Research Article

A New Approach to Constant Slope Surfaces with Quaternions

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Received 5 November 2011; Accepted 8 December 2011

Academic Editor: M. Przanowski

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We show a new method to construct constant slope surfaces with quaternions. Moreover, we give some results and illustrate an interesting shape of constant slope surfaces by using Mathematica.

1. Introduction

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex number in 1843. Hamilton's defining relation is most succinctly written as

$$i^{2} = j^{2} = k^{2} = i \times j \times k = -1.$$
 (1.1)

Computing rotations is a common problem in both computer graphics and character animation. Shoemake [1] introduced an algorithm using quaternions, spherical linear interpolation (SLERP), and Bezier curves to solve this. Quaternions are used as a powerful tool for describing rotations about an arbitrary axis. Many physical laws in classical, relativistic, and quantum mechanics can be written nicely using them. They are also used in aerospace applications, flight simulators, computer graphics, navigation systems, visualizations, fractals, and virtual reality.

Kinematic describes the motion of a point or a point system depending on time. If a point moves with respect to one parameter, then it traces its 1-dimensional path, orbit curve. If a line segment or a rectangle moves with respect to one parameter, then they sweep their two- and three-dimensional paths, respectively [2]. So we are thinking of a curve as the path

traced out by a particle moving in Euclidean 3-space. The position vector of the curve is very important to determine behaviour of the curve.

The Serret-Frenet formulae for a quaternionic curve in \mathbb{R}^3 and \mathbb{R}^4 were given by Bharathi and Nagaraj in [3].

In the last few years, the study of the geometry of surfaces in 3-dimensional spaces, in particular of product type $\mathbf{M}^2 \times \mathbf{R}$, was developed by a large number of mathematicians. Recently, constant angle surfaces were studied in product spaces $\mathbf{S}^2 \times \mathbf{R}$ in [4] and $\mathbf{H}^2 \times \mathbf{R}$ in [5, 6], where \mathbf{S}^2 and \mathbf{H}^2 represent the unit 2-sphere and hyperbolic plane, respectively. The angle is considered between the unit normal of the surface \mathbf{M} and the tangent direction to \mathbf{R} .

Boyadzhiev [7] explored three-dimensional versions of these two properties: surfaces that are equiangular and those that are self-similar. He investigated the relationships among these surfaces and gave some examples. Thereafter, Munteanu [8] defined constant slope surfaces. Such surfaces are those whose position vectors make a constant angle with the normals at each point on the surface. Munteanu showed that they can be constructed by using an arbitrary curve on the sphere S^2 or an equiangular spiral.

There is also a kinematic generation of these surfaces as follows. Take a logarithmic spiral and roll its plane along a general cone such that the eye of the spiral sits in the vertex of the cone. Then the spiral sweeps out a surface with the required property.

More recently, we [9] gave some characterizations of constant slope surfaces and Bertrand curves in Euclidean 3-space. We found parametrization of constant slope surfaces for the tangent indicatrix, principal normal indicatrix, binormal indicatrix, and the Darboux indicatrix of a space curve. Furthermore we investigated Bertrand curves corresponding to parameter curves of constant slope surfaces.

By the definition of surfaces of revolution, we can see that such surfaces can be obtained by rotation matrices. Similarly, in this study, we show that constant slope surfaces can be obtained by quaternion product and the matrix representations *M*. Subsequently, we give some results and an example of constant slope surfaces.

2. Preliminaries

The algebra $H = \{\mathbf{q} = a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbf{R}\}$ of quaternions is defined as the four-dimensional vector space over **R** having a basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with the following properties:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} \times \mathbf{j} \times \mathbf{k} = -\mathbf{1}, \quad \mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}.$$
(2.1)

It is clear that H is an associative and not commutative algebra and **1** is identity element of H.

We use the following four-tuple notation to represent a quaternion:

$$q = (a_0, a_1, a_2, a_3) = (a_0, \mathbf{w}) = a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k},$$
(2.2)

where $S\mathbf{q} = a_0$ is the scalar component of \mathbf{q} and $V\mathbf{q} = \{a_1, a_2, a_3\}$ form the vector part and entire set of \mathbf{q} 's is spanned by the basis quaternions:

We also write $\mathbf{q} = S\mathbf{q}+V\mathbf{q}$. The conjugate of $\mathbf{q} = S\mathbf{q}+V\mathbf{q}$ is then defined as $\overline{\mathbf{q}} = S\mathbf{q}-V\mathbf{q}$. We call a quaternion as pure if its scalar part vanishes. Summation of two quaternions $\mathbf{q} = S\mathbf{q} + V\mathbf{q}$ and $\mathbf{p} = S\mathbf{p} + V\mathbf{p}$ is defined as $\mathbf{q} + \mathbf{p} = (S\mathbf{q} + S\mathbf{p}) + (V\mathbf{q} + V\mathbf{p})$. Multiplication of a quaternion $\mathbf{q} = S\mathbf{q}+V\mathbf{q}$ with a scalar $\lambda \in \mathbf{R}$ is defined as $\lambda \mathbf{q} = \lambda S\mathbf{q} + \lambda V\mathbf{q}$. Quaternion product is defined in the most general form for two quaternions $\mathbf{q} = S\mathbf{q} + V\mathbf{q}$ and $\mathbf{p} = S\mathbf{p} + V\mathbf{p}$ as

$$\mathbf{q} \times \mathbf{p} = S\mathbf{q}S\mathbf{p} - \langle V\mathbf{q}, V\mathbf{p} \rangle + S\mathbf{q}V\mathbf{p} + S\mathbf{p}V\mathbf{q} + V\mathbf{q} \wedge V\mathbf{p}, \tag{2.4}$$

where $\langle V\mathbf{q}, V\mathbf{p} \rangle$ and $V\mathbf{q} \wedge V\mathbf{p}$ denote the familiar dot and cross-products, respectively, between the three-dimensional vectors $V\mathbf{q}$ and $V\mathbf{p}$. Quaternionic multiplication satisfies the following properties: for any two quaternions \mathbf{q} and \mathbf{p} we have $\overline{\mathbf{q} \times \mathbf{p}} = \overline{\mathbf{p}} \times \overline{\mathbf{q}}$ and the formula for the dot product $\langle \mathbf{q}, \mathbf{p} \rangle = (\overline{\mathbf{q}} \times \mathbf{p} + \overline{\mathbf{p}} \times \mathbf{q})/2$. In particular, if $\mathbf{q} = \mathbf{p}$, we obtain $|\mathbf{q}|^2 = \langle \mathbf{q}, \mathbf{q} \rangle = \overline{\mathbf{q}} \times \mathbf{q}$. If $|\mathbf{q}| = 1$, then the quaternion \mathbf{q} is unitary. The inverse of a quaternion \mathbf{q} is given by

$$\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|^2} \overline{\mathbf{q}}, \quad |\mathbf{q}| \neq 0,$$
(2.5)

and it satisfies the relation $\mathbf{q} \times \mathbf{q}^{-1} = \mathbf{q}^{-1} \times \mathbf{q} = 1$ [10]. If **q** is a unitary quaternion, we may write **q** in the trigonometric form as $(\cos \theta, \sin \theta \mathbf{v})$, where $|\mathbf{v}| = 1$.

The most important property of quaternions is that they can characterize rotations in a three-dimensional space. The conventional way of representing three-dimensional rotations is by using a set of Euler angles $\{\theta, \varphi, u\}$, which denote rotations about independent coordinate axes. Any general rotation can be obtained as the result of a sequence of rotations, as given by

$$\begin{bmatrix} x'\\ y'\\ z' \end{bmatrix} = \begin{bmatrix} \cos u & -\sin u & 0\\ \sin u & \cos u & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \varphi & 0 & \sin \varphi\\ 0 & 1 & 0\\ -\sin \varphi & 0 & \cos \varphi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos \theta & -\sin \theta\\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix}.$$
 (2.6)

Let $S \subset \mathbb{R}^3$ be the set obtained by rotating a regular plane curve *C* about an axis in the plane which does not meet the curve; we shall take *xz* plane as the plane of the curve and the *z* axis as the rotation axis. Let x = g(v), z = h(v), a < v < b, g(v) > 0, be a parametrization for *C* and denote by *u* the rotation angle about the *z* axis. Thus, we obtain a map

$$x(u,v) = \begin{bmatrix} \cos u & -\sin u & 0\\ \sin u & \cos u & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g(v)\\ 0\\ h(v) \end{bmatrix} = \begin{bmatrix} g(v)\cos u\\ g(v)\sin u\\ h(v) \end{bmatrix}$$
(2.7)

from the open set $U = \{(u, v) \in \mathbb{R}^3 : 0 < u < 2\pi, a < v < b\}$ into *S*. We can see that *x* satisfies the conditions for a parametrization in the definition of a regular surface. Thus *S* is a regular surface which is called a surface of revolution.

A one-parameter homothetic motion of a rigid body in Euclidean 3-space is given analytically by

$$\mathbf{x}' = hA\mathbf{x} + \mathbf{C},\tag{2.8}$$

in which \mathbf{x}' and \mathbf{x} are the position vectors, represented by column matrices, of a point X in the fixed space \mathbf{R}' and the moving space \mathbf{R} , respectively; A is an orthogonal 3 \times 3-matrix, \mathbf{C} is a translation vector, and h is the homothetic scale of the motion. Also h, A, and \mathbf{C} are continuously differentiable functions of a real parameter t [2].

The map ϕ acting on a pure quaternion **w** :

$$\phi : \mathbf{R}^3 \longrightarrow \mathbf{R}^3, \quad \phi(\mathbf{w}) = \mathbf{q} \times \mathbf{w} \times \mathbf{q}^{-1}$$
 (2.9)

is linear. Without loss of generality we choose $|\mathbf{q}| = 1$ and if $\mathbf{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then

$$\boldsymbol{\phi}(\mathbf{i}) = (a_0^2 + a_1^2 - a_2^2 - a_3^2)\mathbf{i} + (2a_0a_3 + 2a_1a_2)\mathbf{j} + (2a_1a_3 - 2a_0a_2)\mathbf{k}, \boldsymbol{\phi}(\mathbf{j}) = (-2a_0a_3 + 2a_1a_2)\mathbf{i} + (a_0^2 + a_2^2 - a_1^2 - a_3^2)\mathbf{j} + (2a_0a_1 + 2a_2a_3)\mathbf{k}, \boldsymbol{\phi}(\mathbf{k}) = (2a_0a_2 + 2a_1a_3)\mathbf{i} + (2a_2a_3 - 2a_0a_1)\mathbf{j} + (a_0^2 + a_3^2 - a_1^2 - a_2^2)\mathbf{k},$$

$$(2.10)$$

so that the matrix representation of the map ϕ is

$$M = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & 2a_2a_3 - 2a_0a_1 \\ 2a_1a_3 - 2a_0a_2 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_1^2 - a_2^2 \end{bmatrix}.$$
 (2.11)

It is not difficult to check that *M* is orthogonal: $MM^T = I$ and det M = 1 so that the linear map $\phi(\mathbf{w}) = \mathbf{q} \times \mathbf{w} \times \mathbf{q}^{-1}$ represents a rotation in \mathbf{R}^3 [11].

Now we give the characterization of constant slope surfaces as the following theorem.

Theorem 2.1. Let $\mathbf{r} : S \to \mathbf{R}^3$ be an isometric immersion of a surface S in the Euclidean 3-space. Then S is a constant slope surface if and only if either it is an open part of the Euclidean 2-sphere centered in the origin, or it can be parametrized by

$$\mathbf{r}(u,v) = u \sin\theta \left(\cos\xi \mathbf{f}(v) + \sin\xi \mathbf{f}(v) \wedge \mathbf{f}'(v)\right), \tag{2.12}$$

where θ is a constant (angle) different from 0, $\xi = \xi(u) = \cot \theta \log u$, and **f** is a unit speed curve on the Euclidean sphere **S**² [8].

3. New Approach

A quaternion function $\mathbf{Q}(u, v) = \cos \xi(u) - \sin \xi(u) \mathbf{f}'(v)$ defines a 2-dimensional surface in $\mathbf{S}^3 \subset \mathbf{R}^4$, where $\mathbf{f}'(v) = (f'_1(v), f'_2(v), f'_3(v))$ and $|\mathbf{f}'| = 1$. Thus, for the unitary quaternion $\mathbf{Q}(u, v)$, the matrix representation of the map $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ is given by

$$M = \begin{bmatrix} \cos^{2}\xi + \sin^{2}\xi(f_{1}^{\prime 2} - f_{2}^{\prime 2} - f_{3}^{\prime 2}) & 2\sin\xi(\cos\xi f_{3}^{\prime} + \sin\xi f_{1}^{\prime} f_{2}^{\prime}) & 2\sin\xi(\sin\xi f_{1}^{\prime} f_{3}^{\prime} - \cos\xi f_{2}^{\prime}) \\ 2\sin\xi(\sin\xi f_{1}^{\prime} f_{3}^{\prime} - \cos\xi f_{3}^{\prime}) & \cos^{2}\xi + \sin^{2}\xi(-f_{1}^{\prime 2} + f_{2}^{\prime 2} - f_{3}^{\prime 2}) & 2\sin\xi(\sin\xi f_{2}^{\prime} f_{3}^{\prime} + \cos\xi f_{1}^{\prime}) \\ 2\sin\xi(\sin\xi f_{1}^{\prime} f_{3}^{\prime} + \cos\xi f_{2}^{\prime}) & 2\sin\xi(\sin\xi f_{2}^{\prime} f_{3}^{\prime} - \cos\xi f_{1}^{\prime}) & \cos^{2}\xi + \sin^{2}\xi(-f_{1}^{\prime 2} - f_{2}^{\prime 2} + f_{3}^{\prime 2}) \end{bmatrix}.$$

$$(3.1)$$

We are now ready to show the main result of this study.

Theorem 3.1. Let $\mathbf{r} : S \to \mathbf{R}^3$ be an isometric immersion of a surface S in the Euclidean 3-space. Then the constant slope surface S can be reparametrized by $\mathbf{r}(u, v) = \mathbf{Q}(u, v) \times \mathbf{Q}_1(u, v)$, where "×" is the quaternion product, $\mathbf{Q}_1(u, v) = u \sin \theta \mathbf{f}(v) \in \mathbf{R}^3$ is a surface and a pure quaternion.

Proof. Since $\mathbf{Q}(u, v) = \cos \xi(u) - \sin \xi(u) \mathbf{f}'(v)$ and $\mathbf{Q}_1(u, v) = u \sin \theta \mathbf{f}(v)$, we obtain

$$\mathbf{Q}(u,v) \times \mathbf{Q}_{1}(u,v) = (\cos \xi(u) - \sin \xi(u)\mathbf{f}'(v)) \times (u \sin \theta \mathbf{f}(v))$$

= $u \sin \theta (\cos \xi(u) - \sin \xi(u)\mathbf{f}'(v)) \times \mathbf{f}(v)$ (3.2)
= $u \sin \theta \cos \xi(u)\mathbf{f}(v) - u \sin \theta \sin \xi(u)\mathbf{f}'(v) \times \mathbf{f}(v).$

By using (2.4), we get

$$\mathbf{f}'(v) \times \mathbf{f}(v) = \mathbf{f}'(v) \wedge \mathbf{f}(v)$$

= $-\mathbf{f}(v) \wedge \mathbf{f}'(v).$ (3.3)

If we substitute this into the last equation, we have

$$\mathbf{Q}(u,v) \times \mathbf{Q}_1(u,v) = u \sin \theta \left(\cos \xi(u) \mathbf{f}(v) + \sin \xi(u) \mathbf{f}(v) \wedge \mathbf{f}'(v) \right). \tag{3.4}$$

Hence applying Theorem 2.1, we find that the constant slope surface is given by

$$\mathbf{Q}(u,v) \times \mathbf{Q}_1(u,v) = \mathbf{r}(u,v). \tag{3.5}$$

This completes the proof.

As a consequence of this theorem, we get the following important corollary.

Corollary 3.2. Let *M* be the matrix representation of the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ for the unitary quaternion $\mathbb{Q}(u, v)$. Then, for the pure quaternion $\mathbb{Q}_1(u, v)$, we get the constant slope surface as

$$\mathbf{r}(u,v) = M\mathbf{Q}_1(u,v). \tag{3.6}$$

We know that a surface of revolution can be obtained by a rotation matrix. Similarly, we view that the constant slope surface $\mathbf{r}(u, v)$ can be obtained by the matrix representation M, too.

Finally we state the following result.

Corollary 3.3. For the homothetic motion $\tilde{\mathbf{Q}}(u, v) = u \sin \theta \mathbf{Q}(u, v)$, the constant slope surface can be written as $\mathbf{r}(u, v) = \tilde{\mathbf{Q}}(u, v) \times \mathbf{f}(v)$. Therefore we have

$$\mathbf{r}(u, v) = u\sin\theta M \mathbf{f}(v). \tag{3.7}$$

Now, we give some remarks regarding our Theorem 3.1 and Corollary 3.3.

Remark 3.4. Theorem 3.1 says that both the points and the position vectors on the surface $\mathbf{Q}_1(u, v)$ are rotated by $\mathbf{Q}(u, v)$ through the angle $\xi(u)$ about the axis $Sp\{\mathbf{f}'(v)\}$.

Remark 3.5. Corollary 3.3 shows that the position vector of the curve f(v) is rotated by $\tilde{Q}(u, v)$ through the angle $\xi(u)$ about the axis $Sp\{f'(v)\}$ and extended through the homothetic scale $u \sin \theta$.

4. Example

We give an example of constant slope surfaces and draw its picture by using Mathematica.

Example 4.1. We consider the unit speed spherical curve

$$f(v) = \frac{1}{2} \Big(\cos 2v, \sqrt{3}, \ \sin 2v \Big). \tag{4.1}$$

If the angle is taken $\theta = \pi/4$ then we have

$$\mathbf{Q}(u,v) = \cos(\log u) + (\sin(\log u)\sin 2v, 0, -\sin(\log u)\cos 2v), \tag{4.2}$$

$$\mathbf{Q}_{1}(u,v) = \left(\frac{\sqrt{2}}{4}u\cos 2v, \frac{\sqrt{6}}{4}u, \frac{\sqrt{2}}{4}u\sin 2v\right).$$
(4.3)

Thus, by using (3.1) and (3.6), we get the following constant slope surface:

$$\mathbf{r}(u,v) = \begin{bmatrix} \cos^{2}(\log u) - \sin^{2}(\log u) \cos 4v & \sin(2\log u) \cos 2v & -\sin^{2}(\log u) \sin 4v \\ -\sin(2\log u) \cos 2v & \cos(2\log u) & -\sin(2\log u) \sin 2v \\ -\sin^{2}(\log u) \sin 4v & \sin(2\log u) \sin 2v & \cos^{2}(\log u) + \sin^{2}(\log u) \cos 4v \end{bmatrix}$$
$$\cdot \begin{bmatrix} \frac{\sqrt{2}}{4}u \cos 2v \\ \frac{\sqrt{6}}{4}u \\ \frac{\sqrt{2}}{4}u \sin 2v \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{2}}{4}u \cos 2v \left(\cos(2\log u) + \sqrt{3}\sin(2\log u)\right) \\ \frac{\sqrt{6}}{4}u \cos(2\log u) - \frac{\sqrt{2}}{4}u \sin(2\log u) \\ \frac{\sqrt{2}}{4}u \sin 2v \left(\cos(2\log u) + \sqrt{3}\sin(2\log u)\right) \end{bmatrix}.$$
(4.4)

Hence, we can give Mathematica code of this constant slope surface as

ParametricPlot3D
$$\left[\left\{ \frac{\sqrt{2}}{4} u \cos[2v] \left(\cos[2\log[u]] + \sqrt{3} \sin[2\log[u]] \right), \\ \frac{\sqrt{6}}{4} u \cos[2\log[u]] - \frac{\sqrt{2}}{4} u \sin[2\log[u]], \\ \frac{\sqrt{2}}{4} u \sin[2v] \left(\cos[2\log[u]] + \sqrt{3} \sin[2\log[u]] \right) \right\}, \\ \left\{ u, 0, \frac{Pi}{2} \right\}, \{v, 0, Pi\} \right]$$
(4.5)

and the picture of $\mathbf{Q}(u, v) \times \mathbf{Q}_1(u, v)$ is drawn as follows (Figure 1).

5. Conclusion

By the definition of surfaces of revolution, we can see that such surfaces can be obtained by rotation matrices for the position vectors of given regular plane curves. Similarly, in this study, we show that constant slope surfaces can be obtained by quaternion product and the matrix representations *M*. Afterwards, we give some results and illustrate an example of constant slope surfaces by using quaternions and draw its picture by using Mathematica computer program.



Figure 1: Constant slope surface $\mathbf{Q}(u, v) \times \mathbf{Q}_1(u, v)$, $\mathbf{f}(v) = (1/2)(\cos 2v, \sqrt{3}, \sin 2v)$, $\theta = \pi/4$.

Acknowledgment

The authors wish to express their gratitude to Marian Ioan Munteanu for suggesting them the subject and for helpful hints given during the preparation of this work.

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