{2\pi s}{m}\right). \tag{2.5}$$

Lemma 2.2 is proved. □

From now and further we will assume that $-\pi < arg z \leqslant \pi$ for any complex z, while $\operatorname{Arg} z$ will be assumed as multivalued function, and the equality $\operatorname{Arg} z = v$ will mean that one of the values of $\operatorname{Arg} z$ equals to v.

The following lemma asserts that some part of the complex plane is free from points of the hodograph $b(\omega)$.

Lemma 2.3. *Let k, m be coprime, $k > m > 1$, $|a| < \rho^m$. Let $\omega_1 \in [0; \pi/m]$ be the least positive root of the equation $\arg b(\omega) = \pi/m$, and let $0 < \omega_2 \leqslant \omega_1$. Then, for any $\omega \in \mathbb{R}$, from*

$$\arg b(\omega) = \arg b(\omega_2), \tag{2.6}$$

it follows that $|b(\omega)| \geqslant |b(\omega_2)|$.

Proof. The function $|b(\omega)|$ is $2\pi/m$-periodic, increasing in $[(2j-2)\pi/m, (2j-1)\pi/m]$ for any $j \in \mathbb{Z}$ and decreasing in $[(2j-1)\pi/m, 2j\pi/m]$. In addition, $|b(\omega + \pi/m)| = |b(-\omega + \pi/m)|$ for any ω. Let us assume, in order to get a contradiction, that (2.6) and $|b(\omega)| < |b(\omega_2)|$ are true for some $\omega > 0$. Then there exists a positive integer $s < m$ and $\delta \in \mathbb{R}$ such that

$$\omega = \frac{2\pi s}{m} + \delta, \quad |\delta| < \omega_2. \tag{2.7}$$

From (2.3), (2.7) it follows that

$$\begin{aligned} \operatorname{Arg} b(\omega) &= (k-m)\omega + \operatorname{Arg}\left(\rho^m \exp(im\omega) - |a|\right) \\ &= \frac{2\pi ks}{m} + \operatorname{Arg}\left(\rho^m \exp(im\delta) - |a|\right) + (k-m)\delta = \frac{2\pi ks}{m} + \operatorname{Arg} b(\delta). \end{aligned} \tag{2.8}$$

Since k, m are coprime and $s < m$, let us find natural numbers j, q such that $ks = mj + q$, $m > q \geqslant 1$. Then (2.8) implies

$$\operatorname{Arg} b(\omega) = \frac{2\pi q}{m} + \operatorname{Arg} b(\delta), \tag{2.9}$$

which contradicts (2.6) and the inequality $|\arg b(\delta)| < \pi/m$ following from (2.7). Lemma 2.3 is proved. □

3. Basic Ovals

For hodograph (2.3) the equality $b'_\omega(0) = i\rho^{k-m}(k\rho^m - |a|(k-m))$ takes place. If

$$|a| < \frac{\rho^m k}{k-m}, \tag{3.1}$$

then let us look at a closed curve on the complex plane, which we call the basic oval. This curve is an image of an interval $[-\omega_1, \omega_1]$ under the map $b(\omega)$ defined by (2.3). Here $\omega_1 \in (0, \pi/k]$ is the least positive root of the equation $\arg b(\omega) = \pi$. We are also interested in those parts of hodograph (2.3) that can be obtained by the rotation of the basic oval by the angles $2\pi j/m$, $j \in \mathbb{Z}$, $0 \leqslant j < m$ (see Lemma 2.2). We also call them the basic ovals. Here is a formal definition.

Definition 3.1. Let k, m be coprime, $k > m \geqslant 1$, $j \in \mathbb{Z}$, $0 \leqslant j < m$, and let (2.4), (3.1) *hold. The basic oval L_j for* (1.1) *is a closed curve given by* (2.3), *where the variable ω runs from $(-\omega_1 + 2\pi js/m)$ to $(\omega_1 + 2\pi js/m)$, where ω_1 is the least positive root of the equation*

$$\arg b(\omega) = \pi. \tag{3.2}$$

From Lemma 2.2 and formula (2.5) it follows that all m basic ovals can be obtained from L_0 by rotation by the angles $2\pi j/m$, $j = 0, 1, \ldots, (m-1)$.

Considering Definition 3.1, we get the following. For existence of the basic oval it is necessary that $|a| < \rho^m k/(k-m)$. If $m > 1$ and $|a| > \rho^m$, then the complex number 0 is outside any oval, and the intersection of all ovals is empty. If $m = 1$, then for fixed k, ρ, $|a| \in [0, \rho^m k/(k-m))$ the basic oval L_0 is unique. That is why the results related to the stability of (1.1) are different for $m = 1$ and $m > 1$.

The basic oval L_j decreases as $|a|$ increases from 0, and it shrinks to the point $b = -\exp(2\pi j/m)\rho^k/(k-1)$ as $|a|$ reaches $\rho^m k/(k-m)$ (Figure 1 for $m > 1$ and Figure 2 for $m = 1$).

Lemma 3.2. *Let k, m be coprime, $k > m \geqslant 1$, $j \in \mathbb{Z}$, $0 \leqslant j < m$, $a \in \mathbb{R}$, and $0 \leqslant a < \rho^m k/(k-m)$. If the complex number b lies outside the basic oval L_j, then characteristic polynomial* (2.1) *has a root λ such that $|\lambda| > \rho$.*

Proof. Let us fix k, m, a, ρ, j, and let the complex number b lie outside the basic oval L_j. Having changed ρ to $R > \rho$ in Definition 3.1 let us consider the system of ovals $L_j(R)$. If $R \to \infty$, then the ovals $L_j(R)$ include a circle of an arbitrarily large radius. Therefore, there exists R_0 such that the point b is inside the oval $L_j(R_0)$. The ovals L_j and $L_j(R_0)$ are homotopic, therefore, there exists $R_1 \in (\rho, R_0]$ such that b lies on the curve $L_j(R_1)$, which means the existence of a root λ of characteristic polynomial (2.1) such that $|\lambda| = R_1 > \rho$. Lemma 3.2 is proved. □

4. Localization of Roots of Characteristic Polynomial (2.1) for Real Nonnegative a and Complex b

For the stability of (1.1) it is required that all the trajectories are bounded. But sometimes one needs to strenghten or weaken the stability requirement. It justifies the following definition.

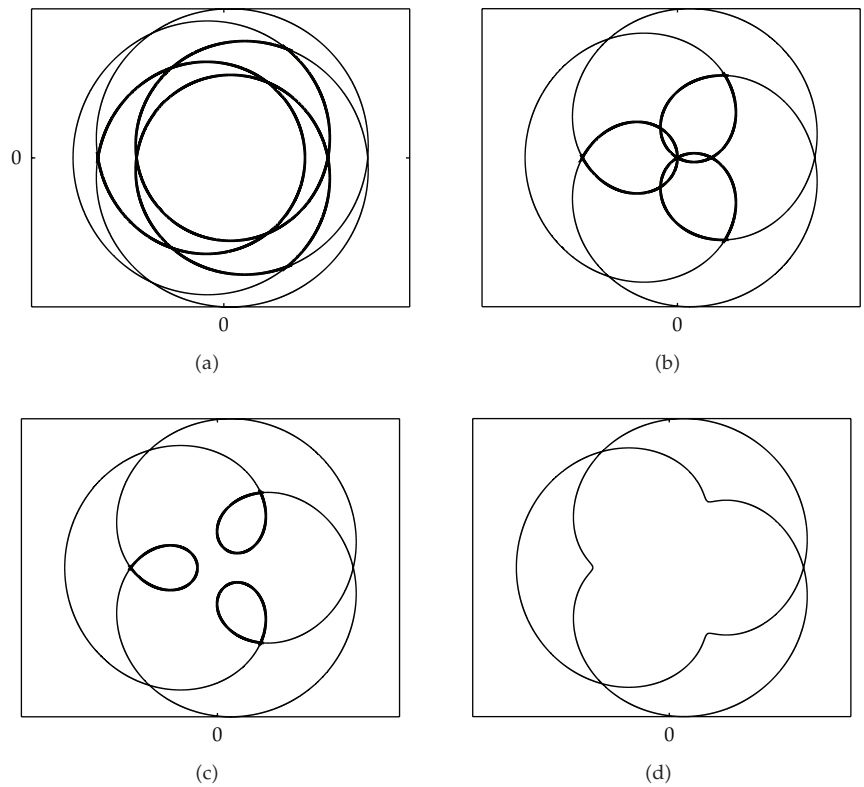

Figure 1: D-decomposition curve and three basic ovals for $m = 3, k = 5$: (a) $|a| < \rho^m$, (b) $|a| = \rho^m$, (c) $\rho^m < |a| < \rho^m k/(k-m)$, and (d) $|a| = \rho^m k/(k-m)$.

Definition 4.1. Equation (1.1) *is said to be ρ-stable if for any of its solutions (x_n) the sequence $(|x_n|/\rho^n)$ is bounded, and asymptotically ρ-stable if for any of its solutions (x_n) one has* $\lim_{n\to\infty}|x_n|/\rho^n = 0$.

If $\rho = 1$, then the concept of (asymptotic) ρ-stability coincides with the concept of usual (asymptotic) stability. Evidently, (1.1) is ρ-stable, if there are no roots of polynomial (2.1) outside the circle of radius ρ centred at the origin, and there are no multiple roots of the polynomial on the boundary of circle. Equation (1.1) is asymptotically ρ-stable if and only if all the roots of its characteristic polynomial (2.1) lie inside the circle of radius ρ with the center at 0.

Let us call the equation (asymptotically) ρ-unstable if it is not (asymptotically) ρ-stable. As we noted in Section 2, if $\rho = 1$, then the proportional change of both delays k, m in (1.1) has no influence on ρ-stability. It is not the case if $\rho \neq 1$. It is easy to see that the equation

$$x_n = ax_{n-md} + bx_{n-kd} \tag{4.1}$$

is asymptotically ρ-stable if and only if (1.1) is asymptotically ρ^d-stable. It implies the following important observation: proportional increase of both delays k, m with the conservation of the coefficients a, b in (1.1) preserves asymptotic ρ-stability if $\rho > 1$ and may not preserve it if $\rho < 1$.

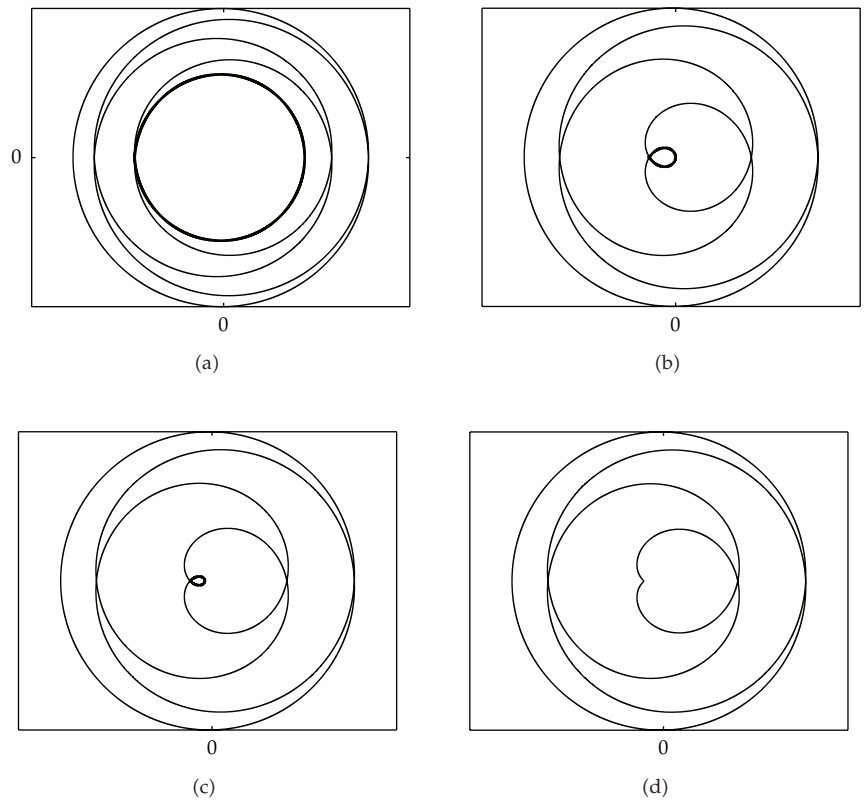

Figure 2: D-decomposition curve and the basic oval L_0 for $m = 1, k = 5$: (a) $|a| < \rho$, (b) $|a| = \rho$, (c) $\rho < |a| < \rho k/(k-1)$, and (d) $|a| = \rho k/(k-1)$.

Definition 4.2. Let k, m be coprime, $k > m \geqslant 1$, $a \in \mathbb{C}$, and $0 \leqslant |a| < \rho^m k/(k-m)$. The stability domain $D(k, m, a, \rho)$ is defined to be a set of all complex numbers b such that for any $j (0 \leqslant j < m)$ the number b lies inside the basic oval L_j.

Under the same conditions if k, m are not coprime and $d = gcd(k, m)$, let us put $D(k, m, a, \rho) = D(k/d, m/d, a, \rho^d)$.

Theorems 4.3–5.2 will justify the name "stability domain" for $D(k, m, a, \rho)$ a little later. Evidently, for coprime k, m such that $k > m > 1$ the domain $D(k, m, a, \rho)$ has the following properties. If $0 \leqslant |a| < \rho^m$, then $D(k, m, a, \rho)$ is the connected domain on the complex plane, containing 0, whose boundary is the D-decomposition curve (2.3). If $|a| = \rho^m$, then the domain $D(k, m, a, \rho)$ degenerates into the point $b = 0$. If $\rho^m < |a| < \rho^m k/(k-m)$, then the domain $D(k, m, a, \rho)$ is empty. If $|a| \geqslant \rho^m k/(k-m)$, then the domain $D(k, m, a, \rho)$ is not defined in view of the fact that the basic ovals (Definition 3.1) are not defined.

If $m = 1$, then the domain $D(k, 1, a, \rho)$ is a set of points lying inside the oval L_0. If $0 \leqslant |a| < \rho$, then $D(k, 1, a, \rho)$ includes 0. If $\rho \leqslant |a| < \rho k/(k-1)$, then $D(k, 1, a, \rho)$ is nonempty and does not contain 0. If $|a| = \rho k/(k-1)$, then $D(k, 1, a, \rho)$ degenerates into the point $b = -\rho^k/(k-1)$. Finally, if $|a| > \rho k/(k-1)$, then the domain $D(k, 1, a, \rho)$ is not defined.

The following theorems are based on the localization of roots of polynomial (2.1) with nonnegative a and complex b with respect to the circle of radius ρ centred at the origin.

Theorem 4.3. *Let k, m be coprime, $k > m > 1$, $a \in \mathbb{R}_+$, $\rho > 0$.*

(1) *If $a > \rho^m$, then for any b (1.1) is ρ-unstable.*

(2) *If $a = \rho^m$, then for any complex $b \neq 0$ (1.1) is ρ-unstable; for $b = 0$ it is ρ-stable (nonasymptotically).*

(3) *If $0 \leqslant a < \rho^m$, then (1.1) is asymptotically ρ-stable if and only if the complex number b lies inside the stability domain $D(k, m, a, \rho)$.*

(4) *If $0 \leqslant a < \rho^m$, then (1.1) is ρ-stable if and only if the complex number b lies either inside or on the boundary of $D(k, m, a, \rho)$.*

Proof. (1) Let $a > \rho^m$. Let us find $R \in \mathbb{R}$ such that

$$\rho^m < R^m < a < \frac{R^m k}{k - m}. \tag{4.2}$$

Taking into account the inequality $a < R^m k/(k - m)$, let us consider m basic ovals $L_j(R)$, $j = 0, 1, \ldots, m - 1$ having ρ replaced by R in Definition 4.1. Since $R^m < a$, the system of ovals $L_j(R)$ has no intersections. Hence, for any complex number b there exists $j \in \mathbb{Z}$, $0 \leqslant j < m$ such that b lies outside the oval $L_j(R)$. By Lemma 3.2 (1.1) is R-unstable. Since $R > \rho$, it is ρ-unstable. Statement 1 is proved.

(2) Let $a = \rho^m$. If $b = 0$, then statement 2 of Theorem 4.3 is obvious. Let $b \neq 0$. If Re $b \geqslant 0$, then b lies outside the oval L_0. If Re $b \leqslant 0$ and m is even, then b lies outside the oval $L_{m/2}$. If Re $b \leqslant 0$ and m is odd, then b lies either outside the oval $L_{(m-1)/2}$ or outside the oval $L_{(m+1)/2}$. In any case (1.1) is ρ-unstable by Lemma 3.2. Statement 2 is proved.

(3) Let $0 \leqslant a < \rho^m$. Let the number b be inside the domain $D(k, m, a, \rho)$. Then for any $j (0 \leqslant j < m)$ the number b lies inside the oval L_j. By Lemma 2.3 the beam drawn on the complex plane from 0 to b does not intersect curve (2.3). Therefore, polynomial (2.1) has the same number of roots inside the circle of radius ρ for given b and for $b = 0$. However if $b = 0$, then all the roots of (2.1) lie inside the circle of radius ρ centred at 0. Therefore, (1.1) is asymptotically ρ-stable for given b.

If b lies on the boundary of the domain $D(k, m, a, \rho)$ or outside it, then b lies either on the boundary of one of the basic ovals L_j or outside one of them, and by Lemma 3.2 (1.1) is asymptotically ρ-unstable.

(4) If b lies outside the domain $D(k, m, a, \rho)$, then the conclusion of statement 4 of Theorem 4.3 is a straightforward consequence of Lemma 3.2. If b lies inside $D(k, m, a, \rho)$, then the conclusion of statement 4 of Theorem 4.3 is a straightforward consequence of statement 3 of Theorem 4.3. Let b lie on the boundary of $D(k, m, a, \rho)$. Then for any root λ of polynomial (2.1) either $|\lambda| < \rho$ or $|\lambda| = \rho$. In the latter case in view of the inequality $0 \leqslant a < \rho^m < \rho^m k/k{-}m$ we have

$$\frac{df}{d\lambda} = \lambda^{k-m-1}(k\lambda^m - a(k - m)) \neq 0, \tag{4.3}$$

hence the root λ such that $|\lambda| = \rho$ is simple. Theorem 4.3 is proved. □

If in (1.1) the least delay m is equal to 1, then the situation is essentially different from the case $m > 1$.

Theorem 4.4. *Let $k > m = 1$, $a \in \mathbb{R}_+$, $\rho > 0$.*

(1) *If $a \geqslant \rho k/(k-1)$, then for all complex numbers b (1.1) is ρ-unstable.*

(2) *If $0 \leqslant a < \rho k/(k-1)$, then (1.1) is asymptotically ρ-stable if and only if the complex number b lies inside the domain $D(k, 1, a, \rho)$.*

(3) *If $0 \leqslant a < \rho k/(k-1)$, then (1.1) is ρ-stable if and only if the complex number b lies inside $D(k, 1, a, \rho)$.*

Proof. (1) Let $a > \rho k/(k-m)$, and let b be a given complex number. Let us find $R > \rho$ such that $\rho k/(k-m) < a < Rk/(k-m)$ and the point b is located outside the oval $L_0(R)$ obtained from Definition 3.1 by substituting R for ρ. By Lemma 3.2 there exists a complex root λ of polynomial (2.1) such that $|\lambda| > R > \rho$, so ρ-instability of (1.1) is proved. Let $a = \rho k/(k-1)$. Then the previous arguments also prove ρ-instability provided that $b \neq -\rho^k/(k-1)$. However, if $b = -\rho^k/(k-1)$, then under the assumption that $a = \rho k/(k-1)$ the number $\lambda = \rho$ is a multiple root of polynomial (2.1), and consequently, (1.1) is also ρ-unstable. Statement 1 of Theorem 4.4 is proved.

(2) Let $0 \leqslant a < \rho k/(k-1)$. Since $D(k, 1, a, \rho)$ is the domain of inner points of the oval L_0, it is connected. The function $|b(\omega)|$ (see (2.3)) increases as ω moves either from 0 to π or from 0 to $(-\pi)$. Therefore, there are no points of hodograph (2.3) inside L_0. To complete the proof of asymptotical ρ-stability of (1.1) at any point of L_0 it is sufficient to prove that there exists at least one point b_0 inside the oval L_0 such that the equation is asymptotically ρ-stable for $b = b_0$.

CASE 1. Let $0 \leqslant a < \rho$. Then the point $b = 0$ lies inside L_0. If $b = 0$, then polynomial (2.1) has the $(k-1)$-multiple root $\lambda = 0$ and the simple root $\lambda = a$. This gives the asymptotic ρ-stability, in view of $a < \rho$.

CASE 2. Let $\rho \leqslant a < \rho k/(k-1)$. Let us consider the point $b = \rho^k - a\rho^{k-1}$ at the boundary of L_0 and consider characteristic polynomial (2.1) with given b:

$$f_2(\lambda) = \lambda^k - a\lambda^{k-1} - \rho^k + a\rho^{k-1}. \tag{4.4}$$

The equation $f_2(\lambda) = 0$ transforms into

$$\left(\left(\frac{\lambda}{\rho}\right) - 1\right)\left(\left(\frac{\lambda}{\rho}\right)^{k-1} - \left(\left(\frac{a}{\rho}\right) - 1\right)\sum_{j=0}^{k-2}\left(\frac{\lambda}{\rho}\right)^{k-j-2}\right) = 0. \tag{4.5}$$

One of roots of (4.5) is equal to ρ, while others lie inside the circle of radius ρ centred at the origin in view of the inequality $0 \leqslant a - \rho < \rho/(k-1)$ [14].

Let us return to (2.1), and let us figure out in what direction the root $\lambda = \rho$ moves as the coefficient b moves from the point $b = \rho^k - a\rho^{k-1}$ toward the interior of L_0 so that $\mathrm{d}b \in \mathbb{R}$, $\mathrm{d}b < 0$. From (2.1) it follows that for $\lambda = \rho$ we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}b} = \frac{\rho^{-k+2}}{k\rho - a(k-1)}, \tag{4.6}$$

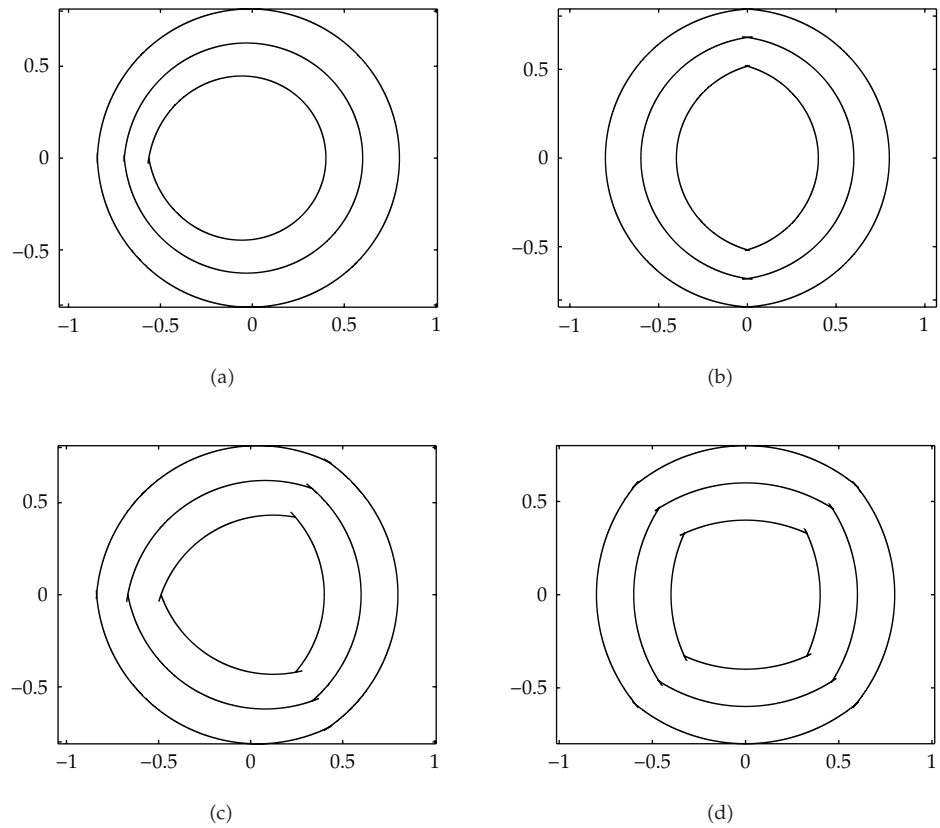

Figure 3: Stability domains $D(k, m, a, \rho)$ for $k = 5$, for different values of the coefficient a: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$, (d) $m = 4$.

and in view of $a < \rho k/(k-1)$ we get $\mathrm{d}\lambda/\mathrm{d}b > 0$. Therefore, $\mathrm{d}b < 0$ implies $\lambda < \rho$. Consequently, there exist values of b inside L_0 providing asymptotic ρ-stability of (1.1), therefore, for any value b inside L_0 (1.1) is asymptotically ρ-stable.

(3) The proof of Statement 3 of Theorem 4.4 is analogous to the proof of Statement 4 of Theorem 4.3. Theorem 4.4 is proved. □

5. Stability of (1.1) with Complex Coefficients a, b

Let us change the variables in (1.1) so that it has no influence on (asymptotic) ρ-stability:

$$x_n = y_n \exp\left(i\frac{n}{m}\arg a\right). \tag{5.1}$$

Equation (1.1) changes to

$$y_n = \alpha y_{n-m} + \beta y_{n-k}, \tag{5.2}$$

where

$$\alpha = |a|, \qquad \beta = b\exp\left(-i\frac{k}{m}\arg a\right). \tag{5.3}$$

The characteristic polynomial for (5.2) has the form

$$\psi(\mu) = \mu^k - \alpha\mu^{k-m} - \beta. \tag{5.4}$$

It is related to (2.1) by the change $\mu = \lambda\exp(-i(1/m)\arg a)$, that saves the absolute values of roots of the equation. It is important for us that new (5.2) has a real nonnegative coefficient at y_{n-m}, in view of (5.3). This allows us to apply the results of the previous section. Therefore, from Theorems 4.3 and 4.4 we immediately derive the following theorems providing an answer to the question on the stability of (1.1) with complex coefficients a, b.

Theorem 5.1. *Let k, m be coprime, $k > m > 1$, $a \in \mathbb{C}$, $\rho > 0$.*

(1) *If $|a| > \rho^m$, then for any complex b (1.1) is ρ-unstable.*

(2) *If $|a| = \rho^m$, then for any $b \neq 0$ (1.1) is ρ-unstable; for $b = 0$ it is ρ-stable (nonasymptotically).*

(3) *If $|a| < \rho^m$, then (1.1) is asymptotically ρ-stable if and only if the complex number $\beta = b\exp(-i(k/m)\arg a)$ lies inside the domain $D(k, m, a, \rho)$.*

(4) *If $|a| < \rho^m$, then (1.1) is ρ-stable if and only if the complex number $\beta = b\exp(-i(k/m)\arg a)$ lies either inside $D(k, m, a, \rho)$ or on its boundary.*

Theorem 5.2. *Let $k > m = 1$, $a \in \mathbb{C}$, $\rho > 0$.*

(1) *If $|a| \geqslant \rho k/(k-1)$, then for any complex b (1.1) is ρ-unstable.*

(2) *If $|a| < \rho k/(k-1)$, then (1.1) is asymptotically ρ-stable if and only if the complex number $\beta = b\exp(-ik\arg a)$ lies inside the domain $D(k, 1, a, \rho)$.*

(3) *If $|a| < \rho k/(k-1)$, then (1.1) is ρ-stable if and only if the complex number $\beta = b\exp(-ik\arg a)$ lies either inside $D(k, 1, a, \rho)$ or on its boundary.*

Example 5.3. Let $m = 6$, $a = 0.8 + 0.9i$, $\rho = 1.15$ in (1.1). Let $6 \leqslant k \leqslant 12$. For every given value k let us find all values of the complex coefficient b for which (1.1) is ρ-stable. The answer is demonstrated by Figure 4. Let us give some comments. First calculate $|a| = 1.204$, $\arg a \simeq 0.844$. If $k = 6$, then to find ρ-stability domain one does not need to use Theorems 4.3–5.2. The domain is a circle given in Figure 4(a). Since 6, 7 are coprime, then for $k = 7$, by Theorem 5.1, (1.1) ρ-stable if and only if $b \in \exp(i(7/6)\arg a)D(7, 6, a, \rho)$. The corresponding "curved hexagon" is shown in Figure 4(a). Similarly for $k = 11$ the condition $b \in \exp(i(11/6)\arg a)D(11, 6, a, \rho)$ is necessary and sufficient for asymptotic stability of (1.1). The corresponding "curved hexagon" is shown in Figure 4(b). For $k = 8$ the stability criterion is the condition $b \in \exp(i(4/3)\arg a)D(4, 3, a, \rho^2)$. The corresponding "curved triangle" is shown in Figure 4(a). Similarly the "digon" $\exp(i(3/2)\arg a)D(3, 2, a, \rho^3)$ for $k = 9$ is shown in Figure 4(a), and the "curved triangle" $\exp(i(5/3)\arg a)D(5, 3, a, \rho^2)$ for $k = 10$ is shown in Figure 4(b). For $k = 12$, according to Theorem 5.2, the stability criterion for (1.1) is $b \in \exp(i \cdot 2\arg a)D(2, 1, a, \rho^6)$ (Figure 4(b)). The corresponding "stability oval" is shown in Figure 4(a).

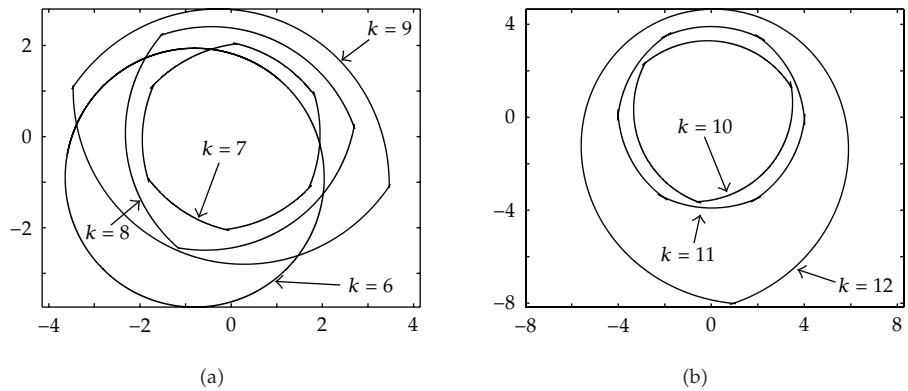

Figure 4: To Example 5.3. Stability domains for $m = 6$, $a = 0.8 + 0.9i$, $\rho = 1.19$; (a) $k = 6, 7, 8, 9$, (b) $k = 10, 11, 12$.

6. Stability Cones for Matrix Equation (1.1) with Simultaneously Triangularizable Matrices

Let us consider a matrix equation

$$x_n = A\ x_{n-m} + Bx_{n-k}, \tag{6.1}$$

$x : \mathbb{Z}_+ \to \mathbb{C}^l$; $A, B \in \mathbb{C}^{l\times l}$. The characteristic equation for (6.1) is

$$\psi(\lambda) = \det\left(I\lambda^k - A\lambda^{k-m} - B\right). \tag{6.2}$$

Definition 6.1. Matrix equation (6.1) *is called ρ-stable if for every solution (x_n) the sequence $(|x_n|/\rho^n)$ is bounded. Equation* (6.1) *is called asymptotically ρ-stable if $\lim_{n\to\infty}|x_n|/\rho^n = 0$ holds for every solution (x_n).*

Obviously, matrix equation (6.1) is asymptotically ρ-stable if and only if all the roots of characteristic polynomial (6.2) lie inside the circle of radius ρ with the center at 0. We also observe that if at least one root of (6.2) lies outside the circle of radius ρ with the center at 0, then (6.1) is ρ-unstable.

In this paper we consider (6.1) only with triangularizable matrices A, B. It is known [12] that if the matrix $AB - BA$ is nilpotent, then A, B can be simultaneously triangularized.

Definition 6.2. If $k > m > 1$, then the ρ-stability cone for given k, m, ρ is a set of points $M = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that $0 \leqslant u_3 \leqslant 1$ and the intersection of the set with any plane $u_3 = a$ $(0 \leqslant a \leqslant 1)$ is the stability domain $D(k, m, a, \rho)$. If $k > m = 1$, then the ρ-stability cone for given k, ρ is a set of points $M = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that $0 \leqslant u_3 \leqslant k/(k-1)$ and the intersection of the set with the plane $u_3 = a(0 \leqslant a \leqslant k/(k-1))$ is the domain $D(k, 1, a, \rho)$.

Let us define a stability cone as the ρ-stability cone for $\rho = 1$.

Returning to Figure 3, we can interpret the figures in Figure 3(a) as sections of the stability cone for $k = 5, m = 1$ at different heights $u_3 = a$, and the ones in Figure 3(b) as sections of the stability cone for $k = 5, m = 2$, and so on.

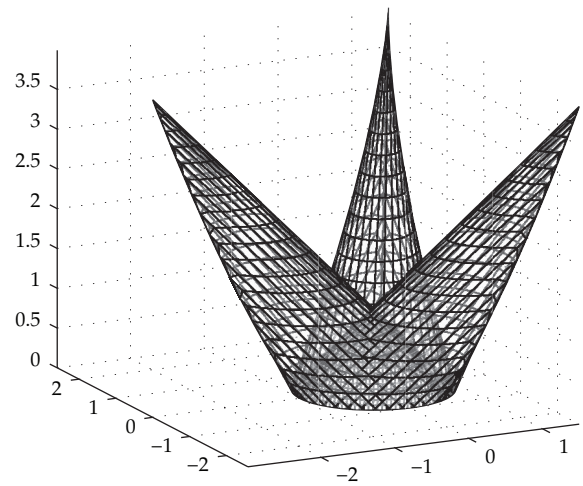

Figure 5: The stability cone as an intersection of three surfaces formed by the basic ovals, $m = 3, k = 4$.

The stability cones for $m > 1$ are the intersections of m conical surfaces formed by the basic ovals as the parameter a changes from 0 to $k/(k-m)$ (Figure 5).

Let us consider the simple case of a diagonal system

$$y_n - \operatorname{diag}(a_{11}, \ldots, a_{ll}) y_{n-m} - \operatorname{diag}(b_{11}, \ldots, b_{ll}) y_{n-k} = 0 \tag{6.3}$$

with complex entries a_{jj}, b_{jj}, $1 \leqslant j \leqslant l$. Let us construct the points $M_j = (u_{1j}, u_{2j}, u_{3j})$ $(1 \leqslant j \leqslant l)$ in $\mathbb{R}^3$ in the following way:

$$u_{1j} = \operatorname{Re}\left(b_{jj} \exp\left(-i \frac{k}{m} \arg a_{jj}\right)\right), \qquad u_{2j} = \operatorname{Im}\left(b_{jj} \exp\left(-i \frac{k}{m} \arg a_{jj}\right)\right), \qquad u_{3j} = |a_{jj}|. \tag{6.4}$$

It follows from the definition of the ρ-stability cone and from Theorems 5.1–5.2 that (6.3) is asymptotically ρ-stable if and only if all the points $M_j (1 \leqslant j \leqslant l)$ lie inside the ρ-stability cone for given k, m. All the points M_j with $u_{3j} = 0$, $u_{1j}^2 + u_{2j}^2 < \rho^k$ are considered as inner points of the ρ-stability cone.

The natural extension of the the class of diagonal systems is that of systems with simultaneously triangularizable matrices. The following theorem is our main result.

Theorem 6.3. *Let $k > m \geqslant 1$, let the numbers k, m be coprime, and $\rho > 0$. Let $A, B, S \in \mathbb{R}^{l \times l}$, and $S^{-1}AS = A_T$, and $S^{-1}BS = B_T$, where A_T and B_T are lower triangle matrices with elements $a_{j\,s}, b_{j\,s}$ $(1 \leqslant j,\ s \leqslant l)$. Let one construct the points $M_j = (u_{1j}, u_{2j}, u_{3j})$, $(1 \leqslant j \leqslant l)$ by the formulas (cf.* (6.4)*)*

$$u_{1j} = |b_{jj}| \cos\left(\arg b_{jj} - \frac{k}{m} \arg a_{jj}\right), \qquad u_{2j} = |b_{jj}| \sin\left(\arg b_{jj} - \frac{k}{m} \arg a_{jj}\right), \qquad u_{3j} = |a_{jj}|. \tag{6.5}$$

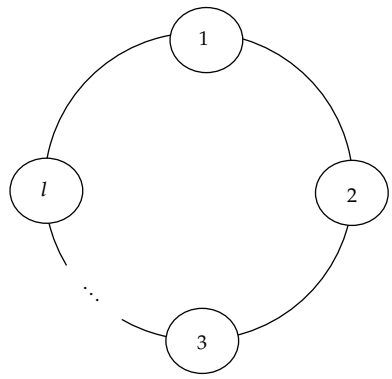

Figure 6: A neural net.

Then (6.1) *is ρ-asymptotically stable if and only if all the points $M_j (1 \leqslant j \leqslant l)$ lie inside the ρ-stability cone for the given k, m, ρ.*

If some point M_j lies outside the ρ-stability cone, then (6.1) *is ρ-unstable.*

Proof. Let us make the change $y_n = Sx_n$. Then (6.1) transforms to the following one:

$$y_n = A_T y_{n-m} + B_T y_{n-k}. \tag{6.6}$$

The characteristic polynomial for (6.6) has the form

$$\varphi(\lambda) = \prod_{j=1}^{l} \left(\lambda^k - a_{jj} \lambda^{k-m} - b_{jj} \right). \tag{6.7}$$

It coincides with the characteristic polynomial of diagonal system (6.3). Therefore, from statement 3 of Theorem 5.1 (for $m > 1$) and from statement 2 of Theorem 5.2 (for $m = 1$) we obtain asymptotic ρ-stability if all the points M_j lie inside the ρ-stability cone. Similarly from statement 4 of Theorem 5.1 (for $m > 1$) and statement 3 of Theorem 5.2 (for $m = 1$) we obtain ρ-instability of (6.1) if some point M_j lies outside the cone. Theorem 6.3 is proved. □

7. Applications to Neural Networks

Let us apply the results of the previous sections to the problem of the stability of discrete neural networks similar to continuous networks studied in [15, 16]. Let us consider a ring configuration of l neurons (Figure 6) interchanging signals with the neighboring neurons.

Let $y_n^{(j)}$ be a signal of the j-th neuron at the n-th moment of time. Let us suppose that the neuron reaction on its state, as well as on that of the previous neuron, is m-units delayed, and reaction on the next neuron is k-units delayed. The neuron chain is closed, and the first neuron is next to the l-th one. Let us assume that the neurons interchange the signals

according to the equations

$$\begin{aligned} y_n^{(1)} &= f\left(y_{n-m}^{(1)}\right) + g\left(y_{n-m}^{(l)}\right) + h\left(y_{n-k}^{(2)}\right), \\ y_n^{(2)} &= f\left(y_{n-m}^{(2)}\right) + g\left(y_{n-m}^{(1)}\right) + h\left(y_{n-k}^{(3)}\right), \\ &\cdots \\ y_n^{(l)} &= f\left(y_{n-m}^{(l)}\right) + g\left(y_{n-m}^{(l-1)}\right) + h\left(y_{n-k}^{(1)}\right), \end{aligned} \tag{7.1}$$

where f, g, h are sufficiently smooth real-valued functions of a real variable. Let us assume that there is a real number y_* such that the stationary sequences $y_n^1 = y_*, \ldots, y_n^l = y_*$ form a solution of (7.1). Let us introduce the variables $x_n^{(j)} = y_n^{(j)} - y_*$ and the vector $x_n = (x_n^{(1)}, \ldots, x_n^{(l)})^T$, and let us linearize system (7.1) in new variables about zero. We get (6.1) with the circulant [17] matrices

$$A = \begin{pmatrix} \alpha & 0 & 0 & \cdots & \beta \\ \beta & \alpha & 0 & \cdots & 0 \\ 0 & \beta & \alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \gamma & 0 & \cdots & 0 \\ 0 & 0 & \gamma & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{7.2}$$

Here $\alpha = df(y_*)/dy$, $\beta = dg(y_*)/dy$, $\gamma = dh(y_*)/dy$. Let us introduce a matrix P (a lines permutation operator):

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{7.3}$$

Then $B = \gamma P$, $A = \alpha I + \beta P^{l-1}$, therefore, diagonalization P generates simultaneous diagonalization of A, B. The eigenvalues of P are $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{l-1}$, where $\varepsilon = \exp(i2\pi/l)$. Therefore,

$$A_T = \alpha I + \beta \operatorname{diag}\left(1, \varepsilon^{l-1}, \varepsilon^{l-2}, \ldots, \varepsilon\right), \qquad B_T = \gamma \operatorname{diag}\left(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{l-1}\right). \tag{7.4}$$

Granting (7.4), by Theorem 6.3, we can build points $M_j = (u_{1j}, u_{2j}, u_{3j})$ in $\mathbb{R}^3$ for system (6.1), (7.2):

$$\begin{aligned} u_{1j} &= \gamma \cos\left(\frac{2\pi(j-1)}{l} - \left(\frac{k}{m}\right) \arg\left(\alpha + \beta \exp\left(-\frac{2\pi i(j-1)}{l}\right)\right)\right), \\ u_{2j} &= \gamma \sin\left(\frac{2\pi(j-1)}{l} - \left(\frac{k}{m}\right) \arg\left(\alpha + \beta \exp\left(-\frac{2\pi i(j-1)}{l}\right)\right)\right), \\ u_{3j} &= \left|\alpha + \beta \exp\left(-\frac{2\pi i(j-1)}{l}\right)\right|. \end{aligned} \tag{7.5}$$

We get the following consequence of Theorem 6.3.

Corollary 7.1. *If for every $j (1 \leqslant j \leqslant l)$ the point $M_j = (u_{1j}, u_{2j}, u_{3j})$ defined by formulas* (7.5) *lies inside the ρ-stability cone for given k, m, ρ, then system* (6.1), (7.2) *is asymptotically ρ-stable. If at least one point M_j lies outside the ρ-stability cone for given k, m, ρ, then system* (6.1), (7.2) *is ρ-unstable.*

Let us proceed to the problem of stability of a neural network with a large number of neurons. The points $M_j = (u_{1j}, u_{2j}, u_{3j})$ defined by (7.5) lie on the closed curve

$$\begin{aligned} u_1(t) &= \gamma \cos\left(t - \left(\frac{k}{m}\right) \arg(\alpha + \beta \exp(-it))\right), \\ u_2(t) &= \gamma \sin\left(t - \left(\frac{k}{m}\right) \arg(\alpha + \beta \exp(-it))\right), \\ u_3(t) &= |\alpha + \beta \exp(-it)|, \quad 0 \leqslant t \leqslant 2\pi. \end{aligned} \tag{7.6}$$

If $l \to \infty$, then the points M_j are dense in the curve (7.6). We get the following consequence of Theorem 6.3.

Corollary 7.2. *Let one consider system* (6.1), (7.2) *with $l \times l$ matrices A, B. If any point of the curve* (7.6) *lies inside the ρ-stability cone for given k, m, ρ, then system* (6.1), (7.2) *is asymptotically ρ-stable for any l. If at least one point of the curve* (7.6) *lies outside the ρ-stability cone for given k, m, ρ, then there exists l_0 such that system* (6.1), (7.2) *is ρ-unstable for any $l > l_0$.*

Example 7.3. Let us consider the ring of neurons shown in Figure 6. Put $k = 4$, $m = 3$, $\rho = 1$, $\beta = 0.1$, $\gamma = 0.4$. Let us pose a question: what are the values of $\alpha > 0$ for which the system of two neurons described by (6.1), (7.2) is stable?

For applications of Corollaries 7.1 and 7.2 we construct the curves (7.6) for six values of α (Figures 7(a) and 7(b)). Assuming $l = 3$, we construct the points M_1, M_2, M_3 in each of the six curves. Then we construct the stability cone for given k, m, ρ (Figure 7(b)). It is the ρ-stability cone for $\rho = 1$. In Figure 7(b) we see that two curves (7.6) corresponding to the values $\alpha = 0.1$, $\alpha = 0.3$ are hidden inside the cone. The point M_1 corresponding to the values $\alpha = 0.5$ is on the surface of the cone, while all other points of the curve (7.6) for $\alpha = 0.5$ lie inside the cone. Therefore, according to Corollaries 7.1 and 7.2, if $\alpha < 0.5$, then system (6.1),

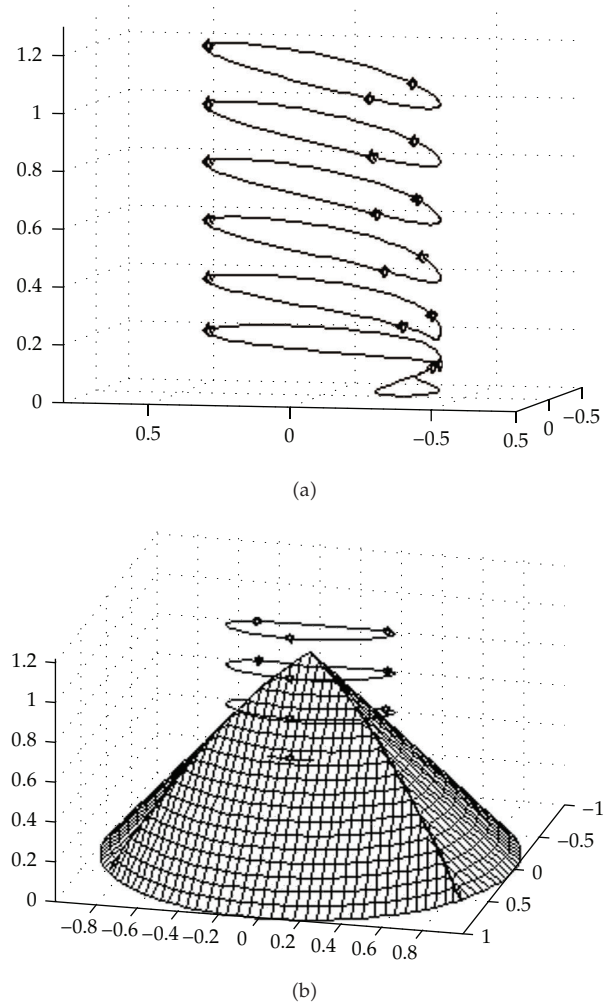

Figure 7: To Example 7.3. (a) The curves (7.6) for $\alpha = 0.1 + rh,\ h = 0.2,\ r = 0, 1, \ldots, 5$. (b) The stability cone covers the curves (7.6) for $\alpha < 0.5$. If $\alpha = 0.5$, then the point M_1 is on the surface of the cone. If $\alpha > 0.5$, then all the curves (7.6) are outside the cone either entirely or partially.

(7.2) is asymptotically stable for any $l \geqslant 2$. In our interpretation it means that the neuron configuration in Figure 6 is stable for any number of neurons. If $\alpha > 0.5$, then all the curves (7.6) lie entirely or partially outside the cone. In view of Corollaries 7.1 and 7.2 system (6.1), (7.2) is unstable. In our interpretation it means that even the configuration of two neurons is unstable.

Now let us consider Example 7.3 without the condition $\rho = 1$. Under assumptions of Example 7.3, for any value α there exists a number $\rho_0(\alpha)$ such that system (6.1), (7.2) is asymptotically ρ-stable if $\rho < \rho_0(\alpha)$ and ρ-unstable if $\rho > \rho_0(\alpha)$. Corollaries 7.1 and 7.2 allow us to find $\rho_0(\alpha)$ by means of construction of different ρ-stability cones. Table 1 shows how ρ_0 depends on α. The value $L(\alpha) = \ln \rho_0(\alpha)$ is a Lyapunov exponent [18] for system (6.1), (7.2).

Example 7.4. Let us consider the neuron chain shown in Figure 6. Let us fix the parameters: $k = 4$, $m = 3$, $\rho = 1$, $\beta = 0.1$, $\alpha = 0.5$. In this example we demonstrate how the change of

Table 1: ρ_0 versus α in Example 7.3.

α	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$\rho_0(\alpha)$	0.870	0.938	1.000	1.056	1.108	1.156	1.201	1.243

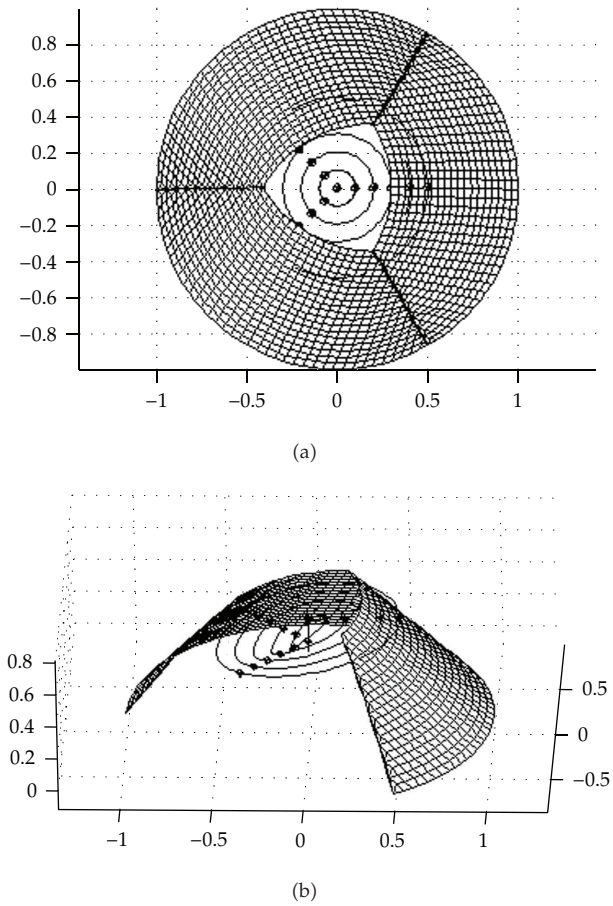

Figure 8: To Example 7.4. The stability cone and the curves (7.6) for $\gamma = rh,\ h = 0.1,\ r = 0, 1, \ldots, 5$ in two projections.

the parameter γ in system (6.1), (7.2) changes the mutual location of the curve (7.6) and the stability cone. In Figure 8(a) we show the curves (7.6) corresponding to the values $\gamma = rh$, $h = 0.1$, $r = 0, 1, \ldots, 5$, and the points M_1, M_2, M_3 for $l = 3$. The upper part of the stability cone corresponding to $u_3 > 0.7$ is removed. In Figure 8(b), one third of the lateral surface of the stability cone is removed too. Figure 8 demonstrates that the curves (7.6) lie inside the cone if $0 \leqslant \gamma < 0.4$. Therefore, if $0 \leqslant \gamma < 0.4$, then system (6.1), (7.2) is asymptotically stable for any $l \geqslant 2$. If $\gamma = 0.4$, then the point $M_1 = (u_1, u_2, u_3)$ defined by (7.5) lies on the cone surface. If $\gamma > 0.4$, then the point M_1 lies outside the cone, and this shows the instability of system (6.1), (7.2). In our interpretation, for $0 \leqslant \gamma < 0.4$, the neuron chain is stable no matter how many neurons are in the chain, and for $\gamma > 0.4$ it is unstable even if it consists only of two neurons.

8. Conclusion

The condition $\|A\| + \|B\| < 1$ is sufficient for the asymptotic stability of matrix (6.1) [19], and it does not require simultaneous triangularization of the matrices A, B. There are sufficient conditions for stability of nonautonomous scalar difference equations in [20–22].

Stability cones for differential matrix equations $\dot{x} = Ax + Bx(t - \tau)$ with one delay τ are introduced in the paper [23] and for some integrodifferential equations in the paper [24].

There are images of the stability domains in the space of parameters of scalar differential equations $\dot{x} = ax(t - \tau_1) + bx(t - \tau_2)$ with delays τ_1, τ_2 [25, 26] and scalar difference equations $x_n = x_{n-1} + ax_{n-m} + bx_{n-k}$ with delays k, m [25]. The results of the papers [25, 26] imply that there is no simple complete description of the stability domains for these equations.

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