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Research Article

Approximation of Solutions of Nonlinear Integral Equations of Hammerstein Type

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Suppose that H is a real Hilbert space and $F, K : H \to H$ are bounded monotone maps with $D(K) = D(F) = H$. Let u^* denote a solution of the Hammerstein equation $u + KFu = 0$. An explicit *iteration process* is shown to converge strongly to u^*. No invertibility or continuity assumption is imposed on K and the operator F is not restricted to be angle-bounded. Our result is a significant improvement on the Galerkin method of Brézis and Browder.

1. Introduction

Let X be a real normed linear space with dual X^*. For $q > 1$, we denote by J_q the generalized duality mapping from X to 2^{X^*} defined by

$$J_q(x) := \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\| \cdot \|f^*\|,\ \|f^*\| = \|x\|^{q-1} \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. J_2 is denoted by J. If X^* is strictly convex, then J_q is single-valued. A map G with domain $D(G)$ in a normed linear space X is said to be *strongly accretive* if there exists a constant $k > 0$ such that for every $x, y \in D(G)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Gx - Gy, j_q(x - y) \rangle \geq k\|x - y\|^q. \tag{1.2}$$

If $k = 0$, G is said to be accretive. If X is a Hilbert space, accretive operators are called *monotone.* The accretive mappings were introduced independently in 1967 by Browder [1]

and Kato [2]. Interest in such mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., Zeidler [3]) that many physically significant problems can be modelled by initial-value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \tag{1.3}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. If in (1.3), $x(t)$ is independent of t, then (1.3) reduces to

$$Au = 0, \tag{1.4}$$

whose solutions correspond to the equilibrium points of the system (1.3). Consequently, considerable research efforts have been devoted, especially within the past 30 years or so, to methods of finding approximate solutions (when they exist) of (1.4). An early fundamental result in the theory of accretive operators, due to Browder [1], states that the initial value problem (1.3) is solvable if A is locally Lipschitzian and accretive on X. Utilizing the existence result for (1.3), Browder [1] proved that if A is locally Lipschitzian and accretive on X, then A is m-accretive, that is, $R(I + A) = X$, where $R(I + A)$ denotes the range of $(I + A)$. Clearly, a consequence of this is that the equation

$$u + Au = 0 \tag{1.5}$$

has a solution. One important generalization of (1.5) is the so-called *equation of Hammerstein type* (see, e.g., Hammerstein [4]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_{\Omega} \kappa(x,y) f(y,u(y))\,dy = h(x), \tag{1.6}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel κ is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by

$$Kv(x) := \int_{\Omega} \kappa(x,y) v(y)\,dy, \quad x \in \Omega, \tag{1.7}$$

and the so-called *superposition* or *Nemytskii* operator by $Fu(y) := f(y,u(y))$ then, the integral equation (1.6) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.8}$$

where, without loss of generality, we have taken $h \equiv 0$.

Interest in (1.8) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess

Greens functions can, as a rule, be transformed into the form (1.8) (see e.g., Pascali and Sburlan [5], Chapter IV). Equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezal [6]).

Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see e.g., Brézis and Browder [7–9], Browder [1], Browder et al. [10], Browder and Gupta [11], Cydotchepanovich [12], and De Figueiredo and Gupta [13]). For the iterative approximation of solutions of (1.4) and (1.5), the *monotonicity/accretivity* of A is crucial. The Mann iteration scheme (see, e.g., Mann [14]) has successfully been employed (see, e.g., the recent monographs of Berinde [15] and Chidume [16]). The recurrence formulas used involved K^{-1} which is also assumed to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not be monotone. In the special case in which the operators are defined on subsets D of X which are compact (or more generally, angle-bounded see e.g., Pascali and Sburlan [5] for definition), Brézis and Browder [7] have proved the strong convergence of a suitably defined Galerkin approximation to a solution of (1.8) (see also Brézis and Browder [9]).

It is our purpose in this paper to prove that an explicit coupled iteration process recently introduced by Chidume and Zegeye [17] which does not involve K^{-1} which is also required to be monotone converges strongly to a solution of (1.8) when K and F are bounded and monotone. Our new method of proof is also of independent interest.

2. Preliminaries

In the sequel, we will need the followings results.

Lemma 2.1 (see Xu [18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0, \tag{2.1}$$

where (i) $\{\alpha_n\} \subset (0,1)$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$, $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 2.2 (see Chidume and Djitte, [19, Lemma 2.5]). *Let H be a real Hilbert space and $A : H \to H$ be a map with $D(A) = H$. Suppose that A is m-accretive, that is,* (i) *for all $u, v \in H$, $\langle Au - Av, u - v\rangle \geq 0$;* (ii) *$R(I + s_0A) = H$ for some $s_0 > 0$. Then A satisfies the range condition, that is, $R(I + sA) = H$ for all $s > 0$.*

We now prove the following result.

Lemma 2.3. *Let H be a real Hilbert space and $F, K : H \to H$ be maps with $D(F) = D(K) = H$. Let $E = H \times H$ and $T : E \to E$ be the map defined by:*

$$Tw = (Fu - v, Kv + u), \quad \forall w = (u, v) \in E. \tag{2.2}$$

Assume that F and K are monotones and satisfy the range condition. Then, T is monotone and also satisfies the range condition.

Proof. On E we have the natural norm $\|\cdot\|_E$ and natural inner product $\langle\cdot,\cdot\rangle_E$ given by:

$$\begin{gathered}\|w\|_E = \left(\|u\|_H^2 + \|v\|_H^2\right)^{1/2}, \quad \text{for } w = (u,v) \in E, \\ \langle w_1, w_2\rangle_E = \langle u_1, u_2\rangle_H + \langle v_1, v_2\rangle_H, \quad \text{for } w_1 = (u_1, v_1),\ w_2 = (u_2, v_2) \in E.\end{gathered} \tag{2.3}$$

Step 1. We prove that T is monotone. Let $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in E$. We have $Tw_1 = (Fu_1 - v_1, Kv_1 + u_1)$ and $Tw_2 = (Fu_2 - v_2, Kv_2 + u_2)$. So, $Tw_1 - Tw_2 = (Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2)$. Therefore, using the fact that F and K are monotone, we obtain,

$$\begin{aligned}\langle Tw_1 - Tw_2, w_1 - w_2\rangle_E &= \langle Fu_1 - Fu_2 + v_2 - v_1, u_1 - u_2\rangle_H \\ &\quad + \langle Kv_1 - Kv_2 + \mathrm{u}_1 - u_2, v_1 - v_2\rangle_H \\ &= \langle Fu_1 - Fu_2, u_1 - u_2\rangle_H + \langle Kv_1 - Kv_2, v_1 - v_2\rangle_H \\ &\geq 0.\end{aligned} \tag{2.4}$$

So, T is monotone.

Step 2. We show that $\mathcal{R}(I_E + rT) = E$ for all r, $0 < r < 1$. In fact let r_0 such that $0 < r_0 < 1$. Since F and K are monotone and satisfy the range condition, then it is known that $(I + r_0F)$ and $(I + r_0K)$ are bijective and moreover, the resolvent $J_{r_0}^F := (I + r_0F)^{-1}$ of F and the resolvent $J_{r_0}^K := (I + r_0K)^{-1}$ of K are nonexpansive.

Let $h = (h_1, h_2) \in E$. Define $G : E \to E$ by

$$Gw = \left(J_{r_0}^F(h_1 + r_0v), J_{r_0}^K(h_2 - r_0u)\right), \quad \forall w = (u,v) \in E. \tag{2.5}$$

Using the fact that $J_{r_0}^F$ and $J_{r_0}^K$ are nonexpansive, we have,

$$\|Gw_1 - Gw_2\|_E \leq r_0\|w_1 - w_2\|_E, \quad \forall w_1, w_2 \in E. \tag{2.6}$$

Therefore G is a contraction. So, by the Banach fixed point theorem, G has a unique fixed point $w^* = (u^*, v^*) \in E$, that is $Gw^* = w^*$ or equivalently,

$$u^* = J_{r_0}^F(h_1 + r_0v^*), \qquad v^* = J_{r_0}^K(h_2 - r_0u^*). \tag{2.7}$$

These imply $(I_E + r_0T)w^* = h$. Therefore, $\mathcal{R}(I_E + r_0T) = E$.

By Lemma 2.2, it follows that T satisfies the range condition. This completes the proof. □

Theorem 2.4 (see Reich [20]). *Let H be a real Hilbert space. Let $A : H \to H$ be monotone with $D(A) = H$ and suppose that A satisfies the range condition: $\mathcal{R}(I + rA) = H$ for all $r > 0$. Let $J_tx := (I + tA)^{-1}x$, $t > 0$ be the resolvent of A, and assume that $A^{-1}(0)$ is nonempty. Then for each $x \in H$, $\lim_{t\to\infty} J_tx \in A^{-1}(0)$.*

3. Main Results

Let H be a real Hilbert space and $F, K : H \to H$ be maps with $D(K) = D(F) = H$ such that the following conditions hold:

(i) F is bounded and monotone, that is,

$$\langle Fu_1 - Fu_2, u_1 - u_2 \rangle \geq 0, \quad \forall u_1, u_2 \in H, \tag{3.1}$$

(ii) K is bounded and monotone, that is,

$$\langle Ku_1 - Ku_2, u_1 - u_2 \rangle \geq 0, \quad \forall u_1, u_2 \in H, \tag{3.2}$$

(iii) F and K satisfy the range condition.

With these assumptions, we prove the following theorem.

Theorem 3.1. *Let H be a real Hilbert space. Let $\{u_n\}$ and $\{v_n\}$ be sequences in H defined iteratively from arbitrary points $u_1, v_1 \in H$ as follows:*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1, \end{aligned} \tag{3.3}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim \theta_n = 0$,

(2) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lambda_n = o(\theta_n)$,

(3) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n) - 1)/\lambda_n\theta_n = 0$.

Suppose that $u + KFu = 0$ has a solution in H. Then, there exists a constant $d_0 > 0$ such that if $\lambda_n \leq d_0\theta_n$ for all $n \geq n_0$ for some $n_0 \geq 1$, then the sequence $\{u_n\}$ converges to u^*, a solution of $u + KFu = 0$.

Proof. Let $E := H \times H$ with the norm $\|z\|_E = (\|u\|_H^2 + \|v\|_H^2)^{1/2}$, where $z = (u, v)$. Define the sequence $\{w_n\}$ in E by: $w_n := (u_n, v_n)$. Let $u^* \in H$ be a solution of $u + KFu = 0$, $v^* := Fu^*$ and $w^* := (u^*, v^*)$. We observe that $u^* = -Kv^*$. It suffices to show that $\{w_n\}$ converges to w^* in E.

For this, let $n_0 \in \mathbb{N}$, there exists $r > 0$ sufficiently large such that $w_1 \in B(w^*, r/2)$, $w_{n_0} \in B(w^*, r)$, where $B(w^*, r)$ denotes the ball of center w^* and radius r. Define $B := \overline{B(w^*, r)}$. Since F and K are bounded, we set $M_1 := \sup\{\|Fx - y\|_H^2 + r^2 : (x, y) \in B\} < \infty$ and $M_2 := \sup\{\|Ky + x\|_H + r^2 : (x, y) \in B\} < \infty$. Let $M := M_1 + M_2$. We split the proof in three steps.

Step 1. We first prove that the sequence $\{w_n\}$ is bounded in E. Indeed, it suffices to show that w_n is in B for all $n \geq n_0$. The proof is by induction. By construction, $w_{n_0} \in B$. Suppose that $w_n \in B$ for $n \geq n_0$. We prove that $w_{n+1} \in B$. Assume for contradiction that $w_{n+1} \notin B$. Then, we have $\|w_{n+1} - w^*\|_E > r$. We compute as follows:

$$\|w_{n+1} - w^*\|^2 = \|u_{n+1} - u^*\|_H^2 + \|v_{n+1} - v^*\|_H^2. \tag{3.4}$$

We have

$$\begin{aligned}\|u_{n+1}-u^*\|_H^2 &= \|u_n-u^*-\lambda_n(Fu_n-v_n)-\lambda_n\theta_n(u_n-u_1)\|^2\\ &= \|u_n-u^*\|_H^2-2\lambda_n\langle Fu_n-v_n+\theta_n(u_n-u_1),u_n-u^*\rangle\\ &\quad+\lambda_n^2\|Fu_n-v_n+\theta_n(u_n-u_1)\|_H^2\\ &\le \|u_n-u^*\|_H^2-2\lambda_n\langle Fu_n-v_n+\theta_n(u_n-u_1),u_n-u^*\rangle+\lambda_n^2M_1.\end{aligned}\tag{3.5}$$

Observing that

$$\begin{aligned}\langle Fu_n-v_n+\theta_n(u_n-u_1),u_n-u^*\rangle &= \langle Fu_n-Fu^*,u_n-u^*\rangle-\langle v_n-v^*,u_n-u^*\rangle\\ &\quad+\ \theta_n\|u_n-u^*\|_H^2+\theta_n\langle u^*-u_1,u_n-u^*\rangle,\end{aligned}\tag{3.6}$$

and using (3.1), we obtain the following estimate:

$$\begin{aligned}\|u_{n+1}-u^*\|_H^2 &\le [1-2\lambda_n\theta_n]\|u_n-u^*\|_H^2+\lambda_n^2M_1\\ &\quad+2\lambda_n\langle v_n-v^*,u_n-u^*\rangle\\ &\quad-2\lambda_n\theta_n\langle u^*-u_1,u_n-u^*\rangle.\end{aligned}\tag{3.7}$$

Following the same argument, we also obtain

$$\begin{aligned}\|v_{n+1}-v^*\|_H^2 &\le [1-2\lambda_n\theta_n]\|v_{\mathrm{n}}-v^*\|_H^2+\lambda_n^2M_2\\ &\quad-2\lambda_n\langle u_n-u^*,v_n-v^*\rangle\\ &\quad-2\lambda_n\theta_n\langle v^*-v_1,v_n-v^*\rangle.\end{aligned}\tag{3.8}$$

Thus, we obtain

$$\begin{aligned}\|w_{n+1}-w^*\|_E^2 &\le [1-2\lambda_n\theta_n]\|w_n-w^*\|_E^2+M\lambda_n^2-2\lambda_n\theta_n\langle u^*-u_1,u_n-u^*\rangle\\ &\quad-2\lambda_n\theta_n\langle v^*-v_1,v_n-v^*\rangle.\end{aligned}\tag{3.9}$$

Using

$$\begin{aligned}0 &\le \|u^*-u_1+(u_n-u^*)\|_H^2=\|u^*-u_1\|_H^2+2\langle u^*-u_1,u_n-u^*\rangle+\|u_n-u^*\|_H^2,\\ 0 &\le \|v^*-v_1+(v_n-v^*)\|_H^2=\|v^*-v_1\|_H^2+2\langle v^*-v_1,v_n-v^*\rangle+\|v_n-v^*\|_H^2,\end{aligned}\tag{3.10}$$

we have

$$\begin{aligned}-2\langle u^*-u_1,u_n-u^*\rangle &\le \|u^*-u_1\|_H^2+\|u_n-u^*\|_H^2,\\ -2\langle v^*-v_1,v_n-v^*\rangle &\le \|v^*-v_1\|_H^2+\|v_n-v^*\|_H^2.\end{aligned}\tag{3.11}$$

Therefore

$$\begin{aligned}\|w_{n+1}-w^*\|_E^2 \le [1-2\lambda_n\theta_n]\|w_n-w^*\|_E^2 + M\lambda_n^2 + \lambda_n\theta_n\|w_n-w^*\|_E^2 \\ + \lambda_n\theta_n\|w^*-w_1\|_E^2.\end{aligned} \tag{3.12}$$

So we obtain the following estimate:

$$\|w_{n+1}-w^*\|_E^2 \le [1-\lambda_n\theta_n]\|w_n-w^*\|_E^2 + \lambda_n\theta_n\|w^*-w_1\|_E^2 + M\lambda_n^2. \tag{3.13}$$

Let $d_0 = r^2/4M$. Then using the induction assumptions, the fact that $w_1 \in B(w^*, r/2)$ and $\lambda_n \le d_0\theta_n$, we obtain

$$\|w_{n+1}-w^*\|_H^2 \le \left[1-\frac{\lambda_n\theta_n}{4}\right]r^2 < r^2, \tag{3.14}$$

a contradiction. Therefore, $w_{n+1} \in B$. Thus by induction, $\{w_n\}$ is bounded and so are $\{u_n\}$ and $\{v_n\}$.

Step 2. We show that there exists a unique sequence $z_n = (x_n, y_n) \in E$ such that

$$\theta_n(x_n-u_1) + Fx_n - y_n = 0, \tag{3.15}$$

$$\theta_n(y_n-v_1) + Ky_n + x_n = 0, \tag{3.16}$$

and $x_n \to x^*$, $y_n \to y^*$, with $x^* + KFx^* = 0$ and $y^* = Fx^*$.

In fact, let $T : E \to E$ be defined by $T(u,v) = (Fu - v, Kv + u)$, for all $(u,v) \in E$. Using the fact that F and K are monotone and satisfy the range condition, it follows from Lemma 2.3 that T is monotone and also satisfies the range condition.

Applying Theorem 2.4, with $t = 1/\theta_n$ and $x = (u_1, v_1)$, we obtain that $\lim_{t\to+\infty} J_t x \in T^{-1}(0)$ implies that

$$\lim_{n\to+\infty}\left(I + \frac{1}{\theta_n}T\right)^{-1}(u_1,v_1) \in T^{-1}(0). \tag{3.17}$$

Set $z_n = (x_n, y_n) := (I + (1/\theta_n)T)^{-1}(u_1, v_1)$. Then $(I + (1/\theta_n)T)(x_n, y_n) = (u_1, v_1)$, for all $n \ge 1$. So we have,

$$\begin{aligned} x_n + \frac{1}{\theta_n}(Fx_n - y_n) &= u_1, \\ y_n + \frac{1}{\theta_n}(Ky_n + x_n) &= v_1. \end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned} \theta_n(x_n-u_1) + Fx_n - y_n &= 0, \\ \theta_n(y_n-v_1) + Kx_n + x_n &= 0. \end{aligned} \tag{3.19}$$

Since T is monotone and satisfies the range condition, then it is known that $(I+rT)$ is bijective for every $r > 0$. So, the sequence $\{z_n\}$ is unique. Using (3.17) and Theorem 2.4, we have, $\lim z_n \in T^{-1}(0)$. Let $x_n \to x^*$ and $y_n \to y^*$. Then $(x^*, y^*) \in T^{-1}(0)$. So, $T(x^*, y^*) = 0$, that is,

$$\begin{aligned} Fx^* - y^* &= 0, \\ Ky^* + x^* &= 0. \end{aligned} \tag{3.20}$$

Therefore, $y^* = Fx^*$ and $x^* + KFx^* = 0$.

Step 3. We show that $\{w_n\} \to (u^*, v^*)$, where $u^* + KFu^* = 0$ and $v^* = Fu^*$.

Claim 1. $w_{n+1} - z_n \to 0$ as $n \to \infty$. We compute as follows:

$$\|w_{n+1} - z_n\|_E^2 = \|u_{n+1} - x_n\|_H^2 + \|v_{n+1} - y_n\|_H^2. \tag{3.21}$$

We have

$$\begin{aligned} \|u_{n+1} - x_n\|_H^2 &= \|u_n - x_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - u_1))\|_H^2 \\ &= \|u_n - x_n\|_H^2 - 2\lambda_n\langle Fu_n - v_n + \theta_n(u_n - u_1), u_n - x_n\rangle \\ &\quad + \lambda_n^2\|Fu_n - v_n + \theta_n(u_n - u_1)\|_H^2. \end{aligned} \tag{3.22}$$

From the boundness of $\{u_n\}$, $\{v_n\}$, and F, there exists $M_3 > 0$ such that $\|Fu_n - v_n + \theta_n(u_n - u_1)\|_H^2 \le M_3$. Using (3.15) and the fact that F is monotone, we obtain

$$\begin{aligned} \|u_{n+1} - x_n\|_H^2 &\le (1 - \lambda_n\theta_n)\|u_n - x_n\|_H^2 - 2\lambda_n\langle Fx_n - v_n, u_n - x_n\rangle \\ &\quad - 2\lambda_n\langle y_n - Fx_n, u_n - x_n\rangle + M_3\lambda_n^2 \\ &= (1 - \lambda_n\theta_n)\|u_n - x_n\|_H^2 - 2\lambda_n\langle y_n - v_n, u_n - x_n\rangle + M_3\lambda_n^2, \end{aligned} \tag{3.23}$$

for some constant $M_3 > 0$. Using (3.16) and similar arguments, we obtain:

$$\begin{aligned} \|v_{n+1} - v_n\|_H^2 &\le (1 - \lambda_n\theta_n)\|v_n - y_n\|_H^2 - 2\lambda_n\langle Ky_n + u_n, v_n - y_n\rangle \\ &\quad + 2\lambda_n\langle x_n + Ky_n, v_n - y_n\rangle + M_4\lambda_n^2 \\ &= (1 - \lambda_n\theta_n)\|v_n - y_n\|_H^2 + 2\lambda_n\langle x_n - u_n, v_n - y_n\rangle + M_4\lambda_n^2, \end{aligned} \tag{3.24}$$

for some constant $M_4 > 0$. Therefore, we have the following estimate:

$$\|w_{n+1} - z_n\|_E^2 \le (1 - \lambda_n\theta_n)\|w_n - z_n\|_E^2 + M'\lambda_n^2, \quad \text{where } M' = M_3 + M_4. \tag{3.25}$$

On the other hand, using the monotonicity of F and K we have

$$\begin{aligned} \|z_{n-1} - z_n\|_E^2 &\le \left\|x_{n-1} - x_n + \theta_n^{-1}(Fx_{n-1} - y_{n-1} - Fx_n + y_n)\right\|_H^2 \\ &\quad + \left\|y_{n-1} - y_n + \theta_n^{-1}(Ky_{n-1} + x_{n-1} - Ky_n - x_n)\right\|_H^2. \end{aligned} \tag{3.26}$$

Using (3.15) and (3.16), we observe that

$$\begin{aligned} x_{n-1} - x_n + \frac{1}{\theta_n}\left(Fx_{n-1} - y_{n-1} - Fx_n + y_n\right) &= \frac{\theta_n - \theta_{n-1}}{\theta_n}(x_{n-1} - u_1), \\ y_{n-1} - y_n + \frac{1}{\theta_n}\left(Ky_{n-1} + x_{n-1} - Ky_n - x_n\right) &= \frac{\theta_n - \theta_{n-1}}{\theta_n}(y_{n-1} - v_1). \end{aligned} \tag{3.27}$$

Therefore,

$$\|z_{n-1} - z_n\|_E \leq \frac{\theta_{\mathrm{n}-1} - \theta_n}{\theta_n}\|z_{n-1} - w_1\|_E. \tag{3.28}$$

Using (3.25) and the boundness of $\{x_n\}$ and $\{y_n\}$, we obtain that there exists $C > 0$ such that:

$$\|w_{n+1} - z_n\|_E^2 \leq (1 - \lambda_n\theta_n)\|w_n - z_{n-1}\|_H^2 + C\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right) + M'\lambda_n^2. \tag{3.29}$$

Thus, by Lemma 2.1, $w_{n+1} - z_n \to 0$. Since $z_n \to (x^*, y^*)$, we obtain that $w_n \to (x^*, y^*)$. But since $w_n = (u_n, v_n)$, this implies that $u_n \to u^*$ and $v_n \to v^*$. This completes the proof.

□

Corollary 3.2. *Let H be a real Hilbert space and $F, K : H \to H$ be maps with $D(K) = D(F) = H$ such that the following conditions hold:*

(i) *F and K are Lipschitz and monotone,*

(ii) *F and K satisfy the range condition.*

Let $\{u_n\}$ and $\{v_n\}$ be sequences in H defined iteratively from arbitrary points $u_1, v_1 \in H$ as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1, \end{aligned} \tag{3.30}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim \theta_n = 0$,

(2) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lambda_n = o(\theta_n)$,

(3) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n) - 1)/\lambda_n\theta_n = 0$.

Suppose that $u + KFu = 0$ has a solution in H. Then, there exists a constant $d_0 > 0$ such that if $\lambda_n \leq d_0\theta_n$ for all $n \geq n_0$ for some $n_0 \geq 1$, then the sequence $\{u_n\}$ converges to u^*, a solution of $u + KFu = 0$.

Let X be a real Banach space with dual space X^* and let $A : X \to X^*$ be a monotone linear operator. The mapping A is said to be *angle-bounded* with constant $\alpha \geq 0$ if

$$\left|(Ax, y) - (Ay, x)\right| \leq 2\alpha(Ax, x)^{1/2}(\mathrm{A}y, y)^{1/2}, \quad \forall x, y \in D(A), \tag{3.31}$$

where $(\cdot,\cdot)$ denotes the duality pairing between elements of X^* and those of X. The class of *angle-bounded* operators is a subclass of the class of monotone operators. The angle-boundness of A with $\alpha = 0$ corresponds to the symmetry of A, that is,

$$(Ax, y) = (Ay, x), \quad \forall x, y \in D(A). \tag{3.32}$$

(See Pascali and Sburlan [5, Chapter IV, page 189]).

Let H be a separable real Hilbert space and C be a closed subspace of H. For a given $f \in C$, consider the Hammerstein equation:

$$(I + KF)u = f, \tag{3.33}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{3.34}$$

where $K_n = P_n^* K P_n : H \to C_n$ and $F_n = P_n F P_n^* : C_n \to H$, where the symbols have their usual meanings (see [5] for the meaning of the symbols). Under this setting, Brézis and Browder (see [9]) proved the following approximation theorem.

Theorem BB. *Let H be a separable real Hilbert space. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be an angle-bounded and weakly compact mapping. Then, for each $n \in \mathbb{N}$, the Galerkin approximation* (3.34) *admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the* (3.33).

Remark 3.3. Theorem BB is the special case of the actual theorem of Brézis and Browder in which the Banach space is a separable real Hilbert space. The main theorem of Brézis and Browder is proved in an arbitrary separable Banach space.

Remark 3.4. The class of mappings considered in our theorem (Theorem 3.1) is larger than that considered in Theorem BB. In particular, in Theorem BB, in addition to assuming that the operator K is bounded and monotone, the authors also required K to be continuous. Furthermore, the operator F is restricted to the class of angle-bounded operators (a subclass of the monotone operators) and is also assumed to be weakly compact. In Theorem 3.1, the operators K and F are only assumed to be bounded and monotone and satisfy the range condition. We remark that continuity of the monotone map K implies that K is m-accretive (see Martin [21]) and it is known that m-accretive implies range condition.

Remark 3.5. Theorem BB guarantees the *existence* of a sequence $\{u_n\}$ which converges strongly to a solution of the Hammerstein equation (3.33). Our theorem provides an *iterative* sequence which converges strongly to a solution of (3.33).

Remark 3.6. Real sequences that satisfy the hypotheses of Theorem 3.1 are $\lambda_n = (n+1)^{-a}$ and $\theta_n = (n+1)^{-b}$ with $0 < b < a$ and $a + b < 1$.

We verify that these choices satisfy, in particular, condition (3) of Theorem 3.1. In fact, using the fact that $(1+x)^p \leq 1+px$, for $x > -1$ and $0 < p < 1$, we have

$$\begin{aligned} 0 \leq \frac{((\theta_{n-1}/\theta_n)-1)}{\lambda_n \theta_n} &= \left[\left(1+\frac{1}{n}\right)^b - 1\right] \cdot (n+1)^{a+b} \\ &\leq b \cdot \frac{(n+1)^{a+b}}{n} = b \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^{1-(a+b)}} \longrightarrow 0, \end{aligned} \tag{3.35}$$

as $n \to \infty$.

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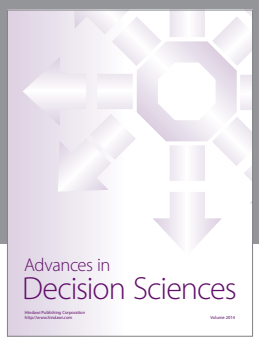

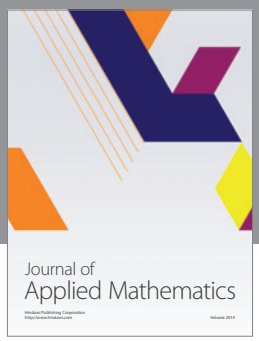

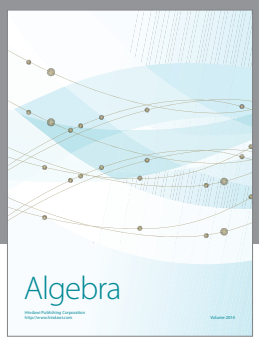

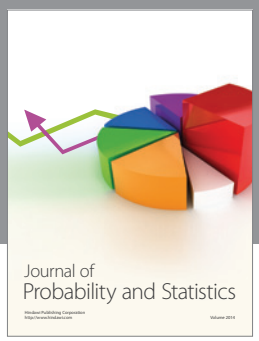

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