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Research Article

On Pointlike Interaction between Three Particles: Two Fermions and Another Particle

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The problem of construction of self-adjoint Hamiltonian for quantum system consisting of three pointlike interacting particles (two fermions with mass 1 plus a particle of another nature with mass $m > 0$) was studied in many works. In most of these works, a family of one-parametric symmetrical operators $\{H_\varepsilon, \varepsilon \in \mathbb{R}^1\}$ is considered as such Hamiltonians. In addition, the question about the self-adjointness of H_ε is equivalent to the one concerning the self-adjointness of some auxiliary operators $\{\mathcal{T}_l, l = 0, 1, \ldots\}$ acting in the space $L_2(\mathbb{R}^1_+, r^2 dr)$. In this work, we establish a simple general criterion of self-adjointness for operators $\mathcal{T}_l$ and apply it to the cases $l = 0$ and $l = 1$. It turns out that the operator $\mathcal{T}_{l=0}$ is self-adjoint for any m, while the operator $\mathcal{T}_{l=1}$ is self-adjoint for $m > m_0$, where the value of m_0 is given explicitly in the paper.

1. Introduction and Statement of the Problem

This paper is continuation of works [1–4] studying the problem of construction of Hamiltonian for a quantum system which consists of two fermions with mass 1 interacting pointwise with a particle of another nature having mass m.

Originally, the construction of such Hamiltonian begins with introduction of the symmetric operator:

$$H_0 = -\frac{1}{2}\left(\frac{1}{m}\Delta_y + \Delta_{x_1} + \Delta_{x_2}\right) \tag{1.1}$$

acting in a Hilbert space $\mathscr{H} = L_2(\mathbb{R}^3) \otimes L_2^{\text{asym}}(\mathbb{R}^3 \times \mathbb{R}^3)$. Here, $x_1, x_2 \in \mathbb{R}^3$ are the positions of fermions, y is the position of a separate particle, and Δ_y, Δ_{x_1}, and Δ_{x_2} are Laplacians with respect to y, x_1, and x_2, respectively. The domain of definition of H_0, $D(H_0) \subset \mathscr{H}$ consists of smooth rapidly decreasing functions $\psi(y, x_1, x_2) \in \mathscr{H}$ on infinity, antisymmetrical with respect to x_1, x_2 and satisfying the following conditions:

$$\psi(y, x_1, x_2)\big|_{x_i=y} = 0, \quad i = 1, 2. \tag{1.2}$$

Usually, some family $\{H_\varepsilon, \varepsilon \in \mathbb{R}^1\}$ of symmetric extensions of the operator H_0 is proposed as a possible "true" Hamiltonian of the system (the so-called Ter-Martirosian-Skornyakov extensions, see [5]). These extensions were constructed in [1–4]. For some values of mass m, the extensions of Ter-Martirosian-Skornyakov are self-adjoint (for all values of the parameter ε); however, for the other values of m they are only symmetric with nonzero deficiency indexes (equal for all ε). It turns out (see [3]) that the self-adjointness of all operators $\{H_\varepsilon\}$ is equivalent to the one for some auxiliary symmetric operator $\mathfrak{T}$ acting in the space $L_2(\mathbb{R}^3)$ (see below). This operator commutes with the operators $\{U_g, g \in O_3\}$ of the representation of the rotation group O_3 that acts in $L_2(\mathbb{R}^3)$ by the usual formula:

$$(U_g f)(k) = f\left(g^{-1}k\right), \quad g \in O_3, \; f \in L_2\left(\mathbb{R}^3\right). \tag{1.3}$$

Let us denote by $\mathscr{H}_l \subset L_2(\mathbb{R}^3)$ the maximal subspace, where the representation (1.3) is multiplied by the irreducible representation of O_3 with weight l, $l = 0, 1, 2, \ldots$ (see [6]). Evidently, the space $\mathscr{H}_l$ is invariant with respect to the operator $\mathfrak{T}$, and the restriction $\mathfrak{T}_l = \mathfrak{T}|_{\mathscr{H}_l}$ of this operator to the space $\mathscr{H}_l$ is symmetric operator. The operator $\mathfrak{T}$ is self-adjoint if all the operators $\{\mathfrak{T}_l, l = 0, 1, \ldots\}$ are self-adjoint. In this paper, we find general simple conditions of self-adjointness of $\mathfrak{T}_l$ and the form of the defect subspaces (with small exclusions) when these conditions are broken. Then, we apply these conditions to the cases $l = 0$ and $l = 1$ and get that the operator $\mathfrak{T}_{l=0}$ is self-adjoint for all values of $m > 0$, while the operator $\mathfrak{T}_{l=1}$ is self-adjoint for $m > m_0$ and has nonzero deficiency indexes for $m \le m_0$, the constant $m_0 > 0$ is indicated below (see (5.4)).

By the way, we note that the value of m_0 obtained in this paper differs from that one given by mistake in [2].

2. A Short Explanation of the Constructions from Papers [1–3]

(1) After Fourier transformation:

$$\begin{aligned}
\psi(y, x_1, x_2) &\longrightarrow \widetilde{\psi}(q, k_1, k_2) \\
&= \frac{1}{2\pi^{9/2}} \int_{(\mathbb{R}^3)^3} \psi(y, x_1 x_2) \exp\{-i(q, y) - i(k_1, x_1) - i(k_2, x_2)\} dy\, dx_1 dx_2 \\
&\equiv (\mathcal{F}\psi)(q, k_1, k_2),
\end{aligned} \tag{2.1}$$

and change of variables:

$$P = q + k_1 + k_2, \qquad p_j = \frac{P}{m+2} - k_j, \quad j = 1, 2, \tag{2.2}$$

the operator

$$\widetilde{H}_0 = \mathcal{F} H_0 \mathcal{F}^{-1}, \tag{2.3}$$

can be represented as a tensor sum:

$$\widetilde{H}_0 = \widetilde{H}_0^{(1)} + \frac{m}{m+1} \widetilde{H}_0^{(2)}, \tag{2.4}$$

where $H_0^{(1)}$ is a self-adjoint operator in $L_2(\mathbb{R}^3)$:

$$\left(\widetilde{H}_0^{(1)} f\right)(P) = \frac{P^2}{m+2} f(P), \quad P \in \mathbb{R}^3,\ f \in L_2\left(\mathbb{R}^3\right), \tag{2.5}$$

and $\widetilde{H}_0^{(2)}$ acts in $L_2^{\text{asym}}\,(\mathbb{R}^3 \times \mathbb{R}^3)$ by formula

$$\left(\widetilde{H}_0^{(2)} g\right)(p_1, p_2) = G(p_1, p_2) g(p_1, p_2), \quad g \in L_2^{\text{asym}}\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \tag{2.6}$$

with

$$G(p_1, p_2) = p_1^2 + p_2^2 + \frac{2}{m+1}(p_1, p_2) > 0. \tag{2.7}$$

The operator $\widetilde{H}_0^{(2)}$ is symmetric, and its domain is

$$D\left(\widetilde{H}_0^{(2)}\right) = \left\{ g \in L_2^{\text{asym}}\left(\mathbb{R}^3 \times \mathbb{R}^3\right) : \int_{\mathbb{R}^3} g(p_1, p_2)\, dp_j = 0, j = 1, 2 \right\}, \tag{2.8}$$

(2) the deficiency subspace $\mathcal{R}_{-1} \subset L_2^{\text{asym}}\,(\mathbb{R}^3 \times \mathbb{R}^3)$ of the operator $\widetilde{H}_0^{(2)}$ consists of the functions of the form:

$$U_{\wp}(p_1, p_2) = \frac{\wp(p_1) - \wp(p_2)}{G(p_1, p_2) + 1}, \tag{2.9}$$

where the function $\wp(p)$ belongs to Hilbert space

$$\mathcal{L} = \left\{ \wp : \int_{\mathbb{R}^3} \frac{|\wp(p)|^2}{\sqrt{p^2 + 1}} dp < \infty \right\}, \tag{2.10}$$

with inner product

$$\langle \wp_1, \wp_2 \rangle = \left(U_{\wp_1}, U_{\wp_2}\right)_{L_2(\mathbb{R}^3 \times \mathbb{R}^3)} \equiv \left(W \wp_1, \wp_2\right)_{L_2(\mathbb{R}^3)}. \tag{2.11}$$

Here W is some positive operator acting in $L_2(\mathbb{R}^3)$ (see [3]). The domain of the operator $(\widetilde{H}_0^{(2)})^*$, that is, a conjugate to $\widetilde{H}_0^{(2)}$, is

$$D\left(\left(\widetilde{H}_0^{(2)}\right)^*\right) = \left\{g \in L_2^{\text{asym}}\left(\mathbb{R}^3 \times \mathbb{R}^3\right) : g(p_1, p_2) = f(p_1, p_2) + U_{\wp}(p_1, p_2) + \frac{U_{\psi}(p_1, p_2)}{G(p_1, p_2) + 1}\right\}, \tag{2.12}$$

where $f \in D(\widetilde{H}_0^{(2)})$, $\wp, \psi \in \mathcal{L}$. In addition, the operator $(\widetilde{H}_0^{(2)})^*$ acts by the formula:

$$\left(\left(\widetilde{H}_0^{(2)}\right)^* g\right)(p_1, p_2) = G(p_1, p_2) g(p_1, p_2) - (\wp(p_1) - \wp(p_2)), \tag{2.13}$$

where $\wp$ is defined by (2.12).

The following asymptotics holds for vectors $g \in D((\widetilde{H}_0^{(2)})^*)N \to \infty$:

$$\int_{|p_1|<N} g(p_1, p_2) dp_1 = 4\pi N \wp(p_2) + b(p_2) + o(1). \tag{2.14}$$

Here

$$b(p) = -(T\wp)(p) + (W\psi)(p), \tag{2.15}$$

where the operator W is defined in (2.11), and $(T\wp)(p)$ is given by the following expression ($\mu = 2/(m+1)$)

$$(T\wp)(p) = 2\pi^2 \sqrt{\left(1 - \frac{\mu^2}{4}\right) p^2 + 1}\wp(p) + \int_{\mathbb{R}^3} \frac{\wp(t)}{G(t, p) + 1} dt, \tag{2.16}$$

defined on the set:

$$D(T) = \left\{\wp \in L_2\left(\mathbb{R}^3\right) : |p|\wp(p) \in L_2\left(\mathbb{R}^3\right)\right\}. \tag{2.17}$$

The above-mentioned Ter-Martirosian-Skornyakov's extension $\widetilde{H}_\varepsilon^{(2)}$ of the operator $\widetilde{H}_0^{(2)}$ is obtained by requiring

$$b(p) = \varepsilon \wp(p), \tag{2.18}$$

where $\varepsilon \in \mathbb{R}^1$ is an arbitrary parameter.

Lemma 2.1. *The operator T defined in the space $L_2(\mathbb{R}^3)$ by* (2.16) *is symmetric, and the self-adjointness of the operators H_ε (for all ε) is equivalent to the self-adjointness of the operator T (see [2, 3, 5]).*

The operator T can be represented as a sum of two operators:

$$T = \mathfrak{T} + T', \tag{2.19}$$

where the symmetric operator $\mathfrak{T}$ (with the domain $D(\mathfrak{T}) = D(T)$) acts as follows:

$$(\mathfrak{T}\wp)(p) = 2\pi^2\sqrt{1 - \frac{\mu^2}{4}}|p|\wp(p) + \int_{\mathbb{R}^3} \frac{\wp(t)dt}{G(t,p)} \tag{2.20}$$

and T' is a bounded self-adjoint operator. Since the deficiency indexes of T coincide with the ones of $\mathfrak{T}$ (see [7]), we shall study the conditions of self-adjointness for the operator $\mathfrak{T}$;

(3) as we said, the space $\mathcal{H}_l \subset L_2(\mathbb{R}^3)$ is invariant with respect to $\mathfrak{T}$; it has the form:

$$\mathcal{H}_l = L_2\left(\mathbb{R}^1_+, r^2 dr\right) \otimes L_2^l(S), \tag{2.21}$$

where $L_2^l(S) \subset L_2(S)$ is the space of spherical functions of weight l (see [6]) on the unit sphere $S \subset \mathbb{R}^3$. In addition, the operator $\mathfrak{T}_l = \mathfrak{T}|_{\mathcal{H}_l}$ has the form

$$\mathfrak{T}_l = M_l \otimes E_l, \tag{2.22}$$

where E_l is the unit operator in $L_2^l(s)$, and M_l acts in $L_2(\mathbb{R}^1_+, r^2 dr)$ by the formula:

$$(M_l f)(r) = 2\pi^2\sqrt{1 - \frac{\mu^2}{4}} r f(r) + 2\pi \int_{-1}^{1} dx P_l(x) \int_0^\infty \frac{(r')^2 f(r')dr'}{r^2 + (r')^2 + \mu r r' x}, \tag{2.23}$$

on the domain

$$D(M_l) \equiv V = \left\{u \in L_2\left(\mathbb{R}^1_+, r^2 dr\right) : r u(r) \in L_2\left(\mathbb{R}^1_+, r^2 dr\right)\right\}. \tag{2.24}$$

Here $P_l(x)$, $l = 0, 1, 2, \ldots$, $x \in [-1, 1]$, are orthogonal polynomials (Legendre polynomials) satisfying $P_l(1) = 1$:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l, \quad x \in (-1, 1). \tag{2.25}$$

The operators $\{M_l, l = 0, 1, \ldots\}$ are symmetric in $L_2(\mathbb{R}^1_+, r^2 dr)$, and the self-adjointness of M_l is equivalent to the self-adjointness of $\mathfrak{T}_l$. Later on, we shall study the operators M_l and derive a condition of self-adjointness.

3. Preparatory Constructions

For every function $u \in V \subset L_2(\mathbb{R}^1_+, r^2 dr)$, we consider the family of functions

$$\mathfrak{h}(u) = \{u_\alpha = r^\alpha u, \alpha \in [0,1],\ u_0 = u\}, \tag{3.1}$$

which we call *a chain* (with initial element $u = u_0$ and the final one u_1). All functions $u_\alpha \in \mathfrak{h}(u)$ belong to $L_2(\mathbb{R}^1_+, r^2 dr)$ and have a uniformly bounded norm:

$$\|u_\alpha\|^2 \le \|u_0\|^2 + \|u_1\|^2, \quad \alpha \in [0,1]. \tag{3.2}$$

Consider the unitary map (Mellin's transformation [8]):

$$\omega : L_2\left(\mathbb{R}^1_+, r^2 dr\right) \longrightarrow L_2\left(\mathbb{R}^1, ds\right) : f(r) \longrightarrow \tilde{f}(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-is+1/2} f(r) dr, \quad s \in \mathbb{R}^1 \tag{3.3}$$

and its inverse:

$$\left(\omega^{-1}\tilde{f}\right)(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty r^{is-3/2} \tilde{f}(s) ds. \tag{3.4}$$

For every set of functions $B \subset L_2(\mathbb{R}^1_+, r^2 dr)$, we denote by $\tilde{B} \subset L_2(\mathbb{R}^1, ds)$ the set of their Mellin's transformations:

$$\tilde{B} = \omega B. \tag{3.5}$$

For every chain $\mathfrak{h}(u)$, we denote by Γ_u the family of functions:

$$\Gamma_u = \widetilde{\mathfrak{h}(u)} = \left\{\gamma_\alpha(s), \alpha \in [0,1]\right\}, \tag{3.6}$$

where $\gamma_\alpha(s) = (\omega u_\alpha)(s)$, $u_\alpha \in \mathfrak{h}(u)$. The family Γ_u can be represented as a function $\Gamma_u(z)$ of a complex variable $z = s + i\alpha$ in the strip:

$$\begin{gathered} I = \left\{z \in \mathbb{C}^1 : \Im z \in [0,1]\right\}, \\ \Gamma_u(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-is-1/2+\alpha} u(r) dr = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-iz} u(r) dr. \end{gathered} \tag{3.7}$$

The function Γ_u is said to be *associated* with the chain $\mathfrak{h}(u)$, and its values $\{\gamma_\alpha(s)\}$ on the lines $\xi_\alpha = \{z = s + i\alpha, s \in \mathbb{R}^1, 0 \le \alpha \le 1\} \subset I$ are called *the sections* of Γ_u.

Proposition 3.1. *For every chain $\mathfrak{h}(u)$, $u \in V$, the associated function $\{\Gamma_u(z), z \in I\}$ is continuous in a closed strip I and analytic inside this strip. Moreover, its sections $\{\gamma_\alpha\}$ satisfy the following inequality:*

$$\sup_{0\leq\alpha\leq 1} \left\|\gamma_\alpha(\cdot)\right\|_{L_2(\mathbb{R}^1)} < \infty. \tag{3.8}$$

Inversely, any function $\{\Gamma(z), z \in I\}$ which possesses these properties is associated with some (unique) chain $\mathfrak{h}(v) : \Gamma = \Gamma_v$, $v \in V$. Let call this chain generated by Γ. In addition, the functions $\{v_\alpha, \alpha \in [0,1]\}$ of the chain $\mathfrak{h}(v)$ are obtained by the inverse Mellin's transformation from the sections of $\Gamma = \{\gamma_\alpha\}$:

$$v_\alpha = \omega^{-1}\gamma_\alpha. \tag{3.9}$$

The proof of this proposition can be obtained by using the arguments given in the book by Paley and Wiener (see [9], Chapter I), which are related to the Fourier transformation of functions analytical in a strip in a complex plane. It is not difficult to reformulate these arguments in terms of Mellin's transformation.

Note that the estimate (3.8) for $\{\gamma_\alpha\}$ follows from the estimate (3.2) and the unitary Mellin's transformation. Denote by $\mathcal{G}$ a linear space of functions Γ satisfying conditions of Proposition 3.1. Let us introduce two maps:

$$\Omega : \mathfrak{h}(u) \longrightarrow \Gamma_u \in \mathcal{G}, \qquad \Omega^{-1} : \Gamma_u \longrightarrow \mathfrak{h}(u). \tag{3.10}$$

Let $N(z)$, $z \in I$, be a bounded, continuous function in the strip I, which is analytic inside I. This function generates the family $\tilde{\kappa}_\alpha^N$, $\alpha \in [0,1]$ of bounded operators in $L_2(\mathbb{R}^1)$ which act as multiplication on the functions $n_\alpha^N(s) = N(z)|_{z=s+i\alpha}$, $s \in \mathbb{R}^1$, $0 \leq \alpha \leq 1$:

$$\left(\tilde{\kappa}_\alpha^N \psi\right)(s) = n_\alpha^N(s)\psi(s), \quad \psi \in L_2\left(\mathbb{R}^1\right). \tag{3.11}$$

Evidently, for any $\Gamma \in \mathcal{G}$, the function $N(z)\Gamma(z)$ belongs to $\mathcal{G}$. If the chain $\mathfrak{h}(u)$ is generated by $\Gamma = \Gamma_u$ and the chain $\mathfrak{h}(v)$ is generated by $N(z)\Gamma(z) = \Gamma_v(z)$, then

$$v_\alpha = \kappa_\alpha^N u_\alpha, \quad \alpha \in [0,1], \quad u_\alpha \in \mathfrak{h}(u), \tag{3.12}$$

where

$$\kappa_\alpha^N = \omega^{-1}\tilde{\kappa}_\alpha^N \omega. \tag{3.13}$$

Denote by Π the following self-adjoint operator in $L_2(\mathbb{R}_+^1, r^2 dr)$:

$$(\Pi f)(r) = r f(r), \tag{3.14}$$

with the domain $D(\Pi) = V$.

It is clear that for any $u \in V$, the power Π^β, $0 \le \beta \le 1$ of the operator Π is applicable to an element $u_\alpha \in \mathfrak{h}(u)$ if $\beta + \alpha \le 1$ and

$$\Pi^\beta u_\alpha = u_{\alpha+\beta}. \tag{3.15}$$

For the function Γ_u that is associated with $\mathfrak{h}(u)$,the action of the operator $\widetilde{\Pi}^\beta = \omega \Pi^\beta \omega^{-1}$ on the sections $\{\gamma_\alpha\}$ of Γ_u has the form:

$$\widetilde{\Pi}^\beta \gamma_\alpha = \gamma_{\alpha+\beta}. \tag{3.16}$$

(again if $\alpha + \beta \le 1$).

4. The Operator M_l

The operator M_l (see (2.23)) can be represented as

$$M_l = \Pi^{1/2} \kappa^l \Pi^{1/2}, \tag{4.1}$$

where $\kappa^l = \kappa^l_{1/2}$ is an operator in $L_2(\mathbb{R}^1_+, r^2 dr)$ acting by the formula:

$$\left(\kappa^l_{1/2} f\right)(r) = 2\pi^2 \sqrt{1 - \frac{\mu^2}{4}} f(r) + 2\pi \int_{-1}^{1} dx\, P_l^0(x) \int_0^\infty \frac{(r')^2 f(r') dr'}{(rr')^{1/2} \left(r^2 + (r')^2 + \mu x r r'\right)}. \tag{4.2}$$

Lemma 4.1. *Operator $\kappa^l_{1/2}$ is bounded and self-adjoint in $L_2(\mathbb{R}^1_+, r^2 dr)$.*

Proof. Pass to the operator:

$$\widetilde{\kappa}^l_{1/2} = \omega \kappa^l_{1/2} \omega^{-1}, \tag{4.3}$$

acting in $L_2(\mathbb{R}^1)$. It follows from calculations in [2, 3] that $\widetilde{\kappa}^l_{1/2}$ is the operator of multiplication on the function:

$$n^l_{1/2}(s) = 2\pi^2 \left(\sqrt{1 - \frac{\mu^2}{4}} + \lambda^l_{1/2}(s) \right), \tag{4.4}$$

where

$$\lambda^l_{1/2}(s) = \begin{cases} \displaystyle\int_0^1 P_l(x) \frac{\operatorname{ch}(sv(x)) dx}{\operatorname{ch}(s\pi/2) \cos(v(x))} & \text{for even } l, \\ \displaystyle -\int_0^1 P_l(x) \frac{\operatorname{sh}(sv(x)) dx}{\operatorname{sh}(s\pi/2) \cos(v(x))} & \text{for odd } l, \end{cases} \tag{4.5}$$

and $v(x) = \arcsin \mu x/2$, $0 \le x \le 1$. As we see the function $n^l_{1/2}(s)$, $s \in \mathbb{R}^1$, is bounded and real. The lemma is proved. □

We see from (4.4) and (4.5) that the functions $n^l_{1/2}(s)$ and $\lambda^l_{1/2}$ are continued up to bounded, analytical functions $N^l(z)$ and $\Lambda^l(z)$ correspondingly, defined in the strip $\widetilde{I} = \{z \in \mathbb{C}^1 : -1/2 \le \Im z \le 1/2\}$. Let us define the functions $\widehat{N^l}(z) = \overline{N}^l(z - i/2)$ which we shall consider in the strip $I = \{z \in \mathbb{C} : 0 \le \Im z \le 1\}$. The operator $\widetilde{\kappa}^l_{1/2}$ coincides with the operator $\widetilde{\kappa}^{\widehat{N^l}}_{1/2}$ from the family $\{\widetilde{\kappa}^{\widehat{N^l}}_\alpha\}$ generated by the function $\widehat{N^l}$ (see (3.11)). Any other operator of this family acts as multiplication on the function:

$$\widehat{n}^l_\alpha(s) = \widehat{N}^l(z)\Big|_{z=s+i\alpha}. \tag{4.6}$$

Denote by κ^l_α the operators

$$\kappa^l_\alpha = \omega^{-1}\widetilde{\kappa}^{N^l}_\alpha \omega, \tag{4.7}$$

acting in $L_2(\mathbb{R}^1_+, r^2 dr)$.

Note that

$$\left(\kappa^l_\alpha\right)^* = \kappa^l_{i-\alpha}. \tag{4.8}$$

It is convenient to represent the operator M_l in form of three sequential maps

$$\begin{aligned} M_l : u_0 \in \mathfrak{h}(u_0) &\longrightarrow \Pi^{1/2} u_0 = u_{1/2} \longrightarrow \kappa^l_{1/2} u_{1/2} = v_{1/2} \\ &\longrightarrow \Pi^{1/2} v_{1/2} = v_1 \in \mathfrak{h}(v), \end{aligned} \tag{4.9}$$

where $v = v_0, v_{1/2}, v_1$ are elements of the chain $\mathfrak{h}(v)$ generated by the function $\Gamma_v = \widehat{N}^l \Gamma_u \in \mathcal{G}$. Note that the chain (4.9) can be rewritten in the following way:

$$u_0 \in \mathfrak{h}(u_0) \xrightarrow{\Omega} \Gamma_{u_0} \longrightarrow \Gamma_v = \widehat{N}^l \Gamma_{u_0} \xrightarrow{\Omega^{-1}} \mathfrak{h}(v) \longrightarrow v_1 \in \mathfrak{h}(v). \tag{4.10}$$

From (4.1) and self-adjointness of $\kappa^l_{1/2}$ it follows that the operator M_l with the domain $D(M_l) = V$ is symmetric. For any $\alpha \in [0,1]$, a representation of M_l similar to (4.1) is valid:

$$M_l = \Pi^{1-\alpha}\kappa^l_\alpha \Pi^\alpha \tag{4.11}$$

as well as decomposition like (4.9).

Let us now describe the domain $D(M^*_l) \supseteq V$ of the operator M^*_l conjugated to M_l. Let $g \in D(M^*_l)$ be a function from $D(M^*_l)$ and $h = M^*_l g \in L_2(\mathbb{R}^1_+, r^2 dr)$. Then for every $u \in V = D(M_l)$, we can write

$$(M_l u, g) = \left(\kappa^l_1 \Pi u, g\right) = \left(\Pi u, \left(\kappa^l_1\right)^* g\right) = \left(u, \Pi \kappa^l_0 g\right) = (u, h). \tag{4.12}$$

Here we use the representation (4.11) for $\alpha = 1$ and the equality (4.8). Denote $f(r) = h(r) - (\Pi\kappa_0^l g)(r)$ and apply the following evident assertion.

Lemma 4.2. *Let a measurable function $f(r)$ satisfies condition*

$$\int_0^\infty f(r)u(r)r^2 dr = 0, \tag{4.13}$$

for any $u \in V$. Then $f = 0$.

From this and (4.12), it follows that

$$\Pi\kappa_0^l g = h. \tag{4.14}$$

Hence

$$w_0 \equiv \kappa_0^l g \in V, \tag{4.15}$$

and $h = w_1 \in \mathfrak{h}(w_0)$ is the final element of the chain $\mathfrak{h}(w_0)$. Thus the domain $D(M_l^*)$ of the operator M_l^* is

$$D(M_l^*) = \left\{ g \in L_2\left(\mathbb{R}_+^1, r^2 dr\right) : \kappa_0^l g \in V \right\}. \tag{4.16}$$

In the case when the operator κ_0^l has the inverse one, $(\kappa_0^l)^{-1}$, which is equivalent to the condition:

$$\widehat{n}_0^l(s) \neq 0, \quad \text{for any } s \in \mathbb{R}^1, \tag{4.17}$$

the following equality is true:

$$D(M_l^*) = \left(\kappa_0^l\right)^{-1} V. \tag{4.18}$$

Let $\widetilde{M}_l^* = \omega M_l^* \omega^{-1}$ be an operator in $L_2(\mathbb{R}^1)$ with domain $D(\widetilde{M}_l^*) = \omega D(M_l^*)$. Then for $\widetilde{g} \in D(\widetilde{M}_l^*)$, the following representation holds true:

$$\widetilde{g}(s) = \left(\widehat{n}_0^l(s)\right)^{-1} \widetilde{w}_0(s) = \left(\widehat{N}^l(z)\right)^{-1} \Gamma_{w_0}(z)\Big|_{z=s}, \tag{4.19}$$

if condition (4.17) is fulfilled. Here $\widetilde{w}_0(s) = (\omega w_0)(s)$ where w_0 is defined in (4.15).

Remarks. (1) Note that the function $\widehat{N}^l(z)$ is invariant with respect to reflection of the complex plane around the point $z = i/2$:

$$z \longrightarrow z^* = -z + i. \tag{4.20}$$

Under this reflection, the strip I is mapped onto itself; hence, for every zero $\overline{z} \in I(\overline{z} \neq i/2)$ of the function $\widehat{N^l}$, there exists another zero, $\overline{z}^* \in I$, of $\widehat{N^l}$ with the same multiplicity. The multiplicity of $\overline{z} = i/2 = \overline{z}^*$ is even;

(2) Since $\widehat{N^l}(z) \to 2\pi^2\sqrt{1-\mu^2/4} > 0$ as $z \to \infty$ inside I, the function $\widehat{N^l}(z)$ has finite number of zeros inside I.

We can now formulate the main criterion of self-adjointness of the operator M_l.

Theorem 4.3. *The operator M_l is self-adjoint if and only if the function $\widehat{N^l}(z)$ has no zeros in the closed strip I.*

Proof. (1) Assume $\widehat{N^l}(z) \neq 0$ in the strip I. Then $(\widehat{N^l})^{-1}(z)$ is bounded and continuous on I and analytical inside I. Let $\widetilde{g} \in \widetilde{D}(\widetilde{M}_l^*)$. Since $\widehat{n}^l(s) \neq 0$ for $s \in \mathbb{R}^1$, the representation (4.19) holds true. Since

$$\left(\widehat{N^l}(z)\right)^{-1}\Gamma_{w_0}(z) = \Gamma_v \in \mathcal{G}, \quad v \in V, \tag{4.21}$$

the element $g = \omega^{-1}\widetilde{g} \in D(M_l^*)$ coincides with $v \in V$, that is, $D(M_l^*) = V = D(M_l)$; it means the self-adjointness of M_l;

(2) assume now the function $N^l(z)$ has zeros $\overline{z}_1, \ldots, \overline{z}_k \in I$. Consider first the case when all zeros are lying inside I and their multiplicities are equal to $p_1, \ldots, p_k$, respectively. Again, let $\widetilde{g} \in \widetilde{D}(\widetilde{M}_l^*)$. Since $\widehat{n}^l(s) \neq 0$, the representation (4.19) holds true. The function $(\widehat{N^l}(z))^{-1}\Gamma_{w_0}(z)$ is meromorphic in I with poles $\overline{z}_1, \ldots, \overline{z}_k$ having the order $p_1, \ldots, p_k$ respectively. For this function the usual canonical representation [10] is true:

$$\left(\widehat{N^l}(z)\right)^{-1}\Gamma_{w_0}(z) = L^{w_0}(z) + \sum_{n=1}^{k}\sum_{m=1}^{p_n} \frac{b_m^{(n)}(w_0)}{(z-z_n)^m}, \tag{4.22}$$

where $L^{w_0}(z)$ is bounded, continuous function on I, and analytical inside I, and the coefficients $b_m^{(n)} = b_m^{(n)}(w_0)$ depend on w_0.

Lemma 4.4. *The function $L^{w_0}(z)$in* (4.22) *belongs to the space $\mathcal{G}$.*

The proof of this lemma is given in The appendix.

From (4.19) and (4.22), for $g = \omega^{-1}\widetilde{g} \in D(M_l^*)$, we have

$$g(r) = v(r) + \sum_{m,n} b_m^{(n)}\left(\omega^{-1}\left(\left(\frac{1}{\cdot - z_n}\right)^m\right)\right)(r), \tag{4.23}$$

where the function $v \in V$ is defined from relation

$$\begin{gathered} L^{w_0}(z) = \Gamma_v(z) \in \mathcal{G}, \\ d_{m,n}(r) := \omega^{-1}\left(\left(\frac{1}{(\cdot - z_n)}\right)^m\right)(r) = A_m^{(n)} r^{-3/2-t_n+is_n}(\ln r)^{m-1}\chi(r), \end{gathered} \tag{4.24}$$

where $A_m^{(n)}$ is an absolute constant, $z_n = s_n + it_n$, $0 < t_n < 1$ and

$$\chi(r) = \begin{cases} 1, & r > 1, \\ 0, & r \leq 1. \end{cases} \tag{4.25}$$

Since linearly independent functions $d_{m,n} \in D(M_l^*)$ do not belong to V, due to (4.23), they form the basis in the defect subspace $\mathfrak{V}$ of the operator M_l (see [7]). Since the dimension of the subspace $\mathfrak{V}$ is equal to $\sum_1^k p_n$ and the operator M_l is real, its deficiency indexes $n_\pm$ are equal and have the form:

$$n_+ = n_- = \frac{1}{2}\sum_1^k p_n. \tag{4.26}$$

(It follows from Remarks that the sum $\sum_1^k p_n$ is even). Consider now the case when one of the zeros of $N^l(z)$, say, $\overline{z}_0 = s_0 \in \mathbb{R}^1$, lies on the boundary of I and has multiplicity p (in addition, there is a zero $\overline{z}_0^* = s_0 + i$). In this case, in a neighborhood of $\overline{z}_0$, the function $\widehat{N^l}(z)$ has the form:

$$\widehat{N^l}(z) = (z - \overline{z}_0)^p Q(z), \tag{4.27}$$

where $Q(z)$ is analytic in this neighborhood. Consider the function,

$$G(z) = \frac{1}{(-i(z - \overline{z}_0))^{1/3}} \frac{1}{(z + 2i)^2}, \tag{4.28}$$

whereby $(-iw)^{1/3}$ for $\Im w > 0$, we mean the branch of a many-valued function $(-iw)^{1/3}$ that takes positive values on the positive part of the imaginary axis. Evidently, the function $G(z)$ is analytic in the strip I and satisfies condition (3.8). However, this function is discontinuous at $\overline{z}_0$ and does not belong to $\mathcal{G}$. In addition, the function $\widehat{N^l_{(z)}}G(z)$ now belongs to $\mathcal{G}$ as follows from (4.27) and (4.28). Thus

$$\widetilde{g}(s) = G(z)\big|_{z=s} \overline{\in} \widetilde{V} = \omega V, \tag{4.29}$$

but

$$\widehat{n}^l(s)\widetilde{g}(s) = \widehat{N^l}(z)G(z)\Big|_{z=s} \in \widetilde{V}. \tag{4.30}$$

Consequently, $g = \omega\widetilde{g}\overline{\in}V$ but $\kappa_0^l g \in V$, that is, $g \in D(M_l^*)$. Thus $D(M_l^*) \neq V$, and the operator M_l has nonzero deficiency indexes. Theorem 4.3 is proved. □

5. The Operators M_l in the Cases $l = 0$ and $l = 1$

Here, we apply Theorem 4.3 to the cases $l = 0$ and $l = 1$.

Theorem 5.1. *(1) For $l = 0$, the operator $M_{l=0}$ is self-adjoint for any $m > 0$;*

(2) the operator $M_{l=1}$ is self-adjoint for $m > m_0$ and has nonzero deficiency indexes for $m \leq m_0$. In addition, for $m < m_0$ these indexes are equal to $(1, 1)$. The constant m_0 is a unique zero of (5.4).

Proof. We need the following properties of the functions $\widehat{\Lambda}^{l=0}(z)$ and $\widehat{\Lambda}^{l=1}(z)$, $z \in I$.

Lemma 5.2. (1) *For any $l = 0, 1, 2, \ldots$ the function $\widehat{\Lambda}^l(z)$ is invariant with respect to reflection* (4.20);

(2) *The point $z = i/2 \in I$ is a nondegenerate critical point for both functions $\widehat{\Lambda}^{l=0}$ and $\widehat{\Lambda}^{l=1}$;*

(3) *These functions take real values on the line*:

$$\widehat{\xi}_{1/2} = \left\{ z = s + \frac{i}{2}, s \in \mathbb{R}^1 \right\}, \tag{5.1}$$

and on the segment:

$$\widehat{\tau} = \{ z = it, 0 \leq t \leq 1 \}. \tag{5.2}$$

Outside the set $B = \widehat{\xi}_{1/2} \cup \widehat{\tau}$, both functions take nonreal values;

(4) *the real values of $\widehat{\Lambda}^l$, $l = 0, 1$, are between 0 and $\widehat{\Lambda}^l(0) = \widehat{\Lambda}^l(i)$. Every value of $\widehat{\Lambda}^l|_B$— except $\widehat{\Lambda}^l(i/2)$* —is taken exactly at two points;

(5) *the extreme values of $\widehat{\Lambda}^l$, $l = 0, 1$, $\widehat{\Lambda}^l(0) = \widehat{\Lambda}^l(i)$ are given by*

$$\begin{aligned} \widehat{\Lambda}^{l=0}(0) &= 8\sqrt{2}\pi^2\mu^{-1}\sin\left(\frac{1}{2}\arcsin\frac{\mu}{2}\right) > 0, \\ \widehat{\Lambda}^{l=1}(0) &= -\frac{32}{3}\sqrt{2}\pi^2\mu^{-2}\sin^3\left(\frac{1}{2}\arcsin\frac{\mu}{2}\right) \equiv -q(\mu) < 0, \end{aligned} \tag{5.3}$$

(6) *the function $q(\mu)$ increases monotonically on the interval $0 < \mu < 2$.*

The proof of this lemma is given in The appendix.

Corollary 5.3. (1) *The zeros of $\widehat{N^l}(z)$, $l = 0, 1$ can only lie in the set B;*

(2) $\widehat{N^{l=0}}(z)|_B > 0$ *for any value of μ, and therefore the operator $M_{l=0}$ is self-adjoint for all $m \in (0, 2)$;*

(3) *The function $\widehat{N^{l=1}}(z)|_B$ is positive if $2\pi^2\sqrt{1 - \mu^2/4} > q(\mu)$ and vanishes at some point $z \in B$ (and also at $z^* \in B$) if $2\pi^2\sqrt{1 - \mu^2/4} \leq q(\mu)$.*

In Figure 1, the curves corresponding to the functions $2\pi^2\sqrt{1 - \mu^2/4}$ and $q(\mu)$ are depicted. We see that they intersect at a unique point with abscissa $\mu_0 \in (0, 2)$ which satisfies the following equation:

$$2\pi^2\sqrt{1 - \frac{\mu_0^2}{4}} = q(\mu_0). \tag{5.4}$$

Thus, for $m > m_0 = 2/\mu_0 - 1$ the operator $M_{l=1}$ is self-adjoint, and for $m < m_0$ it has deficiency indexes $(1, 1)$. For $m = m_0$, the operator $M_{l=1}$ is not self-adjoint as well. Theorem 5.1 is proved. □

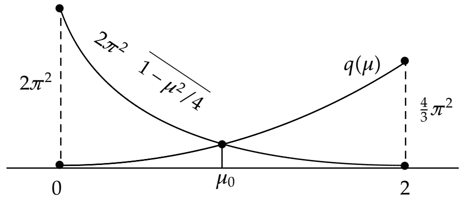

Figure 1

Appendix

Proof of Lemma 4.4. The function $(\widehat{N}^l(z))^{-1}$, $z \in I$ admits the canonical representation (see [10])

$$\left(\widehat{N}^l(z)\right)^{-1} = Q_l(z) + \sum_{n=1}^{k}\sum_{m=1}^{p_n} \frac{a_m^{(n)}}{(z-\overline{z}_n)^m}, \tag{A.1}$$

where $\overline{z}_1,\ldots,\overline{z}_k \in I$ are zeros of $\widehat{N}^l(z)$ (with multiplicities $p_1,\ldots,p_k$), $a_m^{(n)}$ are constants, $a_{p_n}^{(n)} \neq 0$, and $Q_l(z)$ is a bounded, continuous analytic function in I. From this, it follows that for any $v \in V$, $Q_l(z)\Gamma_v(z) \in \mathcal{G}$. Consider some term of the sum (A.1) and write

$$\frac{a_m^{(n)}}{(z-\overline{z}_n)^m}\Gamma_v(z) = \left(P_{m,v}^{(n)}(z) + \sum_{d=1}^{m}\frac{c_{m-d}^{(n)}}{(z-\overline{z}_n)^d}\right)a_m^{(n)}, \tag{A.2}$$

where

$$\begin{gathered} P_{m,v}^{(n)}(z) = \frac{1}{(z-\overline{z}_n)^m}\left(\Gamma_v(z) - \sum_{d=1}^{m} c_{m-d}^{(n)}(z-\overline{z}_n)^{m-d}\right), \\ c_t^{(n)} = c_t^{(n)}(v) = \frac{1}{t!}\Gamma_v^{(t)}(\overline{z}_n), \quad t = 0,1,\ldots. \end{gathered} \tag{A.3}$$

It is clear that $P_{m,v}^{(n)}(z)$ is bounded, continuous analytic function in I. We are going to show that this function belongs to $\mathcal{G}$. Let $O \in I$ be a small neighborhood of $\overline{z}_n$ and $\chi_O(z)$ the characteristic function of O. Obviously, the bounded function $\chi_O P_{m,v}^{(n)}$ satisfies condition (3.8). Every term of the sum

$$(1-\chi_O)P_{m,v}^{(n)}(z) = \frac{\Gamma_v(z)}{(z-\overline{z}_n)^m}(1-\chi_O) - \sum_{d=1}^{m}\frac{c_{m-d}^{(n)}(v)}{(z-\overline{z}_n)^d}(1-\chi_O) \tag{A.4}$$

satisfies this condition as well.

Thus for fixed $\overline{z}_n$ and $v \in V$,

$$\sum_{m=1}^{p_n} \frac{a_m^{(n)} \Gamma_v(z)}{(z-\overline{z}_n)^m} = K_v^{(n)}(z) + \sum_{d=1}^{p_n} \frac{b_d^{(n)}(v)}{(z-\overline{z}_n)^d}, \tag{A.5}$$

where

$$K_v^{(n)}(z) = \sum_{m=1}^{p_n} a_m^{(n)} P_{m,v}^{(n)}(z), \tag{A.6}$$

$$b_d^{(n)}(v) = \sum_{m=1}^{p_n} a_m^{(n)} c_{m-d}^{(n)}(v), \quad d = 1, \ldots, p_n. \tag{A.7}$$

Thus, we get the representation (4.22) where

$$L^{(w_0)}(z) = Q_l(z)\Gamma_{w_0}(z) + \sum_{n=1}^{k} K_{w_0}^{(n)}(z) \in \mathcal{G}, \tag{A.8}$$

and the coefficients $b_m^{(n)}(w_0)$ are given by formula (A.7). Lemma 4.4 is proved. □

Proof of Lemma 5.2. (1) It is more convenient to consider the functions $N^l(z)$ and $\Lambda^l(z)$ in the strip $\widetilde{I} = \{z : |\Im z| < 1/2\}$ instead of the functions $\widehat{N}^l(z)$ and $\widehat{\Lambda}^l(z)$ in the strip I. Similarly, instead of the reflection $z \to z^*$ we consider the reflection $z \to -z$ around the point $z_0 = 0$. It is clear that the functions $\Lambda^l(z)$, $l = 0, 1, 2, \ldots$ are invariant with respect to the change $z \to -z$, and it means the invariance of $\widehat{\Lambda}^l$ with respect to reflection (4.20);

(2) it follows from (4.5) that $z = 0$ is a nondegenerated critical point of $\Lambda^{l=0}$ and $\Lambda^{l=1}$, if we note that $0 < v(x) \le \pi/2$. Correspondingly, $z = i/2$ is a nondegenerated critical point for $\widehat{\Lambda}^l(z)$, $l = 0, 1$. The real axis $\xi_0 = \{z = s; s \in \mathbb{R}^1\}$ coincides with the saddle-point line at $z = 0$ (see [10]) for $\Lambda^{l=0}$ and $-\Lambda^{l=1}$. More precisely, these functions take real values on ξ_0 and decrease monotonically to zero as $|s|$ increases from zero to infinity. On the contrary, $\Lambda^{l=0}$ and $-\Lambda^{l=1}$ increase monotonically along imaginary axis as $|t|$ increases from zero to $1/2$. The monotonicity of $\Lambda^{l=0}$ along real axis follows from (4.5), equality $P_0(x) \equiv 1$, and inequality

$$\left(\frac{\operatorname{ch}(v(x)s)}{\operatorname{ch}((\pi/2)s)}\right)_s' < -\frac{\pi}{2} \frac{\operatorname{sh}(\pi/2 - v(x))s}{(\operatorname{ch}((\pi/2)s))^2} < 0, \tag{A.9}$$

for $s > 0$ and a similar inequality for $s < 0$. The proof of monotonicity of $\Lambda^{l=1}$ along real axis, and also monotonicity of both functions along imaginary axis is analogous if we note that $P_{l=1}(x) \equiv x$ on $(0, 1)$. Thus the functions Λ^l, $l = 0, 1$, take all values between 0 and $\Lambda^l(i/2) = \Lambda^l(-i/2)$ and every value except $\Lambda^l(0)$ which is taken exactly twice;

(3) we will show now that the values of functions $\Lambda^l(z)$, $l = 0, 1$, on the set $\widetilde{I} \setminus B$ are nonreal. Let us represent this set as a union of four sets, $\widetilde{I}_i$, $i = 1, 2, 3, 4$ as shown in Figure 2.

We consider the case $l = 0$; the case $l = 1$ is similar. Figure 3 shows the disposition of lines of levels for function $K_0(z) = \Re\Lambda^{l=0}(z)$ which pass through the points i and $-i$ between lines β and β^*, $\beta = \{z : K_0(z) = 0, \Im z > 0\}$, $\beta^* = \{z : K_0(z) = 0, \Im z < 0\}$.

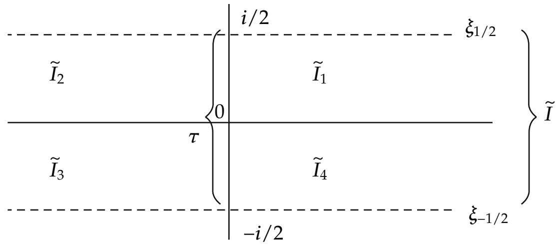

Figure 2

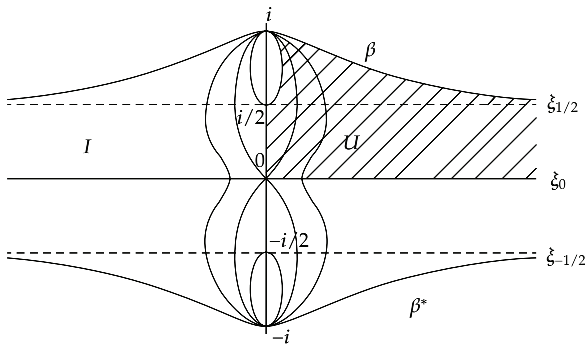

Figure 3

All these lines have common tangents at points i and $-i$, and the line β (resp. β^*) lies above (resp., below) the strip $\widetilde{I}$. The picture represented in Figure 3 is obtained by detailed study of the explicit formula for $\Lambda^{l=0}$:

$$\Lambda^{l=0}(z) = \frac{4\pi^2}{\mu}\frac{\operatorname{sh}(z\arcsin(\mu/2))}{z\operatorname{ch}(z\cdot\pi/2)}, \tag{A.10}$$

together with the proof that the lines β and β^* do not intersect the strip $\widetilde{I}$. This proof is given below.

From Figure 3, we see that the set I_1 lies inside the shaded domain U that is bounded by the real semiaxis $\xi_0^+ = \{z : z = s, s > 0\}$, the segment $(0, i/2)$ on the imaginary axis and the part of line β which lies in the right half-plane. From (A.10), it is easy to see that the function $w = \Lambda^{l=0}(z)$ maps the boundary ∂U of the domain U into the boundary of the right lower quadrant $M = \{w : \Re w > 0, \Im w < 0\}$ of the plain w. Hence, the domain U is mapped inside this quadrant, that is, all values of the function $\Lambda^{l=0}$ in U are nonreal. It means the absence of real values of $\Lambda^{l=0}$ in $\widetilde{I}_1$. For the domains $\widetilde{I}_2$, $\widetilde{I}_3$, and $\widetilde{I}_4$, the proof is similar. Let us now prove that β and β^* do not intersect the line $\xi_{1/2}$. It is sufficient to prove that $\Re\Lambda^{l=0} > 0$ on the line $\xi_{1/2} = \{z : z = s + i/2, s \in \mathbb{R}^1\}$ or, which is the same, that

$$\Re\frac{\operatorname{ch}(zv(x))}{\operatorname{ch}(z\pi/2)}\bigg|_{z=s+i/2} > 0, \tag{A.11}$$

for any $s \in \mathbb{R}^1$ and $x \in (0,1)$. Write

$$\frac{\mathrm{ch}[(s+i/2)v(x)]}{\mathrm{ch}[(s+i/2)\pi/2]} = \frac{\mathrm{ch}(sv(x))\cos(v(x)/2) + i\,\mathrm{sh}(sv(x))\sin(v(x)/2)}{\mathrm{ch}(s\pi/2)\cos(\pi/4) + i\,\mathrm{sh}(s\pi/2)\sin(\pi/4)} = D(s,x). \tag{A.12}$$

Let $s > 0$. Then the values of numerator and denominator of $D(s,x)$ lie in the right upper quadrant of a complex plain, and hence $-\pi/2 < \arg D(s,x) < \pi/2$, that is, $\Re D(s,x) > 0$. Similarly (A.11) can be proved in the case $s < 0$ and for $\Lambda^{l=0}|_{z=s-i/2}$;

(4) let us find the values $\Lambda^l(i/2)$, $l = 0,1$:

(I) the case $l = 0$:

$$\Lambda^{l=0}(i/2) = 2\pi^2 \int_0^1 \frac{\cos(v(x)/2)}{\cos v(x)\cos(\pi/4)}dx. \tag{A.13}$$

After the change $v(x) = \xi$, the integral (A.13) becomes

$$\frac{4\sqrt{2}\pi^2}{\mu}\int_0^{\arcsin\mu/2} \cos\left(\frac{\xi}{2}\right)d\xi = \frac{8\sqrt{2}}{\mu}\pi^2 \sin\left(\frac{1}{2}\arcsin\frac{\mu}{2}\right); \tag{A.14}$$

(II) The case $l = 1$:

$$\Lambda^{l=1}\left(\frac{i}{2}\right) = -2\pi^2 \int_0^1 x\frac{\sin(v(x)/2)dx}{\cos v(x)\sin(\pi/4)}. \tag{A.15}$$

The same change $v(x) = \xi$ reduces to the integral

$$-\frac{8\sqrt{2}\pi^2}{\mu^2}\int_0^{\arcsin\mu/2} \sin\xi\sin\left(\frac{\xi}{2}\right)d\xi = -\frac{32\sqrt{2}}{3}\frac{\pi^2}{\mu^2}\sin^3\left(\frac{1}{2}\arcsin\frac{\mu}{2}\right); \tag{A.16}$$

(5) let us show that the function:

$$q(\mu) = 2\pi^2 \int_0^1 x\frac{\sin(v(x)/2)}{\cos v(x)\sin(\pi/4)}dx \tag{A.17}$$

decreases monotonically as μ changes from 0 to 2. We have

$$\left(\frac{\sin(v(x)/2)}{\cos v(x)}\right)'_\mu \geq 0 \tag{A.18}$$

because the numerator of (A.18) increases, while the denominator decreases with the growth of μ. This implies that

$$q'(\mu) \geq 0, \tag{A.19}$$

that is, $q(\mu)$ increases monotonically. Lemma 5.2 is proved. □

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