International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 987595, 15 pages
doi:10.5402/2011/987595

Research Article

On Some Nonlinear Integrodifferential Inequalities for Functions of n Independent Variables

Hassane Khellaf and Mohamed El-Hadi Smakdji

Department of Mathematics, Faculty of Exact Sciences, University of Mentouri, Constantine 25000, Algeria

Correspondence should be addressed to Hassane Khellaf, khellafhassane@yahoo.fr

Received 23 December 2010; Accepted 11 January 2011

Academic Editors: R. Avery, S. S. Ravindran, and R. Spigler

The object of this paper is to establish some nonlinear integrodifferential integral inequalities in n independent variables. These new inequalities represent a generalization of the results obtained by Pachpatte in the case of a function with one and two variables. Our results can be used as tools in the qualitative theory of a certain class of partial integrodifferential equation.

1. Introduction

It is well known that the integral inequalities involving functions of one and more than one independent variables, which provide explicit bounds on unknown functions, play a fundamental role in the development of the theory of differential equations. In the past few years, a number of integral inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to (see [1–5]) and the references therein.

The study of integrodifferential inequalities for functions of one or n independent variables is also a very important tool in the study of stability, existence, bounds, and other qualitative properties of differential equation solutions, integrodifferential equations, and in the theory of hyperbolic partial differential equations (see [6–9]).

One of the most useful inequalities is given in the following lemma (see [1, 10]).

Lemma 1.1 (see [1]). *Let $\Phi(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for $x \geq 0$, $y \geq 0$, for which the inequality*

$$\Phi(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t)\Phi(s,t)ds\,dt \tag{1.1}$$

holds for $x \geq 0$, $y \geq 0$, *where* $a(x), b(y) > 0$; $a'(x), b'(y) \geq 0$ *are continuous functions defined for* $x \geq 0$, $y \geq 0$. *Then*

$$\Phi(x,y) \leq \frac{[a(0)+b(y)][a(x)+b(0)]}{[a(0)+b(0)]} \exp\left(\int_0^x \int_0^y c(s,t)ds\,dt\right), \tag{1.2}$$

for $x \geq 0$, $y \geq 0$.

Wendroff's inequality has recently evoked a lively interest, as may be seen from the papers of Pachpatte [10]. In [10], Pachpatte considered some new integrodifferential inequalities of the Wendroff type for functions of two independent variables. Our aim in this paper is to establish some integrodifferential inequalities in n independent variables, an application of our results is also given.

2. Results

Throughout this paper, we will assume that S in any bounded open set in the dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n$ $(n \geq 1)$.

For $x = (x_1, x_2, \ldots x_n)$, $t = (t_1, t_2, \ldots t_n)$, $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in S$, we will denote

$$\int_{x^0}^{x} dt = \int_{x_1^0}^{x_1} \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} \cdots dt_n \cdots dt_1. \tag{2.1}$$

Furthermore, for $x, t \in \mathbb{R}^n$, we will write $t \leq x$ whenever $t_i \leq x_i$, $i = 1, 2, \ldots, n$ and $x \geq x_0 \geq 0$, for $x, x^0 \in S$.

We note $D = D_1 D_2 \cdots D_n$, where $D_i = \partial/\partial x_i$, for $i = 1, 2, \ldots, n$.

We use the usual convention of writing $\sum_{s \in \Psi} u(s) = 0$ if Ψ is the empty set.

Our main results are given in the following theorems.

Theorem 2.1. *Let* $\Phi(x)$ *and* $c(x)$ *be nonnegative continuous functions defined on* S, *for which the inequality*

$$\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)\Phi(t)dt \tag{2.2}$$

holds for all $x \in S$ *with* $x \geq x^0 \geq 0$, *where* $a_i(x_i) > 0$, $a_i'(x_i)$ *are continuous functions defined for* $x_i \geq 0$ *for all* $i = 1, 2, \ldots, n$. *Then*

$$\Phi(x) \leq A(x) \exp\left(\int_{x^0}^{x} c(t)dt\right), \tag{2.3}$$

for $x \in S$ *with* $x \geq x^0 \geq 0$, *where*

$$A(x) = \frac{\left[a_1(x_1) + a_2(x_2^0) + \sum_{s=3}^{n} a_s(x_s)\right]\left[a_1(x_1^0) + a_2(x_2) + \sum_{s=3}^{n} a_s(x_s)\right]}{\left[a_1(x_1^0) + a_2(x_2^0) + \sum_{s=3}^{n} a_s(x_s)\right]}. \tag{2.4}$$

Proof. We define the function $u(x)$ by the right member of (2.2), Then

$$Du(x) = c(x)\Phi(x), \tag{2.5}$$

$$u\left(x_1^0, x_2, \ldots, x_n\right) = a_1\left(x_1^0\right) + a_2(x_2) + \sum_{s=3}^{n} a_s(x_s), \tag{2.6}$$

$$u\left(x_1, x_2^0, x_3, \ldots, x_n\right) = a_1(x_1) + a_2\left(x_2^0\right) + \sum_{s=3}^{n} a_s(x_s). \tag{2.7}$$

Using $\Phi(x) \leq u(x)$ in (2.5), we have

$$Du(x) \leq c(x)u(x). \tag{2.8}$$

From (2.8), we observe that

$$\frac{u(x)Du(x)}{u^2(x)} \leq c(x), \tag{2.9}$$

that is

$$\frac{u(x)Du(x)}{u^2(x)} \leq c(x) + \frac{(D_n u(x))(D_1 \cdots D_{n-1}u(x))}{u^2(x)}, \tag{2.10}$$

hence

$$D_n\left(\frac{D_1 \cdots D_{n-1}u(x)}{u(x)}\right) \leq c(x). \tag{2.11}$$

Integrating (2.11) with respect to x_n from x_n^0 to x_n, we have

$$\frac{(D_1 \cdots D_{n-1}u(x))}{u(x)} \leq \int_{x_n^0}^{x_n} c(x_1, \ldots, x_{n-1}, t_n)dt_n, \tag{2.12}$$

thus

$$\frac{u(x)D_1 \cdots D_{n-1}u(x)}{u^2(x)} \leq \int_{x_n^0}^{x_n} c(x_1, \ldots, x_{n-1}, t_n)dt_n + \frac{(D_{n-1}u(x))(D_1 \cdots D_{n-2}u(x))}{u^2(x)}, \tag{2.13}$$

that is

$$D_{n-1}\left(\frac{D_1 \cdots D_{n-2}u(x)}{u(x)}\right) \leq \int_{x_n^0}^{x_n} c(x_1, \ldots, x_{n-1}, t_n)dt_n. \tag{2.14}$$

Integrating (2.14) with respect to x_{n-1} from x_{n-1}^0 to x_{n-1}, we have

$$\frac{D_1 \cdots D_{n-2} u(x)}{u(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}} \int_{x_n^0}^{x_n} c(x_1, \ldots x_{n-2}, t_{n-1}, t_n) dt_n \, dt_{n-1}. \tag{2.15}$$

Continuing this process, we obtain

$$\frac{D_1 D_2 u(x)}{u(x)} \leq \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, x_2, t_3, \ldots, t_n) dt_n \, dt_{n-1} \cdots dt_3, \tag{2.16}$$

from this we obtain

$$D_2 \left(\frac{D_1 u(x)}{u(x)} \right) \leq \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, x_2, t_3, \ldots, t_n) dt_n \, dt_{n-1} \cdots dt_3. \tag{2.17}$$

Integrating (2.17) with respect to x_2 from x_2^0 to x_2 and by (2.7), we have

$$\frac{D_1 u(x)}{u(x)} \leq \frac{a_1'(x_1)}{a_2(x_0^0) + a_1(x_1) + \sum_{s=3}^n a_s(x_s)} + \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} c(x_1, t_2, t_3, \ldots, t_n) dt_n \, dt_{n-1} \cdots dt_2. \tag{2.18}$$

Integrating (2.18) with respect to x_1 from x_1^0 to x_1 and by (2.6), we have

$$\log \frac{u(x)}{u(x_1^0, x_2, \ldots, x_n)} \leq \int_{x_1^0}^{x_1} \frac{a_1'(t_1)}{a_2(x_2^0) + a_1(t_1) + \sum_{s=3}^n a_s(x_s)} dt_1 + \int_{x^0}^{x} c(t) dt, \tag{2.19}$$

that is

$$u(x) \leq A(x) \exp\left(\int_{x^0}^{x} c(t) dt \right). \tag{2.20}$$

By (2.20) and $\Phi(x) \leq u(x)$, we obtain the desired bound in (2.3). □

Remark 2.2. We note that in the special case $n = 2$, $x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 2.1. our estimate reduces to Lemma 1.1 (see [10]).

Theorem 2.3. *Let $\Phi(x)$, $c(x)$, $D_i\Phi(x)$, and $D\Phi(x)$ be nonnegative continuous functions for all $i = 1, 2, \ldots, n$ defined for $x \in S$, $\Phi(x_1^0, x_2, x_3, \ldots, x_n) = 0$ and $\Phi(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots x_n) = 0$ for any $i = 2, 3, \ldots, n$. If*

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t) [\Phi(t) + D\Phi(t)] dt \tag{2.21}$$

holds for $x \in S$, where $a_i(x_i) > 0$; $a_i'(x_i) \geq 0$ are continuous functions defined for $x_i \geq 0$ for all $i = 1, 2, \ldots, n$. Then

$$D\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)\left[A(t)\exp\left(\int_{x^0}^{t}[1+c(\tau)]d\tau\right)\right]dt. \tag{2.22}$$

For $x \in S$ with $x \geq t \geq \tau \geq x^0 \geq 0$, where $A(x)$ is defined in (2.4).

Proof. We define the function

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + \int_{x_0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt, \tag{2.23}$$

$$u\left(x_1^0, x_2, x_3, \ldots, x_n\right) = a_1\left(x_1^0\right) + \sum_{i=2}^{n} a_i(x_i). \tag{2.24}$$

Then, (2.21) can be restated as

$$D\Phi(x) \leq u(x). \tag{2.25}$$

Differentiating (2.23),

$$Du(x) = c(x)[\Phi(x) + D\Phi(x)]. \tag{2.26}$$

Integrating both sides of (2.26) to x from x^0 to x, we have

$$\Phi(x) \leq \int_{x_0}^{x} u(t)dt. \tag{2.27}$$

Now, using (2.27) and (2.25) in (2.26) we obtain

$$Du(x) \leq c(x)\left(u(x) + \int_{x^0}^{x} u(t)dt\right). \tag{2.28}$$

If we put

$$v(x) = u(x) + \int_{x^0}^{x} u(t)dt, \tag{2.29}$$

$$v\left(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots, x_n\right) = u\left(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots, x_n\right), \tag{2.30}$$

then by (2.29), we have

$$Dv(x) = Du(x) + u(x). \tag{2.31}$$

Using the facts that $Du(x) \leq c(x)v(x)$ and $u(x) \leq v(x)$, we have

$$Dv(x) \leq [1 + c(x)]v(x). \tag{2.32}$$

Which, by following an argument similar to that in the proof of Theorem 2.1, yields the estimate for $v(x)$ such that

$$v(x) \leq A(x) \exp\left(\int_{x^0}^{x} [1 + c(t)]dt\right). \tag{2.33}$$

By (2.33) and (2.28), we have

$$Du(x) \leq c(x)A(x) \exp\left(\int_{x^0}^{x} [1 + c(t)]dt\right), \tag{2.34}$$

$$D_1 D_2 \cdots D_{n-1} u\left(x_1, \ldots, x_{n-1}, x_n^0\right) = 0. \tag{2.35}$$

Integrating both sides of (2.34) to x_n from x_n^0 to x_n and by (2.35), we have

$$D_1 D_2 \cdots D_{n-1} u(x) \leq \int_{x_n^0}^{x_n} c(x_1, \ldots x_{n-1}, t_n) A(x_1, \ldots x_{n-1}, t_n) \exp\left(\int_{x^0}^{t} [1 + c(\tau)]d\tau\right) dt_n. \tag{2.36}$$

By (2.23), we have

$$D_1 D_2 u\left(x_1, x_2, x_3^0, x_4, \ldots, x_n\right) = 0. \tag{2.37}$$

Continuing this process, and by (2.37), we obtain

$$D_1 D_2 u(x) \leq \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, x_2, t_3, \ldots, t_n) A(x_1, x_2, t_3, \ldots, t_n) \exp\left(\int_{x^0}^{t} [1 + c(\tau)]d\tau\right) dt_n \cdots dt_3. \tag{2.38}$$

By (2.23), we have

$$D_1 u\left(x_1, x_2^0, x_3, x_4, \ldots, x_n\right) = a_1'(x_1). \tag{2.39}$$

Integrating both sides of (2.38) to x_2 from x_2^0 to x_2 and by (2.39), we have

$$\begin{aligned} D_1 u(x) \leq a_1'(x_1) + \int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, t_2, \ldots, t_n) A(x_1, t_2, \ldots, t_n) \\ \times \exp\left(\int_{x^0}^{t} [1 + c(\tau)]d\tau\right) dt_n \cdots dt_2. \end{aligned} \tag{2.40}$$

Integrating (2.40) with respect to x_1 from x_1^0 to x_1, and by (2.24), we have

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} c(t)A(t)\exp\left(\int_{x^0}^{t} [1 + c(\tau)]d\tau\right)dt. \tag{2.41}$$

By (2.41) and (2.25), we obtain the desired bound in (2.22). □

Remark 2.4. We note that in the special case $n = 2$, $x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 2.3, then our result reduces to Theorem 1 obtained in [10].

Theorem 2.5. *Let $\Phi(x)$, $c(x)$, $D_i\Phi(x)$, and $D\Phi(x)$ be nonnegative continuous functions for all $i = 1,2,\ldots,n$ defined for $x \in S$, $\Phi(x_1^0, x_2, x_3, \ldots, x_n) = 0$ and $\Phi(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots x_n) = 0$ for any $i = 2,3,\ldots,n$. If*

$$D\Phi(x) \le \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right] \tag{2.42}$$

holds for $x \in S$, where $a_i(x_i) > 0$; $a_i'(x_i) \ge 0$ are continuous functions defined for $x_i \ge 0$ for all $i = 1,2,\ldots,n$. and $M \ge 0$ is constant. Then

$$D\Phi(x) \le A(x)\exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right), \tag{2.43}$$

for $x \in S$, with $x \ge t \ge x^0 \ge 0$, where $A(x)$ is defined in (2.4).

Proof. We define the function

$$u(x) = \sum_{i=1}^{n} a_i(x_i) + M\left[\Phi(x) + \int_{x^0}^{x} c(t)[\Phi(t) + D\Phi(t)]dt\right] \tag{2.44}$$

with

$$u\left(x_1^0, x_2, x_3, \ldots, x_n\right) = a_1\left(x_1^0\right) + \sum_{i=2}^{n} a_i(x_i). \tag{2.45}$$

Differentiating (2.44), we have

$$Du(x) = M[D\Phi(x) + c(x)[\Phi(x) + D\Phi(x)]]. \tag{2.46}$$

Using the fact that $D\Phi(x) \le u(x)$ and $M\Phi(x) \le u(x)$, we have

$$Du(x) \le [M + c(x) + Mc(x)]u(x), \tag{2.47}$$

by (2.47), we have

$$u(x) \leq A(x) \exp\left(\int_{x^0}^{x} [M + c(t) + Mc(t)]dt\right), \tag{2.48}$$

where $A(x)$ is defined in (2.4).

By (2.48) and using the fact that $D\Phi(x) \leq u(x)$ from (2.42), we obtain the desired bound in (2.43). □

Remark 2.6. We note that in the special case $n = 2$, $x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 2.5, then our result reduces to Theorem 2 obtained in [10].

Theorem 2.7. *Let $\Phi(x)$, $p(x)$, and $q(x)$ be nonnegative continuous functions defined for $x \in S$. If*

$$\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} p(t)\Phi(t)dt + \int_{x^0}^{x} p(t)\left(\int_{x^0}^{t} q(s)\Phi(s)ds\right)dt \tag{2.49}$$

holds for $x \geq x^0 \geq 0$, where $a_i(x_i) > 0$; $a_i'(x_i) \geq 0$ are continuous functions defined for $x_i \geq 0$ for all $i = 1, 2, \ldots, n$. Then

$$\Phi(x) \leq \sum_{i=1}^{n} a_i(x_i) + \int_{x^0}^{x} p(t)Q(t)dt, \tag{2.50}$$

for all $x \geq x^0 \geq 0$, where

$$Q(x) = A(x) \exp\left(\int_{x^0}^{x} (p(t) + q(t))dt\right), \tag{2.51}$$

with $A(x)$ defined in (2.4).

Proof. The proof of this Theorem follows by an argument similar to that in Theorem 2.1, We omit the details. □

Remark 2.8. We note that in the special case $n = 2$, $x \in \mathbb{R}_+^2$ and $x^0 = (x_1^0, x_2^0) = (0,0)$ in Theorem 2.7, our result reduces to Theorem 2 obtained in [10].

3. Nonlinear Integrodifferential in n Independents Variables

In this section, we will give some new nonlinear integrodifferential inequalities for the functions of n-independent variables.

We can also give the following lemma.

Lemma 3.1 (see [2, 11]). *Let $u(x)$, $a(x)$, and $b(x)$ be nonnegative continuous functions, defined for $x \in S$.*

Assume that $a(x)$ is a positive, continuous function and nondecreasing in each of the variables $x \in S$. If

$$u(x) \leq a(x) + \int_{x^0}^{x} b(t)u(t)dt \tag{3.1}$$

holds for all $x \in S$, with $x \geq x^0 \geq 0$. Then

$$u(x) \leq a(x)\exp\left(\int_{x^0}^{x} b(t)dt\right). \tag{3.2}$$

Theorem 3.2. *Let $\Phi(x)$, $a(x)$, $b(x)$, $c(x)$, $f(x)$, $D_i\Phi(x)$, and $D\Phi(x)$ be nonnegative continuous functions for all $i = 1,2,\ldots,n$ defined for $x \in S$, $\Phi(x_1^0, x_2, x_3, \ldots, x_n) = 0$ and $\Phi(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots x_n) = 0$ for any $i = 2,3,\ldots,n$. Let $K(\Phi(x))$ be a real-valued, positive, continuous, strictly nondecreasing, subadditive, and submultiplicative function for $\Phi(x) \geq 0$, and let $H(\Phi(x))$ be a real-valued, continuous positive, and nondecreasing function defined for $x \in S$. Assume that $a(x)$ and $f(x)$ are positive and nondecreasing in each of the variables $x \in S$. If*

$$D\Phi(x) \leq a(x) + f(x)H\left(\int_{x^0}^{x} c(t)K(\Phi(t))dt\right) + \int_{x^0}^{x} b(t)D\Phi(t)dt \tag{3.3}$$

holds, for $x \in S$ with $x \geq x^0 \geq 0$. Then

$$D\Phi(x) \leq \left\{a(x) + f(x)H\left(G^{-1}\left[G(\xi) + \int_{x_0}^{x} c(t)K\big(p(t)f(t)\big)dt\right]\right)\right\}\exp\left(\int_{x^0}^{x} b(t)dt\right), \tag{3.4}$$

for $x \in S$, where

$$\begin{aligned}
p(x) &= \int_{x^0}^{x} \exp\left(\int_{x^0}^{t} b(s)ds\right)dt, \\
\xi &= \int_{x_0}^{\infty} c(t)K\big(a(t)p(t)\big)dt, \\
G(z) &= \int_{z^0}^{z} \frac{ds}{K(H(s))}, \quad z \geq z^0 > 0,
\end{aligned} \tag{3.5}$$

where G^{-1} is the inverse function of G, and

$$G(\xi) + \int_{x_0}^{x} c(t)K\big(p(t)f(t)\big)dt \tag{3.6}$$

is in the domain of G^{-1} for $x \in S$.

Proof. We define the function

$$z(x) = a(x) + f(x)H\left(\int_{x^0}^{x} c(t)K(\Phi(t))dt\right), \tag{3.7}$$

then (2.4) can be restated as

$$D\Phi(x) \leq z(x) + \int_{x^0}^{x} b(t)D\Phi(t)dt. \tag{3.8}$$

Clearly, $z(x)$ is a positive, continuous function and nondecreasing in each of the variables $x \in S$, using (3.1) of Lemma 3.1 to (3.8), we have

$$D\Phi(x) \leq z(x)\exp\left(\int_{x^0}^{x} b(t)dt\right). \tag{3.9}$$

Integrating to x from x^0 to x, we have

$$\Phi(x) \leq z(x)p(x), \tag{3.10}$$

where

$$p(x) = \int_{x^0}^{x} \exp\left(\int_{x^0}^{t} b(s)ds\right)dt. \tag{3.11}$$

By (3.7), we have

$$z(x) = a(x) + f(x)H(v(x)), \tag{3.12}$$

where

$$v(x) = \int_{x^0}^{x} c(t)K(\Phi(t))dt. \tag{3.13}$$

By (3.10) and (3.13), we have

$$\Phi(x) \leq \{a(x) + f(x)H(v(x))\}p(x). \tag{3.14}$$

From (3.14) and (3.13) and since K is a subadditive and submultiplicative function, we notice that

$$\begin{aligned} v(x) &\leq \int_{x^0}^{x} c(t) K\left[\{a(t) + f(t) H(v(t))\} p(t)\right] dt, \\ &\leq \int_{x^0}^{x} c(t) K\left(a(t) p(t)\right) dt + \int_{x^0}^{x} c(t) K\left(f(t) p(t)\right) K(H(v(t))) dt, \\ &\leq \int_{x^0}^{\infty} c(t) K\left(a(t) p(t)\right) dt + \int_{x^0}^{x} c(t) K\left(f(t) p(t)\right) K(H(v(t))) dt. \end{aligned} \tag{3.15}$$

We define $\Psi(x)$ as the right side of (3.14), then

$$\Psi\left(x_1^0, x_2, x_3, \ldots, x_n\right) = \int_{x^0}^{\infty} c(t) K\left(a(t) p(t)\right) dt, \tag{3.16}$$

$$v(x) \leq \Psi(x). \tag{3.17}$$

$\Psi(x)$ is positive and nondecreasing in each of the variables $x_2, \ldots, x_n \in R_+^{n-1}$, then

$$\begin{aligned} D_1 \Psi(x) = &\int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, t_2, \ldots, t_n) K\left(p(x_1, t_2, \ldots, t_n) f(x_1, t_2, \ldots, t_n)\right) \\ &\times K(H(v(x_1, t_2, \ldots, t_n))) dt_n \cdots dt_2, \\ \leq &\int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} d(x_1, t_2, \ldots, t_n) K\left(p(x_1, t_2, \ldots, t_n) f(x_1, t_2, \ldots, t_n)\right) \\ &\times K(H(\Psi(x_1, t_2, \ldots, t_n))) dt_n \cdots dt_2, \\ \leq &K(H(\Psi(x))) \int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, t_2, \ldots, t_n) \\ &\times K\left(p(x_1, t_2, \ldots, t_n) f(x_1, t_2, \ldots, t_n)\right) dt_n \cdots dt_2. \end{aligned} \tag{3.18}$$

Dividing both sides of (3.18) by $K(H(\Psi(x)))$, we get

$$\frac{D_1 \Psi(x)}{K(H(\Psi(x)))} \leq \int_{x_2^0}^{x_2} \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} c(x_1, t_2, \ldots, t_n) K\left(p(x_1, t_2, \ldots, t_n) f(x_1, t_2, \ldots, t_n)\right) dt_n \cdots dt_2. \tag{3.19}$$

We note that

$$G(z) = \int_{z^0}^{z} \frac{ds}{K(H(s))}, \quad z \geq z^0 > 0. \tag{3.20}$$

Thus, it follows that

$$D_1 G(\Psi(x)) = \frac{D_1\Psi(x)}{K(H(\Psi(x)))}. \tag{3.21}$$

From (3.19), (3.20), and (3.21), we have

$$D_1 G(\Psi(x)) \le \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\cdots\int_{x_n^0}^{x_n} c(x_1,t_2,\ldots,t_n)K\big(p(x_1,t_2,\ldots,t_n)f(x_1,t_2,\ldots,t_n)\big)dt_n\cdots dt_2. \tag{3.22}$$

Now, setting $x_1 = s$ in (3.22) and then integrating with respect from x_1^0 to x_1, we obtain

$$G(\Psi(x)) \le G\Big(\Psi\Big(x_1^0, x_2,\ldots,x_n\Big)\Big) + \int_{x^0}^{x} c(t)K\big(p(t)f(t)\big)dt, \tag{3.23}$$

by (3.23), we have

$$\Psi(x) \le G^{-1}\left[G\Big(\Psi\Big(x_1^0, x_2,\ldots,x_n\Big)\Big) + \int_{x_0}^{x} c(t)K\big(p(t)f(t)\big)dt\right]. \tag{3.24}$$

The required inequality in (3.4) follows from the fact (3.9), (3.12), (3.17), and (3.24). □

Many interesting corollaries can be obtained from Theorem 3.2.

Corollary 3.3. *Let $\Phi(x)$, $a(x)$, $b(x)$, $c(x)$, $D_i\Phi(x)$, $D\Phi(x)$, and $K(\Phi(x))$ be as defined in Theorem 3.2. If*

$$D\Phi(x) \le a(x) + \int_{x^0}^{x} c(t)g(\Phi(t))dt + \int_{x^0}^{x} b(t)D\Phi(t)dt \tag{3.25}$$

holds, for $x \in \mathbb{R}_+^n$ with $x \ge x^0 \ge 0$. Then

$$D\Phi(x) \le \left\{a(x) + T^{-1}\left[T(\xi) + \int_{x_0}^{x} c(t)K\big(p(t)\big)dt\right]\right\}\exp\left(\int_{x^0}^{x} b(t)dt\right), \tag{3.26}$$

for $x \in \mathbb{R}_+^n$ with $x \ge x^0 \ge 0$, where

$$p(x) = \int_{x^0}^{x}\exp\left(\int_{x^0}^{t} b(s)ds\right)dt,$$
$$\xi = \int_{x_0}^{\infty} c(t)K\big(a(t)p(t)\big)dt, \tag{3.27}$$
$$T(z) = \int_{z^0}^{z}\frac{ds}{K(s)}, \quad z \ge z^0 > 0,$$

where T^{-1} is the inverse function of T and

$$T(\xi) + \int_{x_0}^{x} c(t)K(p(t))dt \tag{3.28}$$

is in the domain of T^{-1} for $x \in \mathbb{R}_+^n$.

Proof. The proof of this Corollary follows by an argument similar to that in Theorem 3.2. We omit the details. □

Corollary 3.4. *Let $\Phi(x)$, $b(x)$, $c(x)$, $D_i\Phi(x)$, and $D\Phi(x)$ be as defined in Theorem 3.2. If*

$$D\Phi(x) \le M + \int_{x^0}^{x} c(t)\Phi(t)dt + \int_{x^0}^{x} b(t)D\Phi(t)dt \tag{3.29}$$

holds, for $x \in \mathbb{R}_+^n$ with $x \ge x^0 \ge 0$, where $M > 0$ is a constant, then

$$\Phi(x) \le M\left\{1 + \exp\left[\log\left(\int_{x_0}^{\infty} c(t)p(t)dt\right) + \int_{x_0}^{x} c(t)p(t)dt\right]\right\}p(x), \tag{3.30}$$

for $x \in \mathbb{R}_+^n$ with $x \ge x^0 \ge 0$, where

$$p(x) = \int_{x^0}^{x} \exp\left(\int_{x^0}^{t} b(s)ds\right)dt. \tag{3.31}$$

Proof. Setting $g(x) = x$ and $a(x) = M$ in Corollary 3.3, we obtain our result in this Corollary. We omit the details. □

Similarly, we can obtain many other kinds of estimates.

4. An Application

In this section, we present an immediate simple example of application of Theorem 3.2 to the study of boundedness of the solution of a partial integrodifferential equation.

Consider the nonlinear partial integrodifferential equation

$$\begin{gathered} Du(x) = f(x) + \int_{0}^{x} h(x, t, u(t), Du(t))dt, \\ u(\ldots, x_i, 0, x_{i+2}, \ldots) = 0, \quad \forall i = 1, 2, \ldots, n, \end{gathered} \tag{4.1}$$

for $x \in \mathbb{R}_+^n$, where $h : \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x) : \mathbb{R}_+^n \to \mathbb{R}$ are continuous functions.

Assume that functions are defined and continuous on their respective domains of definition, such that

$$\left|f(x)\right| \leq M,$$
$$|h(x,t,u(t),Du(t))| \leq c(t)|u(t)| + b(t)|Du(t)| \tag{4.2}$$

for $x \in \mathbb{R}^n_+$, where $M > 0$ is a constant and $c(x)$ and $b(x)$ are nonnegative, continuous functions defined for $x \in \mathbb{R}^n_+$. If $\Phi(x)$ is any solution of boundary value problem (4.1), then

$$D\Phi(x) = f(x) + \int_0^x h(x,t,\Phi(t),D\Phi(t))dt \tag{4.3}$$

for $x \in \mathbb{R}^n_+$, by (4.2), we have

$$|D\Phi(x)| = M + \int_0^x c(t)|\Phi(x)| + b(t)|D\Phi(x)|dt. \tag{4.4}$$

Now, by a suitable application of Corollary 3.4 of Theorem 3.2, we obtain the bound on the solution $\Phi(x)$ of (4.1).

$$|\Phi(x)| \leq Mp(x)\left\{1 + \exp\left[\log\left(\int_0^\infty c(t)p(t)dt\right) + \int_0^x c(t)p(t)dt\right]\right\} \tag{4.5}$$

or $x \in \mathbb{R}^n_+$, where

$$p(x) = \int_0^x \exp\left(\int_0^t b(s)ds\right)dt. \tag{4.6}$$

Remark 4.1. Using a similar method of those in the proof of the theorems above, we can also obtain *new reversed inequalities* of our results. Our results also can be generalized to integrodifferential inequalities with a time delay for functions of one or n independent variables, this is under study and will be addressed in a forthcoming work. Among these integrodifferential inequalities with a delay, we can quote:

$$D\Phi(x) \leq a(x) + \int_{\alpha(x^0)}^{\alpha(x)} c(t)K(\Phi(t))dt + \int_{\beta(x^0)}^{\beta(x)} b(t)D\Phi(t)dt.$$
$$D\Phi(x) \leq a(x) + f(x)\int_{\alpha(x^0)}^{\alpha(x)} c(t)\Phi(t)K(\Phi(t))dt + \int_{\beta(x^0)}^{\beta(x)} b(t)D\Phi(t)dt. \tag{4.7}$$
$$D\Phi(x) \leq a(x) + f(x)H\left(\int_{\alpha(x^0)}^{\alpha(x)} c(t)n(\Phi(t))K(\Phi(t))dt\right) + \int_{\beta(x^0)}^{\beta(x)} b(t)g(D\Phi(t))dt.$$

References

[1] D. Baĭnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[2] M. Denche and H. Khellaf, "Integral inequalities similar to Gronwall inequality," *Electronic Journal of Differential Equations*, vol. 2007, no. 176, pp. 1–14, 2007.

[3] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Annals of Mathematics*, vol. 20, no. 4, pp. 292–296, 1919.

[4] B. G. Pachpatte, "On some fundamental integrodifferential and integral inequalities," *Analele Ştiintifice ale Universităţii "Al.I.Cuza" din Iaşi*, vol. 23, no. 1, pp. 77–86, 1977.

[5] D. R. Snow, "Gronwall's inequality for systems of partial differential equations in two independent variables," *Proceedings of the American Mathematical Society*, vol. 33, pp. 46–54, 1972.

[6] J. A. Oguntuase, "On an inequality of Gronwall," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 2, no. 1, article 9, pp. 1–6, 2001.

[7] B. G. Pachpatte, "Bounds on certain integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 3, article 47, pp. 1–10, 2002.

[8] C. C. Yeh, "On some integral inequalities in n independent variables and their applications," *Journal of Mathematical Analysis and Applications*, vol. 86, no. 2, pp. 387–410, 1982.

[9] C. C. Yeh and M. H. Shih, "The Gronwall-Bellman inequality in several variables," *Journal of Mathematical Analysis and Applications*, vol. 86, no. 1, pp. 157–167, 1982.

[10] B. G. Pachpatte, "On some new integro-differential inequalities of the Wendroff type," *Journal of Mathematical Analysis and Applications*, vol. 73, no. 2, pp. 491–500, 1980.

[11] H. Khellaf, "On integral inequalities for functions of several independent variables," *Electronic Journal of Differential Equations*, vol. 2003, no. 123, pp. 1–12, 2003.