Research Article

Notes on the Union of Weakly Primary Submodules

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Received 30 August 2011; Accepted 18 October 2011

Academic Editor: U. A. Rozikov

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Let R be a commutative ring with identity, and let M be an R -module. A proper submodule N of M is said to be weakly primary if $0 \neq rm \in N$ for $r \in R$ and $m \in M$, which implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n . In this paper, we study weakly primary submodules, and we investigate the union of weakly primary submodules of R -modules.

1. Introduction

Let R be a commutative ring with identity, and let M be a unital R -module. A commutative ring R is called a quasilocal ring if it has a unique maximal ideal P and denoted by (R, P) . Let N be a submodule of M and the ideal $\{r \in R : rM \subseteq N\}$ denoted by $(N : M)$. Let I be an ideal of R and the radical of I denoted by $\sqrt{I}$ and defined $\sqrt{I} = \{r \in R : r^n \in I \text{ for some positive integer } n\}$. A proper submodule N of M is said to be prime (weakly prime) if $rm \in N$ ($0 \neq rm \in N$), then either $m \in N$ or $rM \subseteq N$ (either $m \in N$ or $rM \subseteq N$), where $r \in R$ and $m \in M$. A proper submodule N of M is said to be primary (weakly primary) if $rm \in N$ ($0 \neq rm \in N$), then either $m \in N$ or $r^n M \subseteq N$ for some positive integer n (either $m \in N$ or $r^n M \subseteq N$ for some positive integer n), where $r \in R$ and $m \in M$. It is clear that every primary submodule is weakly primary. However, since 0 is always weakly primary (by definition), so a weakly primary submodule need not be primary. A proper submodule N of an R -module M said to be maximal if there is no submodule K of M such that $N \subsetneq K \subsetneq M$. A submodule N of M is called u -submodule of M , provided that N contained in a finite union of submodules must be contained one of those submodules. M is called u -module if every submodule of M is a u -submodule. A submodule N of M is called upr -submodule of M , provided that N contained in a finite union of primary submodules that must be contained in one of those primary submodules. M is called upr -module if every submodule of M is a upr -submodule. A submodule N of M is called um -submodule of M , provided that N contained in a finite union of maximal submodules that must be contained in one of those

submodules. M is called *um*-module if every maximal submodule of M is a *um*-submodule. An R -module M is called a multiplication module, provided that for each submodule N of M , there exists an ideal I of R such that $N = IM$. If R is a ring and M an R -module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Obviously, if R is an integral domain, then $T(M)$ is a submodule of M . In this paper, we investigate finite unions of weakly primary submodules of R -modules.

2. On Weakly Primary Submodules

It is clear that every primary submodule is a weakly primary submodule. However, since 0 is always weakly primary (by definition), a weakly primary submodule need not be primary, but we have the following results.

Proposition 2.1. *Let M be an R -module with $T(M) = 0$. Then, every weakly primary submodule of M is primary.*

Proof. Let N be a weakly primary submodule of M . Suppose that $rm \in N$, where $r \in R$, $m \in M$. If $0 \neq rm \in N$, N weakly primary gives $m \in N$ or $r^n M \subseteq N$ for some positive integer n . If $rm = 0$, then $r = 0$ or $m = 0$, since $T(M) = 0$. So, N is primary. $\square$

Proposition 2.2. *Let M be a module over a quasilocal ring (R, P) with $PM = 0$. Then, every proper submodule of M is weakly primary.*

Proof. Let N be a proper submodule of M and $0 \neq rm \in N$, where $r \in R$ and $m \in M$. If r is a unite, then $m \in N$. Let r is not a unite, so $rm \in PM = 0$, a contradiction. Hence, N is weakly primary. $\square$

Lemma 2.3. *Let M be an R -module. Assume that N and K are submodules of M such that $K \subseteq N$ with $N \neq M$. Then, the following hold:*

- (i) *if N is a weakly primary submodule of M , then N/K is a weakly primary submodule of M/K ,*
- (ii) *if K and N/K are weakly primary submodules, then N is weakly primary.*

Proof. (i) Let $0 \neq r(m + K) = rm + K \in N/K$, where $r \in R$ and $m \in M$. If $rm = 0$, then $r(m + K) = 0$, which is a contradiction. If $rm \neq 0$, N weakly primary gives either $m \in N$ or $r^n \in (N :_R M)$ for some positive integer n , hence either $m + K \in N/K$ or $r^n \in (N/K :_R M/K)$ (since we have $(N :_R M) = (N/K :_R M/K)$), as required.

(ii) Let $0 \neq rm \in N$, where $r \in R$ and $m \in M$, so $r(m + K) = rm + K \in N/K$. If $rm \in K$, then K weakly primary gives either $m \in K \subseteq N$ or $r^n \in (K :_R M) \subseteq (N :_R M)$. So, we may assume that $rm \notin K$. Then, $0 \neq r(m + K) \in N/K$. Since N/K is weakly primary, we get either $m + N \in N/K$ or $r^n \in (N/K :_R M/K) = (N : M)$ for some positive integer n . Thus, $m \in N$ or $r^n \in (N :_R M)$ for some positive integer n , as required. $\square$

Theorem 2.4. *Let M be a secondary R -module and N a nonzero weakly primary R -submodule of M . Then, N is secondary.*

Proof. Let $r \in R$. If $r^n M = 0$ for some $n \in \mathbb{N}$. Then, $r^n N \subseteq r^n M = 0$, so r is nilpotent on N . Suppose that $rM \neq 0$; we show that r divides N . Assume that $n \in N$. So, $n = rm$ for some

$m \in M$. We may assume that $0 \neq rm$. Hence, $0 \neq rm \in N$ and $r^n M \not\subseteq N$ for any positive integer n (since $rM = M$), then N weakly primary gives $m \in N$. Thus, $rN = N$, as needed. $\square$

Theorem 2.5. *Let M be an R -module, N a secondary R -submodule of M , and K a weakly primary submodule of M . Then, $N \cap K$ is secondary.*

Proof. The proof is straightforward. $\square$

Proposition 2.6. *Let M be a module over a commutative ring R and S a multiplicatively closed subset of R . Let N be a weakly primary submodule of M such that $\sqrt{(N : M)} \cap S = \emptyset$. Then, $S^{-1}N$ is a weakly primary submodule of $S^{-1}R$ -module $S^{-1}M$.*

Proof. Let $0/1 \neq r/s \cdot m/t \in S^{-1}N$, where $r/s \in S^{-1}R$ and $m/t \in S^{-1}M$. So, $0/1 \neq rm/st = n/t'$ for some $n \in N$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'rm = s'stn \in N$ (because if $s't'rm = 0$, $rm/st = s't'rm/s't'st = 0/1$, a contradiction) and $s't' \notin \sqrt{(N : M)}$, so N weakly primary gives $0 \neq rm \in N$. Hence, $m \in N$ or $r^n \in (N : M)$, thus $(r/s)^n \in S^{-1}(N : M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $m/t \in S^{-1}N$, as needed. $\square$

Lemma 2.7. *Let M be a module over a quasilocal ring (R, P) and N a weakly primary submodule of M . Then, $(N :_R M)_P = (N_{P:P_P} M_P)$.*

Proof. Let $r/s \in (N_{P:P_P} M_P)$ and $m \in M$. We show that $rm \in N$. We may assume that $rm \neq 0$. We have $r/s \cdot m/1 \in S^{-1}N$, so $rm/s = n/t$ for some $t \in S$ and $n \in N$. There exists $t' \in S$ such that $t'trm = t'sn \in N$. If $t'trm = 0$, then $tt' \in (0 : rm) \cap S \subseteq P \cap S = \emptyset$, a contradiction. So, $0 \neq t'trm \in N$ and $tt' \notin \sqrt{(N : M)}$, then $rm \in N$. Thus, $(N_{P:P_P} M_P) \subseteq (N :_R M)_P$. Clearly, $(N :_R M)_P \subseteq (N_{P:P_P} M_P)$, so the proof is complete. $\square$

Theorem 2.8. *Let M be a module over a quasilocal ring (R, P) . Then, there exists a one to one correspondence between the weakly primary submodules of M and the weakly primary submodules of R_P -module M_P .*

Proof. Let K be a weakly primary submodule of M_P . So, $K = N_P$ for some submodule N of M . We show that N is weakly primary submodule of M . Let $0 \neq rm \in N$, so $0/1 \neq rm/1 \in N_P$ (if $rm/1 = 0/1$, then $srM = 0$ for some $s \in S$, $s \in (0 : rm) \cap S \subseteq P \cap S = \emptyset$, a contradiction). Hence, $(r/1)^n \in (N_{P:P_P} M_P) \subseteq (N :_P M)_P$ for some positive integer n by Lemma 2.7 or $m/1 \in N_P$, since N_P is weakly primary. Thus, $r^n \in (N : M)$ for some positive integer n or $m \in N$, as required. Let N be a weakly primary submodule of M . Then, by Proposition 2.6, N_P is weakly primary submodule of M_P . $\square$

3. Unions of Weakly Primary Submodules

Definition 3.1. Let M be a module over a commutative ring R and N a submodule of M ; N is called a *uwpr*-submodule of M , provided that N contained in a finite weakly primary submodules of M must be contained in one of those weakly primary submodules. M is called a *uwpr*-module if every submodule of M is a *uwpr*-submodule.

Clearly, every *u*-module is *uwpr*-module, and every *uwpr*-module is *upr*-module and also, every *upr*-module is *um*-module.

Theorem 3.2. Let M be a module over a commutative ring R and K a submodule of M . Then,

- (i) if N is a $uwpr$ -submodule of M and K is a weakly primary submodule of M such that $K \subseteq N$, then N/K is a $uwpr$ -submodule of R -module M/K ,
- (ii) if N/K is a $uwpr$ -submodule of M/K , then N is a $uwpr$ -submodule of M .

Proof. (i) Let $N/K \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$, where L_i 's are weakly primary submodules of M/K . Then, by Lemma 2.3, there exists weakly primary submodules N_i of M such that $L_i = N_i/K$ for $i = 1, 2, \dots, n$. So, $N/K \subseteq N_1/K \cup \cdots \cup N_n/K = N_1 \cup \cdots \cup N_n/K$, so $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. Hence, $N \subseteq N_k$ for some k , since N is $uwpr$ -submodule. Thus $N/K \subseteq N_k/K$, for some k , as needed.

(ii) Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, where N_i 's are weakly primary submodule of M . So, $N/K \subseteq N_1 \cup \cdots \cup N_n/K = N_1/K \cup \cdots \cup N_n/K$. Therefore, N_i/K 's are weakly primary submodules of M/K , so $N/K \subseteq N_k/K$ for some k . Hence, $N \subseteq N_k$ for some k , as required. $\square$

Theorem 3.3. Let M be an R -module, N a weakly primary submodule of M , and M a $uwpr$ -module. Then, M/N is a $uwpr$ -module.

Proof. By Theorem 3.2. $\square$

Theorem 3.4. Let M be a module over quasi local ring (R, P) . Then, M is a $uwpr$ -module if and only if R_P module M_P is a $uwpr$ -module.

Proof. Let M be a $uwpr$ -module, and let K be a submodule of R_P -module M_P such that $K \subseteq K_1 \cup \cdots \cup K_n$, where K_i 's are weakly primary submodules of M_P . So, $K = N_P$ for some submodule N of M and $K_i = (N_i)_P$ for some weakly primary submodules N_i of M by Theorem 2.8. Hence, $N_P \subseteq (N_1)_P \cup \cdots \cup (N_n)_P$, so $N_P \subseteq (N_1 \cup \cdots \cup N_n)_P$, thus $N \subseteq N_1 \cup \cdots \cup N_n$ by [1, Theorem 2.8]. Therefore, by hypothesis $N \subseteq N_k$ for some $1 \leq k \leq n$. So $N_P \subseteq (N_k)_P$ for some $1 \leq k \leq n$ as needed.

Conversely, let R_P -module M_P is a $uwpr$ -module, and let N be a submodule of M such that $N \subseteq N_1 \cup \cdots \cup N_n$, where N_i 's are weakly primary submodules of M . So, $N_P \subseteq (N_1 \cup \cdots \cup N_n)_P = (N_1)_P \cup \cdots \cup (N_n)_P$. Thus, $N_P \subseteq (N_k)_P$ for some $1 \leq k \leq n$ by hypothesis. Then, $N \subseteq N_k$ for some $1 \leq k \leq n$. So M is a $uwpr$ -module. $\square$

Theorem 3.5. Let M be a finitely generated R -module. Then, M is a um -module if and only if every submodule N in M such that $N \subseteq \bigcup_{i=1}^n P_i$, where P_i 's are weakly primary submodules implies that $N + P_i \neq M$ for some i .

Proof. Let M be a finitely generated um -module. Suppose that N be a submodule of M such that $N \subseteq \bigcup_{i=1}^n P_i$, where P_i 's are weakly primary submodules of M . For each P_i , M_i is a maximal submodule containing P_i . Then, $N \subseteq \bigcup_{i=1}^n M_i$, and so, $N \subseteq M_i$ for some i by hypothesis. Since $P_i \subseteq M_i$, we have $N + P_i \subseteq M_i \neq M$.

Conversely, let N be a submodule of M such that $N \subseteq \bigcup_{i=1}^n M_i$, where M_i 's are maximal submodules of M . Since every maximal submodule is weakly primary submodule, then $N + M_i \neq M$ for some i by hypothesis. Therefore, since $M_i \subseteq N + M_i \subsetneq M$, then $N + M_i = M_i$, so $N \subseteq M_i$ for some i . The proof is complete. $\square$

Definition 3.6. By a chain of weakly primary submodules of an R -module M , we mean a finite strictly increasing sequence $P_1 \subseteq \cdots \subseteq P_n$; the weakly primary dimension of this chain is n . We

define the weakly primary dimension of M to be the supremum of the lengths of all chains of weakly primary submodules in M .

Theorem 3.7. *Let M be a finitely generated R -module with weakly primary dimension 1. Then, M is a $uwpr$ -module if and only if M is a um -module.*

Proof. Let M be a $uwpr$ -module. Since every $uwpr$ -module is a um -module, so M is a um -module.

Conversely, let M be a um -module. Let N be a nonzero submodule of M such that $N \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$, where P_i 's are weakly primary submodules of M . We may assume that $P_i \neq 0$ for all $i \in \{1, 2, \dots, n\}$. By Theorem 3.5, $N + P_i \neq M$ for some i . There exists a maximal submodule P_i of M such that $N + P_i \subseteq M_i$. Since 0 is a weakly primary submodule of M , so we have $0 \subsetneq N_i \subseteq M_i$. Hence, since weakly primary dimension of M is 1; $P_i = M_i$. Consequently, $N \subseteq P_i$ for some i , as needed. $\square$

Theorem 3.8. *Let M be an R -module with $T(M) = 0$. Then, M is a upr -module if and only if M is a $uwpr$ -module.*

Proof. Let M be a $uwpr$ -module. Then, M is a upr -module since every primary submodule is weakly primary. Let M be a up -module and N a submodule of M such that $N \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$ where P_i 's are weakly primary submodules of M . By Proposition 2.1, P_i 's are primary submodule, so $N \subseteq P_i$ for some i , as needed. $\square$

References

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