International Scholarly Research Network
ISRN Algebra
Volume 2011, Article ID 926191, 10 pages
doi:10.5402/2011/926191

Research Article

The Cycle-Complete Graph Ramsey Number $r(C_9, K_8)$

M. S. A. Bataineh,[1] M. M. M. Jaradat,[1,2] and L. M. N. Al-Zaleq[1]

[1] *Department of Mathematics, Yarmouk University, Irbid 21163, Jordan*
[2] *Department of Mathematics, Statistics, and Physics, Qatar University, Doha, Qatar*

Correspondence should be addressed to M. M. M. Jaradat, mmjst4@yu.edu.jo

Received 11 May 2011; Accepted 14 June 2011

Academic Editor: R. L. Soto

It has been conjectured by Erdős, Faudree, Rousseau, and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $(m,n) \neq (3,3)$ satisfying that $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). In this paper, we prove this for the case $m = 9$ and $n = 8$.

1. Introduction

All graphs considered in this paper are undirected and simple. C_m, P_m, K_m and S_m stand for cycle, path, complete, and star graphs on m vertices, respectively. The graph $K_1 + P_n$ is obtained by adding an additional vertex to the path P_n and connecting this new vertex to each vertex of P_n. The number of edges in a graph G is denoted by $\mathcal{E}(G)$. Further, the minimum degree of a graph G is denoted by $\delta(G)$. An independent set of vertices of a graph G is a subset of the vertex set $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set. The neighborhood of the vertex u is the set of all vertices of G that are adjacent to u, denoted by $N(u)$. $N[u]$ denote to $N(u) \cup \{u\}$. For vertex-disjoint subgraphs H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$. Let H be a subgraph of the graph G and $U \subseteq V(G)$, $N_H(U)$ is defined as $(\bigcup_{u \in U} N(u)) \cap V(H)$. Suppose that $V_1 \subseteq V(G)$ and V_1 is nonempty, the subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $\langle V_1 \rangle_G$.

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order N, G contains C_m or $\alpha(G) \geq n$. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m-1)(n-1)+1$. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [1] proved the following result: for all $m \geq n^2 - 2$, $r(C_m, K_n) = (m-1)(n-1)+1$. The above restriction was improved by Nikiforov [2] when he proved the equality for $m \geq 4n+2$. Erdős et al. [3] gave the following conjecture.

Conjecture 1. $r(C_m, K_n) = (m-1)(n-1)+1$, *for all* $m \geq n \geq 3$ *except* $r(C_3, K_3) = 6$.

The conjecture was confirmed by Faudree and Schepl [4] and Rosta [5] for $n = 3$ in early work on Ramsey theory. Yang et al. [6] and Bollobás et al. [7] proved the conjecture for $n = 4$ and $n = 5$, respectively. The conjecture was proved by Schiermeyer [8] for $n = 6$. Jaradat and Baniabedalruhman [9, 10] proved the conjecture for $n = 7$ and $m = 7, 8$. Later on, Chena et. al. [11] proved the conjecture for $n = 7$. Recently, Jaradat and Al-Zaleq [12] and Y. Zhang and K. Zhang [13], independently, proved the conjecture in the case $n = m = 8$. In a related work, Radziszowski and Tse [14] showed that $r(C_4, K_7) = 22$ and $r(C_4, K_8) = 26$. In [15] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Also, Schiermeyer [16] proved that $r(C_5, K_7) = 25$. For more results regarding the Ramsey numbers, see the dynamic survey [17] by Radziszowski.

Until now, the conjecture is still open. Researchers are interested in determining all the values of the Ramsey number $r(C_m, K_8)$. In this paper our main purpose is to determine the values of $r(C_9, K_8)$ which confirm the conjecture in the case $m = 9$ and $n = 8$. The follwoing known theorem will be used in the sequel.

Theorem 1.1. *Let G be a graph of order n without a path of length k ($k \geq 1$). Then*

$$\mathcal{E}(G) \leq \frac{k-1}{2} n. \tag{1.1}$$

Further, equality holds if and only if its components are complete graphs of order k.

2. Main Result

In this paper we confirm the Erdős, Faudree, Rousseau, and Schelp conjecture in the case C_9 and K_8. In fact, we prove that $r(C_9, K_8) = 57$. It is known, by taking $G = (n-1)K_{m-1}$, that $r(C_m, K_n) \geq (m-1)(n-1)+1$. In this section we prove that this bound is exact in the case $m = 9$ and $n = 8$. Our proof depends on a sequence of 8 lemmas.

Lemma 2.1. *Let G be a graph of order ≥ 57 that contains neither C_9 nor an 8-elemant independent set. Then $\delta(G) \geq 8$.*

Proof. Suppose that G contains a vertex of degree less than 8, say u. Then $|V(G - N[u])| \geq 49$. Since $r(C_9, K_7) = 49$, as a result $G - N[u]$ has independent set consists of 7 vertices. This set with the vertex u is an 8-elemant independent set of vertices of G. That is a contradiction. □

Throughout all Lemmas 2.2 to 2.8, we let G be a graph with minimum degree $\delta(G) \geq 8$ that contains neither C_9 nor an 8-elemant independent set.

Lemma 2.2. *If G contains K_8, then $|V(G)| \geq 72$.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of K_8, Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. Since $\delta(G) \geq 8$, $U_i \neq \emptyset$ for all $1 \leq i \leq 8$. Since there is a path of order 8 joining any two vertices of U, as a result $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$ (otherwise, if $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 8$, then the concatenation of the $u_i u_j$-path of order 8 with $u_i w u_j$, is a cycle of order 9, a contradiction). Similarly, since there is a path of order 7 joining any two vertices of U, as a result for all $1 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$

we have that $xy \notin E(G)$ (otherwise, if there are $1 \leq i < j \leq 8$ such that $x \in U_i$, $y \in U_j$ and $xy \in E(G)$, then the concatenation of the u_iu_j-path of order 7 with u_ixyu_j, is a cycle of order 9, a contradiction). Also, since there is a path of order 6 joining any two vertices of U, as a result, $N_R(U_i) \cap N_R(U_j) = \emptyset$, $1 \leq i < j \leq 8$ (otherwise, if there are $1 \leq i < j \leq 8$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the u_iu_j-path of order 6 with u_ixwyu_j, is a cycle of order 9 where $x \in U_i$, $y \in U_j$ and $xw, wy \in E(G)$, a contradiction). Therefore $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 8(\delta(G) + 1) \geq (8)\ (9) = 72$. □

Lemma 2.3. *If G contains $K_8 - S_6$, then G contains K_8.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of $K_8 - S_6$ where the induced subgraph of $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ is isomorphic to K_7. Without loss of generality we may assume that $u_1u_8, u_2u_8 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. Then, as in Lemma 2.2, we have the following:

(1) $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$ except possibly for $i = 1$ and $j = 2$.

(2) $E(U_i, U_j) = \emptyset$ for all $1 \leq i < j \leq 8$.

(3) $N_R(U_i) \cap N_R(U_j) = \emptyset$ for all $1 \leq i < j \leq 8$.

(4) $E(N_R(U_i), N_R(U_j)) = \emptyset$ for all $1 \leq i < j \leq 8$.

Since $\alpha(G) \leq 7$, as a result at least five of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $3 \leq i \leq 8$ are complete. Since $\delta(G) \geq 8$, it implies that these complete graphs contain K_8. Hence, G contains K_8. □

Lemma 2.4. *If G contains K_7, then G contains $K_8 - S_6$ or K_8.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of K_7. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Since $\delta(G) \geq 8$, $U_i \neq \emptyset$ for all $1 \leq i \leq 7$. Now we consider the following two cases.

Case 1. $U_i \cap U_j \neq \emptyset$. for some $1 \leq i < j \leq 7$, say $w \in U_i \cap U_j$. Then it is clear that G contains $K_8 - S_6$. In fact, the induced subgraph $\langle U \cup \{w\}\rangle_G$ contains $K_8 - S_6$.

Case 2. $U_i \cap U_j = \emptyset$. for all $1 \leq i < j \leq 7$. Note that between any two vertices of U there are paths of order 5, 6 and 7. Thus, as in Lemma 2.2, for all $1 \leq i < j \leq 7$, we have the following.

(1) $E(U_i, U_j) = \emptyset$.

(2) $N_R(U_i) \cap N_R(U_j) = \emptyset$.

(3) $E(N_R(U_i), N_R(U_j)) = \emptyset$.

Since $\alpha(G) \leq 7$, we have that the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $1 \leq i \leq 7$ are complete. Since $\delta(G) \geq 8$, it implies that these complete graphs contain K_8. Hence, G contains K_8. □

Lemma 2.5. *If G contains $K_1 + P_7$, then G contains K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of $K_1 + P_7$ where $K_1 = u_1$ and $P_7 = u_2u_3u_4u_5u_6u_7u_8$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \le i \le 8$. Now we have the following two cases.

Case 1. $U_4 \cap U_6 = \emptyset$. Since $\delta(G) \ge 8, U_i \ne \emptyset$ for all $1 \le i \le 8$. Now we have the following.

(1) $U_i \cap U_j = \emptyset$ for all $2 \le i < j \le 8$ except possibly for $(i,j) \in \{(3,5),(3,6),(3,7),(4,7),(5,7)\}$ since otherwise a cycle of order 9 is produced, a contradiction.

(2) $E(U_i, U_j) = \emptyset$ for all $2 \le i < j \le 8$.

(3) $N_R(U_i) \cap N_R(U_j) = \emptyset$ for all $2 \le i < j \le 8$.

(4) $E(N_R(U_i), N_R(U_j)) = \emptyset$ for all $2 \le i < j \le 8$.

((2), (3), and (4) follows easily from being that $K_1 + P_7$ contains paths of order 7, 6, and 5 between any two vertices u_i and u_j, $2 \le i < j \le 8$). Since $\alpha(G) \le 7$, as a result at least three of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2, 4, 5, 6, 8$ are complete graphs. Now we have the following two assertions.

(*i*) $|N_R(U_i)| \ge 7$ and so $|U_i \cup N_R(U_i)| \ge 8$ for each $i = 2,\ 8$. The following is the proof of assertion (i) for $i = 8$.

Since $\delta(G) \ge 8$, $|U_8| \ge 1$. Let $y \in U_8$ and y is adjacent to $x \in \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$. Then we have the following.

(i) If $x = u_1$, then $u_8yu_1u_2u_3u_4u_5u_6u_7u_8$ is a C_9, this is a contradiction.

(ii) If $x = u_2$, then $u_8yu_2u_3u_4u_5u_6u_7u_1u_8$ is a C_9, this is a contradiction.

(iii) If $x = u_3$, then $u_8yu_3u_2u_1u_4u_5u_6u_7u_8$ is a C_9, this is a contradiction.

(iv) If $x = u_4$, then $u_8yu_4u_3u_2u_1u_5u_6u_7u_8$ is a C_9, this is a contradiction.

(v) If $x = u_5$, then $u_8yu_5u_4u_3u_2u_1u_6u_7u_8$ is a C_9, this is a contradiction.

(vi) If $x = u_1$, then $u_8yu_6u_5\mathrm{u}_4u_3u_2u_1u_7u_8$ is a C_9, this is a contradiction.

(vii) If $x = u_7$, then $u_8yu_7u_6u_5u_4u_3u_2u_1u_8$ is a C_9, this is a contradiction.

Since $\delta(G) \ge 8$, $|N_R(y)| \ge 7$, and so $|\{y\} \cup N_R(y)| \ge 8$. Hence, $|U_8 \cup N_R(U_8)| \ge 8$. By a similar argument as above and using the symmetry of $P_7 + K_1$, one can easily show that $|U_2 \cup N_R(U_2)| \ge 8$.

(*ii*) If there is $i \in \{4,5,6\}$ such that $|N_R(U_i)| < 6$, then $|N_R(U_j)| \ge 6$ and so $|U_j \cup N_R(U_j)| \ge 7$ for any $j \in \{4,5,6\}$ with $i \ne j$. The following is the proof of assertion (ii).

Assume that $|N_R(U_4)| < 6$. By (1) $U_4 \cap U_i = \emptyset$ for all $2 \le i \le 8$ except possibly $i = 4, 7$. Thus, for $y \in U_4$, y is adjacent to u_4 and to at most u_1 and u_7. Now we show that $|N_R(U_5)| \ge 6$. Assume $|N_R(U_5)| < 6$. By (1) $U_5 \cap U_i = \emptyset$ for all $2 \le i \le 8$ except possibly $i = 3, 5, 7$. Thus, for any $w \in U_5$, w is adjacent to u_5 and to at most u_1, u_3 and u_7. Now, we have the following.

(A) If w adjacent to both u_1 and u_3, then $u_2u_3wu_5u_6u_7yu_4u_1u_2$ is a C_9.

(B) If w adjacent to both u_1 and u_7, then $u_2u_3u_4yu_5wu_7u_6u_1u_2$ is a C_9.

(C) If w adjacent to both u_3 and u_7, then $u_2u_3wu_7u_6u_5u_4yu_1u_2$ is a C_9.

Thus, w is adjacent to at most one of u_1, u_3, and u_7, and so $|N_R(U_5)| \ge 6$. We now show that $|N_R(U_6)| \ge 6$. As above assume $|N_R(U_6)| < 6$. By (1), $U_6 \cap U_i = \emptyset$ for all $2 \le i \le 8$ except possibly $i = 3, 6$. Thus, for $w \in U_6$, w is adjacent to u_1, u_3, and u_6. Hence, $u_8u_7yu_4u_3wu_6u_5u_1u_8$ is a C_9, which implies that w is adjacent to at most one of u_1 and u_3 and so $|N_R(U_6)| \geq 6$.

Now, by using the same argument as above and taking into account that $P_7 + K_1$ is symmetry, we can easily see that if $|N_R(U_6)| < 6$, then both of $|N_R(U_4)|$ and $|N_R(U_5)|$ are greater than or equal 6. So we need to consider the case when $|N_R(U_5)| < 6$. As above, $U_5 \cap U_i = \emptyset$ for all $2 \leq i \leq 8$ except possibly $i = 5, 3$ and 7. Thus, for any $w \in U_5$, w is adjacent to u_5 and to at most u_1, u_3 and u_7. Now, assume that $|N_R(U_4)| < 6$. By using (A), (B) and (C) as above and using the same arguments to get the same contradiction. Similarly, by symmetry we get that $|N_R(U_6)| \geq 6$.

Therefore, from *(i)* and *(ii)*, at least four of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2, 4, 5, 6, 8$ contain 7 vertices and so at least two of them contain K_7. Thus, G contains K_7.

Case 2. $U_4 \cap U_6 \neq \emptyset$, say $u_9 \in U_4 \cap U_6$. For simplicity, in the rest of this case we consider $U_i' = N(u_i) \cap V(R')$ where $R' = G - U \cup \{u_9\}$ and let $J = \{2, 3, 5, 7, 8, 9\}$. Then $u_2u_9, u_3u_9, u_5u_9, u_7u_9, u_8u_9 \notin E(G)$ (otherwise, G contains C_9) and $\delta(G) \geq 8$. Hence $U_i' \neq \emptyset$, for all $i \in J$. Now we have the following assertions (see the Appendix).

(1) $U_i' \cap U_j' = \emptyset$ for all $i, j \in J$ and $i \neq j$.

(2) $E(U_i', U_j') = \emptyset$ for all $i, j \in J$ and $i \neq j$.

(3) $N_R(U_i') \cap N_R(U_j') = \emptyset$ for all $i, j \in J$ and $i \neq j$.

(4) $E(N_R(U_i'), N_R(U_j')) = \emptyset$ for all $i, j \in J$ and $i \neq j$.

Since $\alpha(G) \leq 7$, as a result at least five of the induced subgraphs $\langle U_i' \cup N_R(U_i')\rangle_G$, $i = 2, 3, 5, 7, 8, 9$ are complete graphs. Since $\delta(G) \geq 8$ and G contains no C_9, $|N_R(U_i')| \geq 6$ and so $|U_i' \cup N_R(U_i')| \geq 7$ for each $i = 2, 5, 8, 9$. Therefore at least three of the induced subgraphs $\langle U_i' \cup N_R(U_i')\rangle_G$, $i = 2, 3, 5, 7, 8, 9$ contain K_7. Thus, G contains K_7. □

Lemma 2.6. *If G contains $K_1 + P_6$, then G contains $K_1 + P_7$ or K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = u_2u_3u_4u_5u_6u_7$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Since $\delta(G) \geq 8$, $|U_i| \geq 2$ for all $1 \leq i \leq 7$. Now we have the following cases.

Case 1. $U_i \cap U_j = \emptyset$ for all $2 \leq i < j \leq 7$. Then we have the following.

(1) $E(U_i, U_j) = \emptyset$ for all $2 \leq i < j \leq 7$ except possibly for $(i, j) \in \{(3, 5), (3, 6), (4, 6)\}$.

(2) $N_R(U_i) \cap N_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 7$.

(3) $E(N_R(U_i), N_R(U_j)) = \emptyset$ for all $2 \leq i < j \leq 7$.

Since $\alpha(G) \leq 7$, as a result at least one of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $i = 2, 4, 5, 7$ is complete. Since $\delta(G) \geq 8$, it implies that this complete graph contains K_7.

Case 2. $U_i \cap U_j \neq \emptyset$ for some $2 \le i < j \le 7$, say $u_8 \in U_r \cap U_s$. In the rest of this case we have the following subcases:

Subcase 2.1. $(r,s) \in \{(6,7),(5,7),(4,7),(7,3),(2,7),(5,6),(4,6),(6,3),(4,5)\}$. For simplicity, in the rest of this subcase we consider $U'_i = N(u_i) \cap V(R')$ where $R' = G - U \cup \{u_8\}$ and let $J = \{m : 2 \le m \le 8$ and $m \notin \{r, s, \lceil (r+s)/s \rceil + 1\}\}$. Since $\delta(G) \ge 8$, then $U'_i \neq \emptyset$, for all $2 \le i \le 8$. Now we have the following assertions.

(1) $U'_i \cap U'_j = \emptyset$ for all $i, j \in J$ with $i \neq j$.

(2) $E(U'_i, U'_j) = \emptyset$ for all $i, j \in J$ with $i \neq j$.

(3) $N_{R'}(U'_i) \cap N_{R'}(U'_j) = \emptyset$ for all $i, j \in J$ with $i \neq j$.

(4) $E(N_{R'}(U'_i), N_{R'}(U'_j)) = \emptyset$ for all $i, j \in J$ with $i \neq j$.

Since $\alpha(G) \le 7$, as a result at least one of the induced subgraphs $\langle U'_i \cup N_{R'}(U'_i)\rangle_G$, $i \in J$ is complete. Since $\delta(G) \ge 8$ and $|U'_i| \ge 2$ for each $i \in J$ (because otherwise G contains $K_1 + P_7$), it implies that this complete graph contains K_7.

Subcase 2.2. $(r,s) \notin \{(6,7),(5,7),(4,7),(7,3),(2,7),(5,6),(4,6),(6,3),(4,5)\}$. Then, by the symmetry, we have a subcase similar to Subcase 2.1. □

Lemma 2.7. *If G contains K_6, then G contains $K_1 + P_6$ or K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of K_6. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \le i \le 6$. Since $\delta(G) \ge 8, |U_i| \ge 3$ for all $1 \le i \le 6$. Now we split our work into the following two cases.

Case 1. There are $1 \le i < j \le 6$ such that $U_i \cap U_j \neq \emptyset$, then G contains $K_1 + P_6$.

Case 2. $U_i \cap U_j = \emptyset$ for all $1 \le i < j \le 6$. Then we consider the following subcases.

Subcase 2.1. $E(U_i, U_j) = \emptyset$ for all $1 \le i < j \le 6$. Since between any two vertices of U there are paths of order 5 and 6, as a result $N_R(U_i) \cap N_R(U_j) = \emptyset$ and $E(N_R(U_i), N_R(U_j)) = \emptyset$ for each $1 \le i < j \le 6$. Therefore, since $\alpha(G) \le 7$, at least five of the induced subgraphs $\langle U_i \cup N_R(U_i)\rangle_G$, $1 \le i \le 6$ are complete graphs. Since $\delta(G) \ge 8$, these complete graphs contain K_7. Thus, G contains K_7.

Subcase 2.2. $E(U_i, U_j) \neq \emptyset$ for some $1 \le i < j \le 6$, say $i = 1$ and $j = 2$ and $u_1u_7u_8u_2$ is a path. For simplicity, in the rest of this subcase we consider $U'_i = N(u_i) \cap V(R')$ where $R' = G - U \cup \{u_7, u_8\}$. Since $\delta(G) \ge 8$, then $U'_i \neq \emptyset$, for all $3 \le i \le 8$. Now we have the following.

(1) $U'_i \cap U'_j = \emptyset$ for all $3 \le i < j \le 8$.

(2) $E(U'_i, U'_j) = \emptyset$ for all $3 \le i < j \le 8$.

(3) $N_R(U'_i) \cap N_R(U'_j) = \emptyset$ for all $3 \le i < j \le 8$.

(4) $E(N_R(U'_i), N_R(U'_j)) = \emptyset$ for all $3 \le i < j \le 8$.

Therefore, since $\alpha(G) \le 7$, at least five of the induced subgraphs $\langle U'_i \cup N_R(U'_i)\rangle_G$, $3 \le i \le 8$ are complete graphs. Since $\delta(G) \ge 8$, it implies that these complete graphs contain K_7. Thus, G contains K_7. □

Lemma 2.8. *If G be a graph of order ≥ 57, then G contains $K_1 + P_6$ or K_6.*

Proof. Suppose that G contains neither $K_1 + P_6$ nor K_6. Then we have the following claims.

Claim 1. $|N(u)| \leq 28$ for any $u \in V(G)$.

Proof. Suppose that u is a vertex with $|\langle N_G(u)\rangle_G| \geq 29$. Let $\langle N_G(u)\rangle_G = \bigcup_{i=1}^{r} G_i$ where G_i is a component for each i. $\langle N_G(u)\rangle_G$ has minimum number of independent vertices if it has a maximum number of edges. Thus, by Theorem 1.1 G_i must be a complete graph for each i. But $\langle N_G(u)\rangle_G$ contains no P_6. Thus, G_i must be a complete graph of order at most 5. Also $\langle N_G(u)\rangle_G$ contains no K_5, thus G_i must be a complete graph of order at most 4. Hence, the minimum number of independent vertices of $\langle N_G(u)\rangle$ occurs only if $\langle N_G(u)\rangle$ contains either a 7 tetrahedrons and an isolated vertex or 6 tetrahedron, a triangle and a K_2 or 6 tetrahedrons and 2 triangles. In any of these cases $\alpha(G) \geq 8$. This is a contradiction. The proof of the claim is complete. □

Claim 2. $\alpha(G) = 7$.

Proof. Since $|V(G)| \geq 57$ and G contains no C_9 and since $r(C_9, K_7) = 49$, $\alpha(G) \geq 7$. But G has no 8-element independent set, so $\alpha(G) \leq 7$. Thus, $\alpha(G) = 7$. The proof of the claim is complete. □

Now, for any 7 independent vertices $u_1, u_2, u_3, u_4, u_5, u_6$, and u_7, set $N_i[u_{i+1}] = N[u_{i+1}] - (\bigcup_{j=1}^{i} N[u_j])$, $1 \leq i \leq 6$. Analogously, we set $N_i(u_{i+1})$, $1 \leq i \leq 6$. Let $A = \bigcup_{i=1}^{6} N_i[u_{i+1}]$, $B = \bigcup_{i=1}^{6} N_i(u_{i+1})$, and $\beta = \alpha(\langle B\rangle_G)$.

Claim 3. $|N(u_1) \cup B| \geq 50$.

Proof. Suppose that $|N(u_1) \cup B| \leq 49$. Then $|N[u_1] \cup A| \leq 56$. And so $|G - (N[u_1] \cup A)| \geq 57 - 56 = 1$. But $r(C_9, K_1) = 1$, so $G - (N[u_1] \cup A)$ contains a vertex, say u_8, which is not adjacent to any of $u_1, u_2, u_3, u_4, u_5, u_6$, and u_7. Thus, $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is an 8-element independent set. Therefore, $\alpha(G) \geq 8$. That is a contradiction. The proof of the claim is complete. □

□

Now, by Lemma 2.1, $\delta(G) \geq 8$ and so by Claim 1, we have that $8 \leq |N(u_1)| \leq 28$. Thus, if $|N(u_1)| = r$, then $|B| \geq 50 - r$. By a similar argument as in Claim 1, we have that $\alpha(\langle N(u_1)\rangle_G) \geq \lceil r/4 \rceil$ and $\beta \geq \lceil (50 - r)/4 \rceil$. Note that for any $8 \leq r \leq 21$, $\lceil (50 - r)/4 \rceil$ is greater than or equal to 8. And so $\alpha(G) \geq 8$. Now we have the following cases.

Case 1. $22 \leq |N(u_1)| \leq 25$, then by a similar argument as in Claim 1, we have that $\alpha(\langle N(u_1)\rangle_G) \geq 6$ and $\beta \geq 7$. Then, $\langle B\rangle_G$ has an independent set which consists of 7 vertices. This set with the vertex u_1 is an 8-element independent set of vertices of G. That is a contradiction.

Case 2. $|N(u_1)| = 26$, then $|B| \geq 24$. By a similar argument as in Claim 1, we have that $\alpha(\langle N(u_1)\rangle_G) \geq 7$ and $\beta \geq 6$. Now we have the following two subcases.

Subcase 2.1. $\beta \geq 7$. Then we have a subcase similar to Case 1.

Subcase 2.2. $\beta = 6$. The best case of such subgraph is the graph that shown in Figure 1 . Now we have the following two subcases.

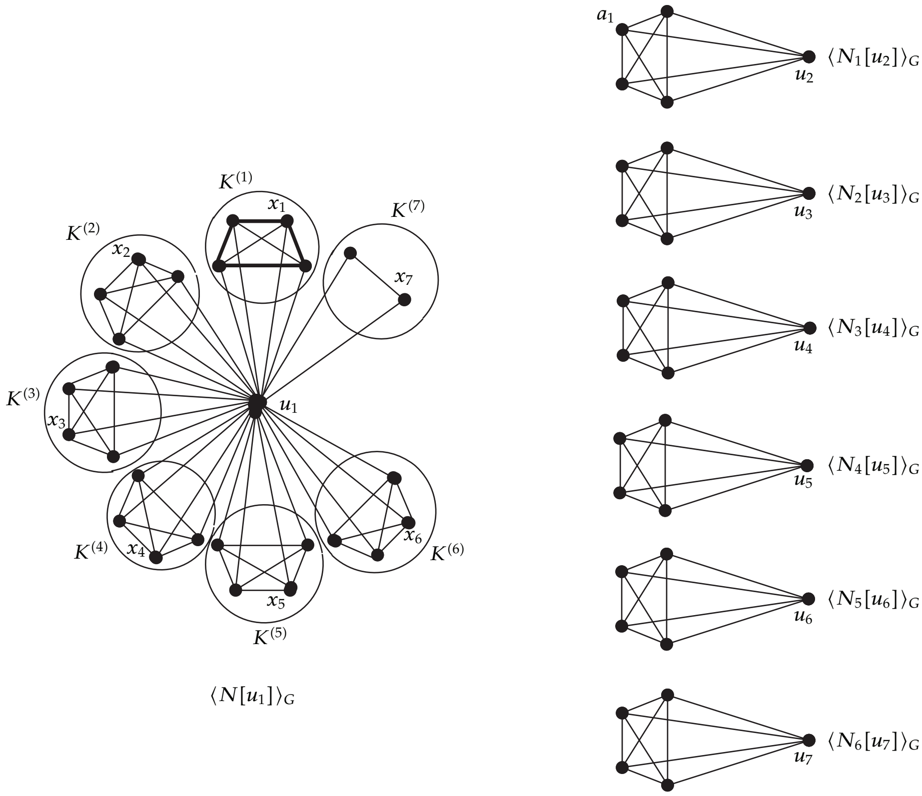

Figure 1: Describes the situation in Subcase 2.2.

Subcase 2.2.1. There is a vertex of $\bigcup_{i=1}^{6} N_i(u_{i+1})$, say a_1, that is not adjacent to at least one vertex of each $K^{(j)}(1 \leq j \leq 7)$, say x_j. Then $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, a_1\}$ is an 8-elemant independent set of vertices of G. And so $\alpha(G) \geq 8$. That is a contradiction.

Subcase 2.2.2. For each vertex of $\bigcup_{i=1}^{6} N_i(u_{i+1})$ there is $1 \leq j \leq 7$ such that this vertex is adjacent to all vertices of $K^{(j)}$. Then G contains $K_1 + P_6$ or C_9. That is a contradiction.

Case 3. $27 \leq |N(u_1)| \leq 28$, Then by using the same argument as in Case 2, we have the same contradiction.

Theorem 2.9. *$r(C_9, K_8) = 57$.*

Proof. Suppose that there exists a graph G of order 57 that contains neither C_9 nor an 8-elements independent set. Then by Lemma 2.1, $\delta(G) \geq 8$ and by Lemma 2.8, G contains $K_1 + P_6$ or K_6. Thus, by Lemmas 2.7, 2.6, 2.5, 2.4, 2.3, and 2.2, we have that $|V(G)| \geq 72$. That is a contradiction. The proof is complete. □

Appendix

To show that the assertions (1)–(4) of Case 2 of Lemma 2.5 are true, it suffices to show that for any two vertices of $\{u_2, u_3, u_5, u_7, u_8, u_9\}$ there are paths of order 8,7,6 and 5. The following are paths of order 8 between vertices of $\{u_3, u_5, u_7, u_8, u_9\}$.

1- u_2-u_3 path: $u_2u_1u_8u_7u_6u_5u_4u_3$, by symmetry we find u_7-u_8 path.

2- u_2-u_5 path: $u_2u_3u_4u_1u_8u_7u_6u_5$, by symmetry we find u_5-u_8 path.

3- u_2-u_7 path: $u_2u_3u_4u_5u_6u_1u_8u_7$, by symmetry we find u_3-u_8 path.

4- u_2-u_8 path: $u_2u_3u_4u_5u_6u_7u_1u_8$.

5- u_2-u_9 path: $u_2u_3u_4u_5u_1u_7u_6u_9$, by symmetry we find u_8-u_9 path.

6- u_3-u_5 path: $u_3u_4u_9u_6u_7u_8u_1u_5$, by symmetry we find u_5-u_7 path.

7- u_3-u_7 path: $u_3u_4u_9u_6u_5u_1u_8u_7$.

8- u_3-u_9 path: $u_3u_4u_5u_1u_8u_7u_6u_9$, by symmetry we find u_7-u_9 path.

3- u_5-u_9 path: $u_5u_4u_3u_2u_1u_7u_6u_9$.

The following are paths of order 7 between vertices of $\{u_3, u_5, u_7, u_8, u_9\}$.

1- u_2-u_3 path: $u_2u_1u_7u_6u_5u_4u_3$, by symmetry we find u_7-u_8 path.

2- u_2-u_5 path: $u_2u_3u_4u_1u_7u_6u_5$, by symmetry we find u_5-u_8 path.

3- u_2-u_7 path: $u_2u_3u_4u_5u_6u_1u_7$, by symmetry we find u_3-u_8 path.

4- u_2-u_8 path: $u_2u_3u_4u_5u_6u_1u_8$.

5- u_2-u_9 path: $u_2u_3u_4u_5u_1u_6u_9$, by symmetry we find u_8-u_9 path.

2- u_3-u_5 path: $u_3u_2u_1u_8u_7u_6u_5$, by symmetry we find u_5-u_7 path.

2- u_3-u_7 path: $u_3u_2u_1u_4u_5u_6u_7$.

2- u_3-u_9 path: $u_3u_4u_5u_1u_7u_6u_9$, by symmetry we find u_7-u_9 path.

3- u_5-u_9 path: $u_5u_4u_3u_2u_1u_6u_9$.

The following are paths of order 6 between vertices of $\{u_3, u_5, u_7, u_8, u_9\}$.

1- u_2-u_3 path: $u_2u_1u_6u_5u_4u_3$, by symmetry we find u_7-u_8 path.

2- u_2-u_5 path: $u_2u_3u_4u_1u_6u_5$, by symmetry we find u_5-u_8 path.

3- u_2-u_7 path: $u_2u_3u_4u_5u_1u_7$, by symmetry we find u_3-u_8 path.

4- u_2-u_8 path: $u_2u_3u_4u_5u_1u_8$.

5- u_2-u_9 path: $u_2u_3u_4u_1u_6u_9$, by symmetry we find u_8-u_9 path.

2- u_3-u_5 path: $u_3u_2u_1u_7u_6u_5$, by symmetry we find u_5-u_7 path.

2- u_3-u_7 path: $u_3u_2u_1u_5u_6u_7$.

2- u_3-u_9 path: $u_3u_2u_1u_7u_6u_9$, by symmetry we find u_7-u_9 path.

3- u_5-u_9 path: $u_5u_1u_2u_3u_4u_9$.

The following are paths of order 5 between vertices of $\{u_3, u_5, u_7, u_8, u_9\}$.

1- u_2-u_3 path: $u_2u_1u_5u_4u_3$, by symmetry we find u_7-u_8 path.

2- u_2-u_5 path: $u_2u_3u_4u_1u_5$, by symmetry we find u_5-u_8 path.

3- u_2-u_7 path: $u_2u_3u_4u_1u_7$, by symmetry we find u_3-u_8 path.

4- u_2-u_8 path: $u_2u_3u_4u_1u_8$.

1- u_2-u_9 path: $u_2u_1u_5u_4u_9$, by symmetry we find u_8-u_9 path.

2- u_3-u_5 path: $u_3u_2u_1u_6u_5$, by symmetry we find u_5-u_7 path.

2- u_3-u_7 path: $u_3u_2u_1u_6u_7$.

2- u_3-u_9 path: $u_3u_4u_5u_{67}u_9$, by symmetry we find u_7-u_9 path.

3- u_5-u_9 path: $u_5u_1u_3u_4u_9$.

References

[1] J. A. Bondy and P. Erdős, "Ramsey numbers for cycles in graphs," *Journal of Combinatorial Theory. Series B*, vol. 14, pp. 46–54, 1973.

[2] V. Nikiforov, "The cycle-complete graph Ramsey numbers," *Combinatorics, Probability and Computing*, vol. 14, no. 3, pp. 349–370, 2005.

[3] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, "On cycle-complete graph Ramsey numbers," *Journal of Graph Theory*, vol. 2, no. 1, pp. 53–64, 1978.

[4] R. J. Faudree and R. H. Schelp, "All Ramsey numbers for cycles in graphs," *Discrete Mathematics*, vol. 8, pp. 313–329, 1974.

[5] V. Rosta, "On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II," *Journal of Combinatorial Theory. Series B*, vol. 15, pp. 94–120, 1973.

[6] J. S. Yang, Y. R. Huang, and K. M. Zhang, "The value of the Ramsey number $r(C_n, K_4)$ is $3(n-1)+1(n \geq 4)$," *The Australasian Journal of Combinatorics*, vol. 20, pp. 205–206, 1999.

[7] B. Bollobás, C. Jayawardene, J. Yang, Y. R. Huang, C. Rousseau, and K. M. Zhang, "On a conjecture involving cycle-complete graph Ramsey numbers," *The Australasian Journal of Combinatorics*, vol. 22, pp. 63–71, 2000.

[8] I. Schiermeyer, "All cycle-complete graph Ramsey numbers $r(C_n, K_6)$," *Journal of Graph Theory*, vol. 44, no. 4, pp. 251–260, 2003.

[9] A. Baniabedalruhman and M. M. M. Jaradat, "The cycle-complete graph Ramsey number $r(C_7, K_7)$," *Journal of Combinatorics, Information & System Sciences*, vol. 35, pp. 293–305, 2010.

[10] M. M. M. Jaradat and A. Baniabedalruhman, "The cycle-complete graph Ramsey number $r(C_8, K_7)$," *International Journal of Pure and Applied Mathematics*, vol. 41, no. 5, pp. 667–677, 2007.

[11] Y. Chen, T. C. E. Cheng, and Y. Zhang, "The Ramsey numbers $R(C_m, K_7)$ and $R(C_7, K_8)$," *European Journal of Combinatorics*, vol. 29, no. 5, pp. 1337–1352, 2008.

[12] M. M. M. Jaradat and B. M. N. Alzaleq, "The cycle-complete graph Ramsey number $r(C_8, K_8)$," *SUT Journal of Mathematics*, vol. 43, no. 1, pp. 85–98, 2007.

[13] Y. Zhang and K. Zhang, "The Ramsey number $r(C_8, K_8)$," *Discrete Mathematics*, vol. 309, no. 5, pp. 1084–1090, 2009.

[14] S. P. Radziszowski and K.-K. Tse, "A computational approach for the Ramsey numbers $r(C_4, K_n)$," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 42, pp. 195–207, 2002.

[15] C. J. Jayawardene and C. C. Rousseau, "The Ramsey number for a cycle of length five vs. a complete graph of order six," *Journal of Graph Theory*, vol. 35, no. 2, pp. 99–108, 2000.

[16] I. Schiermeyer, "The cycle-complete graph Ramsey number $r(C_5, K_7)$," *Discussiones Mathematicae. Graph Theory*, vol. 25, no. 1-2, pp. 129–139, 2005.

[17] S. P. Radziszowski, "Small ramsey numbers," *Electronic Journal of Combinatorics*, pp. 1–72, 2009.