International Scholarly Research Network
ISRN Computational Mathematics
Volume 2012, Article ID 139514, 5 pages
doi:10.5402/2012/139514

Research Article

On Some Volterra and Fredholm Problems via the Unified Integrodifferential Quadrature Method

Abdelwahab Zerarka, Abdesselam Soukeur, and Hanane Saidi

Laboratory of Applied Mathematics, Mohamed Khider University of Biskra, BP145, 07000 Biskra, Algeria

Correspondence should be addressed to Abdelwahab Zerarka, abzerarka@yahoo.fr

Received 6 September 2011; Accepted 20 October 2011

Academic Editors: V. Rai and K. Wang

We present a new approach based on the formulation of the integrodifferential quadrature method (hereafter called IDQ) to handle Volterra's and Fredholm's equations. This approach is constructed and tested with some realistic numerical examples using the basic computational aspects.

1. Introduction

In the present work, we present a formulation of a new numerical approach which is based on the generalized integrodifferential quadrature method and applied to weakly singular Volterra and Fredholm integrodifferential equations in the linear case. This method studies the situation in which the unknown function is identified as the Lagrange polynomial and the interpolating points of the Tchebychev type are used. The accuracy and efficiency of solving integrodifferential equations are still an ongoing research in numerical analysis. Several different methods have also been modeled for the solutions of linear and nonlinear problem [1–27].

The central argument for the interest of the type of this problem comes naturally from its wide applications almost in any branches of science and engineering described by systems of ODEs and PDEs [28–38] which in some situations can be transformed into a set of integral equations. Much attention has been devoted to the investigation of new mathematical models and numerical approaches to evaluate the solutions of the integral equations.

New calculations are performed for the construction of the solution by a suitable choice of the interpolating points using the unified integrodifferential quadrature method. Based on this fact, this technique is a very promising and powerful methodology in successfully locating best solutions of the problem. Our main purpose is to develop a general numerical schema for the integrodifferential equations that is universally applicable.

The contents of this paper are organized as follows. In Section 2, a formulation adapted to the problem of construction of the solution of the one-dimensional weakly singular integrodifferential equation by using a summary limited to some aspects of the unified integrodifferential quadrature method is qualitatively presented. Section 3 exposes some essential examples for the construction of the unknown solution. The comments and conclusion are given in Section 4.

2. Unified Integrodifferential Quadrature Method

The description of generalized integrodifferential quadrature method is summarized as follows: the starting point of the weakly singular integrodifferential equation is written as

$$\sum_{k=1}^{l} \gamma_k(x) \frac{d^k f(x)}{dx^k} = \alpha(x) f(x) + \beta(x) + \lambda W(x,z), \quad a \le x \le T, \tag{1}$$

where $W(x,z)$ is given by

$$W(x,z) \equiv \int_a^z (x-s)^{-\gamma} K(x,s) f(s) ds, \tag{2}$$

and we set

$$W(x,z)=\begin{cases}W(x,x), & \text{for Volterra type,}\\ W(x,T), & \text{for Fredholm type,}\end{cases} \tag{3}$$

with prescribed boundary condition $f(a)=f_a$.

λ and ν are parameters with $0<\nu<1$. $\alpha(x)$, $\beta(x)$, and $\gamma_k(x)$ $(k=0,\dots,m)$ are given functions, and $K(x,s)$ is a regular kernel of the integrodifferential equation. It is assumed that the function involved in (1) is sufficiently regular. The above equations, for $\nu=0$, become regular integrodifferential equations.

In order to avoid the singularity, (1) is integrated by part and with little effort; we obtain

$$\begin{aligned}W(x,z)=&\frac{1}{(\nu-1)}\Big[(x-z)^{1-\nu}K(x,z)f(z)\\&-(x-a)^{1-\nu}K(x,a)f(a)\Big]\\&-\frac{1}{(\nu-1)}\int_a^z(x-s)^{1-\nu}\frac{d[K(x,s)f(s)]}{ds}ds.\end{aligned} \tag{4}$$

Now the relation (1) becomes

$$\begin{aligned}\sum_{k=1}^{l}\gamma_k(x)\frac{d^kf(x)}{dx^k}=&\alpha(x)f(x)+\beta_\nu(x,z)\\&+\frac{\lambda}{(1-\nu)}\int_a^z(x-s)^{1-\nu}\frac{d[K(x,s)f(s)]}{ds}ds,\end{aligned} \tag{5}$$

where $\beta_\nu(x,z)=\beta(x)+(\lambda/(\nu-1))((x-z)^{1-\nu}K(x,z)f(z)-(x-a)^{1-\nu}K(x,a)f(a))$, and $z=x$ and $z=T$ for Volterra type and Fredholm type, respectively.

At this stage, we introduce the quadrature aspect of solutions: the derivatives and integral in expression (6) are approximated by

$$\frac{d^kf(x_m)}{dx^k}=\sum_{j=0}^{N}D_{mj}^kf\left(x_j\right), \tag{6}$$

$$\int_a^{x_m}(x_m-s)^{1-\nu}\frac{d[K(x_m,s)f(s)]}{ds}ds=\sum_{j=0}^{N}I_{mj}f\left(x_j\right), \tag{7}$$

for Volterra type,

$$\int_a^{T}(x_m-s)^{1-\nu}\frac{d[K(x_m,s)f(s)]}{ds}ds=\sum_{j=0}^{N}J_{mj}f\left(x_j\right), \tag{8}$$

for Fredholm type,

where $\{x_k\}_{k=0}^N$ are interpolating points, taken as the points of Tchebychev of the form $x_j=(1/2)T[1-\cos((2j+1)/(2N+2)\pi)]$, $0\le j\le N$. D_{ij} and I_{mj} are the weighting coefficients linked with Lagrange interpolated polynomials $P_{N,j}(x)$ and are explicitly given by

$$D_{mj}^k=\frac{d^kP_{N,j}(x_m)}{dx^k}, \tag{9}$$

$$\begin{aligned}I_{mj}&=\int_0^{x_m}(x_m-s)^{1-\nu}\frac{d\left[K(x_m,s)P_{N,j}(s)\right]}{ds}ds,\\J_{mj}&=\int_0^{T}(x_m-s)^{1-\nu}\frac{d\left[K(x_m,s)P_{N,j}(s)\right]}{ds}ds.\end{aligned} \tag{10}$$

With little effort the original problems (1) can be converted to an algebraic system and take, respectively, the following compact forms:

$$\alpha(x_m)f(x_m)+\beta_\nu(x_m,x_m)=\sum_{j=0}^{N}U_{mj}^If\left(x_j\right),\quad m=0,\dots,N, \tag{11}$$

for Volterra type, and

$$\alpha(x_m)f(x_m)+\beta_\nu(x_m,T)=\sum_{j=0}^{N}U_{mj}^Jf\left(x_j\right),\quad m=0,\dots,N, \tag{12}$$

for Fredholm type, where

$$U_{mj}^I=\sum_{k=0}^{l}\gamma_k(x_m)D_{mj}^k-\frac{\lambda}{(1-\nu)}I_{mj}, \tag{13}$$

$$U_{mj}^J=\sum_{k=0}^{l}\gamma_k(x_m)D_{mj}^k-\frac{\lambda}{(1-\nu)}J_{mj}. \tag{14}$$

The relations (12) and (13) are real systems of $M\times M$ equations in the $M\times M$ unknown real coefficients U_{mj}^I and U_{mj}^J, respectively, where $M=N+1$. In order to avoid unnecessary calculation and to optimize the computational aspect, it is therefore more convenient to get the desired coefficients D_{ii}, I_{ii}, and J_{ii} in the following forms:

$$D_{ii}^k=-\sum_{j=0,j\neq i}^{N}D_{ij}^k,\quad \text{for } i=0,\dots,N,\ k=1,\dots,l, \tag{15}$$

$$I_{ii}=\int_a^{x_i}(x_i-s)^{1-\nu}\frac{dK(x_i,s)}{ds}ds-\sum_{j=0,j\neq i}^{N}I_{ij},\quad \text{for } i=0,\dots,N, \tag{16}$$

$$J_{ii}=\int_a^{T}(x_i-s)^{1-\nu}\frac{dK(x_i,s)}{ds}ds-\sum_{j=0,j\neq i}^{N}J_{ij},\quad \text{for } i=0,\dots,N. \tag{17}$$

Now the expressions (14), (15), (16), (17), and (18) provide the charmingly practical formulae for the weighting coefficients. Once the function values at all grid points are

obtained, it is then easy to determine the function values in the overall domain in terms of polynomial approximation, such that

$$f(x) = \sum_{j=0}^{N} f\left(x_j\right) P_{N,j}(x). \tag{18}$$

The practical part of this study is examined in the following section.

3. Worked Examples

We shall consider the solution of some crack situations in order to show the application and the effectiveness of the method described in Section 2. Four examples are considered for this purpose.

We write (1) in the form

$$\alpha_p f_p + \beta_{\nu,pp} = \sum_{j=0}^{N} U_{pj}^{I} f_j, \quad p = 0, \ldots, N, \tag{19}$$

$$\alpha_p f_p + \beta_{\nu,pT} = \sum_{j=0}^{N} U_{pj}^{J} f_j, \quad p = 0, \ldots, N, \tag{20}$$

where f_j and α_i stand for the values of $f(x_i)$ and $\alpha(x_i)$, respectively. $\beta_{\nu,pp}$ and $\beta_{\nu,pT}$ denote the values of $\beta_\nu(x_p, x_p)$ and $\beta_\nu(x_p, T)$, respectively.

Evaluating (20) and (21) at x_p, the linear systems of algebraic equations follow, respectively,

$$[IA - U_1]F = B_1, \tag{21}$$

$$[IA - U_2]F = B_2, \tag{22}$$

where the vectors F and $B_1(B_2)$ have components f_p and $\beta_{\nu,pp}(\beta_{\nu,pT})$, respectively, $U_1(U_2)$ is a matrix with the elements $U_{1,ij} = U_{ij}^{I}(U_{2,ij} = U_{ij}^{J})$, and the components of U^I and U^I are given by (14) and (15) together with (16), (17), and (18). The equations (20) and (21) are explicit prescriptions that give the solutions in $O((N+1)^2)$ operations.

The accuracy of solutions can be checked by using the error tolerance $E(x_i) = |f(x_i) - f_{\text{exact}}(x_i)| = 10^{-j}$, where "$j$" is a positive integer. We continue evaluating the solution until the above relation was satisfied.

In order to verify the efficiency of the method developed in the previous section, the following examples have been selected to provide a comparison with previously published work.

3.1. Example 1. In this example we use Volterra's integrodifferential equation [39] given by

$$\frac{df(x)}{dx} = f(x) + \beta(x) - \int_0^x (x-s)^{-1/2} f(s)ds, \qquad 0 \le x \le 1, f(0) = 0, \tag{23}$$

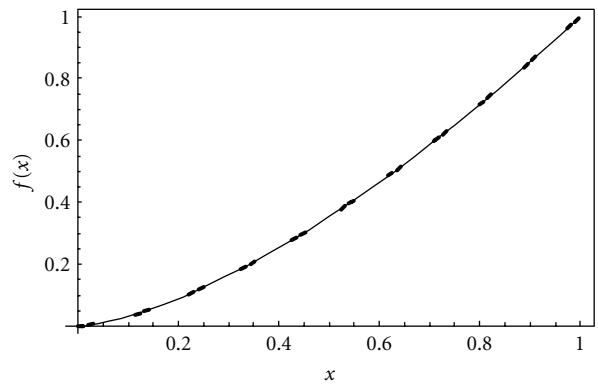

Figure 1: Solution $f(x)$ of (23). Solid curve: exact solution: $f(x) = x^{3/2}$, dash curve: present work.

where $\beta(x) = (3/2)\sqrt{x} + x\sqrt{x} + (3\pi/8)x^2$, from the governing equation (1) and (24) subject to $K(x,s) = 1$, we deduce the following: $l = 1, \gamma_1(x) = 1, \alpha(x) = 1, \lambda = -1, \nu = 1/2, a = 0, T = 1$, and $f_0 = 0$. The exact solution is $f(x) = x^{3/2}$.

Illustrated in Figure 1 is the basic application of the problem (24). The numerical result displayed in Figure 1 is qualitatively in good agreement with the exact solution in which the error tolerance, TOLV $= 10^{-5}$.

3.2. Example 2. We consider the numerical solution of the following Volterra's integrodifferential equation [40]:

$$\frac{d^2 f(x)}{dx^2} = f(x) + \beta(x) + \int_0^x xsf(s)ds, \qquad 0 \le x \le 1, f(0) = 1, \tag{24}$$

where $\beta(x) = -x(x+1) + x(1-x)\exp(x) + 2 - x^5/4$, and from this equation we deduce the following: $l = 2$, $\gamma_1(x) = 0$, $\gamma_2(x) = 1$, $\alpha(x) = 1$, $\lambda = 1, \nu = 0$, $a = 0$, $T = 1$, $f_0 = 1$, and the kernel $K(x,s) = xs$.

The above equation has the exact solution $f(x) = x^2 + \exp(x)$.

The result is displayed in Figure 2, where the error tolerance, TOLV $= 10^{-6}$. With this tolerance and for the number of the interpolation points $N = 10$, the results under consideration are very satisfactory.

3.3. Example 3. Consider the Fredholm integrodifferential equation [41]:

$$\frac{d^2 f(x)}{dx^2} + x\frac{df(x)}{dx} = xf(x) + \beta(x) + \int_{-1}^{1} e^{-s}\sin(x) f(s)ds, \qquad -1 \le x \le 1, f(-1) = \frac{1}{e}, \tag{25}$$

where $\beta(x) = \exp(x) - 2\sin(x)$, and from this equation we deduce the following: $l = 2$, $\gamma_1(x) = x$, $\gamma_2(x) = 1$, $\alpha(x) = x$, $\lambda = 1, \nu = 0$, $a = -1$, $T = 1$, $f_{-1} = 1/e$, and the kernel $K(x,s) = e^{-s}\sin(x)$. The exact solution has the form $f(x) = \exp(x)$.

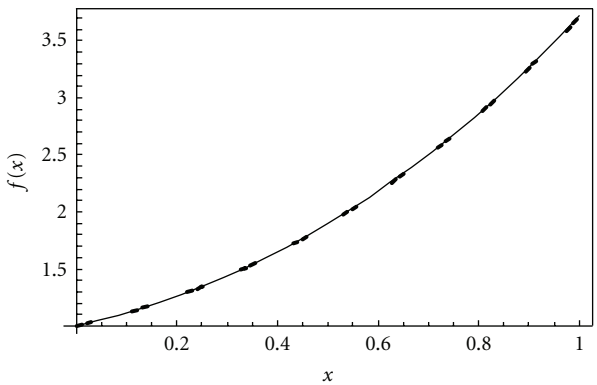

Figure 2: Solution $f(x)$ of (24). Solid line: exact solution: $f(x) = x^2 + \exp(x)$, dash curve: present work.

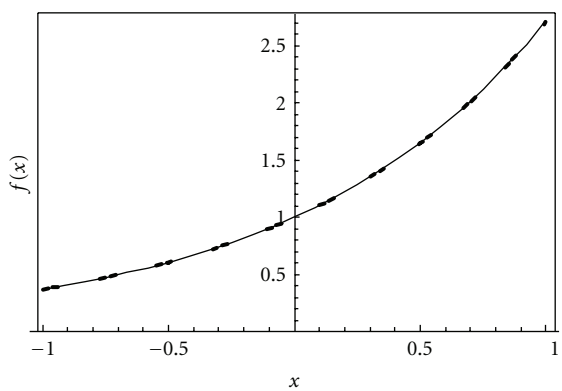

Figure 3: Solution $f(x)$ of (25). Solid curve: exact solution: $f(x) = \exp(x)$, dash curve: present work.

In Figure 3, the number of interpolation points is taken equal to $N = 10$ to solve this problem with TOLV $= 10^{-8}$. We see from this Figure that the method provides accurate results in the full range, where the error tolerance, TOLV $= 10^{-6}$.

3.4. Example 4. We apply the mentioned technique to solve the fifth-order Fredholm integrodifferential equation [42]:

$$\frac{d^5 f(x)}{dx^5} - x^2 \frac{d^3 f(x)}{dx^3} - \frac{df(x)}{dx} = x f(x) + \beta(x) + \int_{-1}^{1} f(s)ds,$$
$$-1 \le x \le 1, f(-1) = -\sin(1), \tag{26}$$

where $\beta(x) = x^2 \cos(x) - x \sin(x)$, and from this equation we deduce the following: $l = 5$, $\gamma_1(x) = -1$, $\gamma_2(x) = \gamma_4(x) = 0$, $\gamma_3(x) = -x^2$, $\gamma_5(x) = 1$, $\alpha(x) = x$, $\lambda = 1, \nu = 0$, $a = -1$, $T = 1, f_{-1} = -\sin(1)$, and the kernel $K(x, s) = 1$. The exact solution has the form $f(x) = \sin(x)$. Figure 4 presents computer simulations of this example, that is generated using the number of interpolating points $N = 8$. The result is compared with the exact solution. From Figure 4, we see that the agreement between the numerical result of the present method and the exact result is very good, with an error tolerance TOLV $= 10^{-6}$.

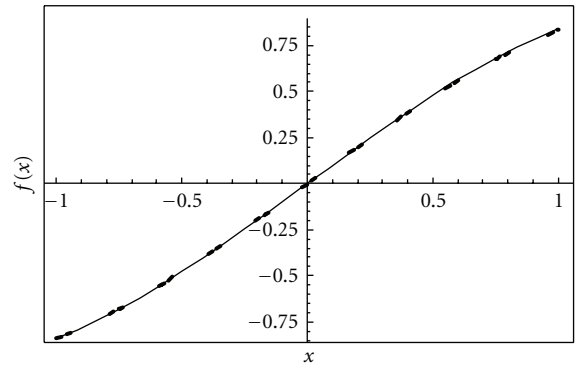

Figure 4: Solution $f(x)$ of the equation (26). Solid curve: exact solution: $f(x) = \sin(x)$, dash curve: present work.

4. Comments and Conclusions

This paper has introduced a new formulation that uses the unified integrodifferential quadrature method to handle some integrodifferential equations. The main advantage of Lagrange polynomial is that the weighting coefficients that have to be computed do not depend on the $f(x_i)$ and it is independent on the manner the discrete points are ordered.

The preliminary results, obtained through the use of this method, show that the resulting solutions are quite good for all examples which have been selected here as a testbed. We can note that similar patterns of convergence are seen for all four examples with a common tolerance TOLV $= 10^{-6}$.

This method seems to be a powerful alternative and consequently gives very accurate solutions to the problems under consideration. It would be interesting to generalize this method to the nonlinear case. This matter deserves a further work.

Acknowledgments

The authors gratefully acknowledge helpful conversations with Professor W. Cramer. This work was sponsored in part by the M.E.R.S (Ministère de l'Enseignement et de la Recherche Scientifique): under contract no. D01420060012.

References

[1] R. K. Miller and A. Feldstein, "Smoothness of solutions of Volterra integral equations with weakly singular kernels," *SIAM Journal on Mathematical Analysis*, vol. 2, pp. 242–258, 1971.

[2] J. Norbury and A. M. Stuart, "Volterra integral equations and a new Gronwall inequality. I. The linear case," *Proceedings of the Royal Society of Edinburgh Section A*, vol. 106, no. 3-4, pp. 361–373, 1987.

[3] J. Norbury and A. M. Stuart, "Volterra integral equations and a new Gronwall inequality. II. The nonlinear case," *Proceedings of the Royal Society of Edinburgh Section A*, vol. 106, no. 3-4, pp. 375–384, 1987.

[4] A. Zerarka, S. Hassouni, H. Saidi, and Y. Boumedjane, "Energy spectra of the Schrödinger equation and the differential quadrature method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, no. 7, pp. 737–745, 2005.

[5] H. Brunner and P. J. van der Houwen, *The Numerical Solution of Volterra Equations*, vol. 3 of *CWI Monographs*, North-Holland Publishing, Amsterdam, The Netherlands, 1986.

[6] C. T. H. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, UK, 1977.

[7] L. M. Delves and J. Walsh, *Numerical Solution of Integral Equations*, Clarendon Press, Oxford, UK, 1974.

[8] J. A. Cochran, *The Analysis of Linear Integral Equations*, McGraw-Hill, New York, NY, USA, 1972.

[9] K. E. Atkinson, *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1976.

[10] P. H. M. Wolkenfelt, "The construction of reducible quadrature rules for Volterra integral and integro-differential equations," *IMA Journal of Numerical Analysis*, vol. 2, no. 2, pp. 131–152, 1982.

[11] H.-J. Dobner, "Bounds of high quality for first kind Volterra integral equations," *Reliable Computing*, vol. 2, no. 1, pp. 35–45, 1996.

[12] D. Bugajewski, "On the Volterra integral equation in locally convex spaces," *Demonstratio Mathematica*, vol. 25, no. 4, pp. 747–754, 1992.

[13] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge, UK, 1985.

[14] H. Brunner, A. Pedas, and G. Vainikko, "The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations," *Mathematics of Computation*, vol. 68, no. 227, pp. 1079–1095, 1999.

[15] J. P. Kauthen and H. Brunner, "Continuous collocation approximations to solutions of first kind Volterra equations," *Mathematics of Computation*, vol. 66, no. 220, pp. 1441–1459, 1997.

[16] Q. Hu, "Multilevel correction for discrete collocation solutions of Volterra integral equations with delay arguments," *Applied Numerical Mathematics*, vol. 31, no. 2, pp. 159–171, 1999.

[17] H. Brunner, Q. Hu, and Q. Lin, "Geometric meshes in collocation methods for Volterra integral equations with proportional delays," *IMA Journal of Numerical Analysis*, vol. 21, no. 4, pp. 783–798, 2001.

[18] V. Karlin, V. G. Maz'ya, A. B. Movchan, J. R. Willis, and R. Bullough, "Numerical solution of nonlinear hypersingular integral equations of the Peierls type in dislocation theory," *SIAM Journal on Applied Mathematics*, vol. 60, no. 2, pp. 664–678, 2000.

[19] M. H. Fahmy, M. A. Abdou, and M. A. Darwish, "Integral equations and potential-theoretic type integrals of orthogonal polynomials," *Journal of Computational and Applied Mathematics*, vol. 106, no. 2, pp. 245–254, 1999.

[20] T. Diogo and P. Lima, "Numerical solution of a nonuniquely solvable Volterra integral equation using extrapolation methods," *Journal of Computational and Applied Mathematics*, vol. 140, no. 1-2, pp. 537–557, 2002.

[21] F. Fagnani and L. Pandolfi, "A recursive algorithm for the approximate solution of Volterra integral equations of the first kind of convolution type," *Inverse Problems*, vol. 19, no. 1, pp. 23–47, 2003.

[22] P. Oja and D. Saveljeva, "Cubic spline collocation for Volterra integral equations," *Computing*, vol. 69, no. 4, pp. 319–337, 2002.

[23] I. Bock and J. Lovíšek, "On a reliable solution of a Volterra integral equation in a Hilbert space," *Applications of Mathematics*, vol. 48, no. 6, pp. 469–486, 2003.

[24] M. Federson, R. Bianconi, and L. Barbanti, "Linear Volterra integral equations," *Acta Mathematicae Applicatae Sinica*, vol. 18, no. 4, pp. 553–560, 2002.

[25] M. Federson, R. Bianconi, and L. Barbanti, "Linear Volterra integral equations as the limit of discrete systems," *Cadernos de Matemática*, vol. 4, pp. 331–352, 2003.

[26] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, vol. 7, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1985.

[27] U. Kulisch and W. L. Miranker, *A New Approach to Scientific Computation*, Academic Press, New York, NY, USA, 1983.

[28] R. Hammer, M. Hocks, U. Kulisch, and D. Ratz, *Numerical Toolbox for Verified Computing I*, Springer, Berlin, Germany, 1995.

[29] D. Bugajewski, "On differential and integral equations in locally convex spaces," *Demonstratio Mathematica*, vol. 28, no. 4, pp. 961–966, 1995.

[30] H.-J. Dobner, "Bounds for the solution of hyperbolic problems," *Computing*, vol. 38, no. 3, pp. 209–218, 1987.

[31] E. T. Copson, *Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1975.

[32] D. Bugajewska, *Topological properties of solution sets of some problems for differential equations*, Ph.D. thesis, Poznan, 1999.

[33] A. Constantin, "On the unicity of solutions for the differential equation $x^{(n)} = f(t, x)$," *Rendiconti del Circolo Matematico di Palermo Serie II*, vol. 42, no. 1, pp. 59–64, 1993.

[34] D. Bugajewski and S. Szufla, "Kneser's theorem for weak solutions of the Darboux problem in Banach spaces," *Nonlinear Analysis, Theory, Methods & Applications*, vol. 20, no. 2, pp. 169–173, 1993.

[35] M. Maron and R. Lopez, *Numerical Analysis*, Wadsworth Publishing Company, Belmont, Calif, USA, 1991.

[36] M. Hukuhara, "Théorèmes fondamentaux de la théorie des équations différentielles ordinaires dans l'espace vectoriel topologique," *Journal of the Faculty of Science, University of Tokyo, Section IA*, vol. 8, pp. 111–138, 1959.

[37] R. K. Miller and A. Feldstein, "Smoothness of solutions of Volterra integral equations with weakly singular kernels," *SIAM Journal on Mathematical Analysis*, vol. 2, pp. 242–258, 1971.

[38] G. Adomian, "Random Volterra integral equations," *Mathematical and Computer Modelling*, vol. 22, no. 8, pp. 101–102, 1995.

[39] H. Brunner, "Polynomial spline collocation methods for Volterra integrodifferential equations with weakly singular kernels," *IMA Journal of Numerical Analysis*, vol. 6, no. 2, pp. 221–239, 1986.

[40] E. Rawashdeh, D. McDowell, and L. Rakesh, "Polynomial spline collocation methods for second-order Volterra integrodifferential equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 56, pp. 3011–3022, 2004.

[41] K. Al-Khaled and F. Allan, "Decomposition method for solving nonlinear integro-differential equations," *Journal of Applied Mathematics & Computing*, vol. 19, no. 1-2, pp. 415–425, 2005.

[42] A. Akyüz-Daşcıoğlu, "A Chebyshev polynomial approach for linear Fredholm-Volterra integro-differential equations in the most general form," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 103–112, 2006.