Research Article

Bayesian Estimation for Burr Distribution Type III Based on Trimmed Samples

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Trimmed samples are widely employed in several areas of statistical practice, especially when some sample values at either or both extremes might have been contaminated. The problem of estimating the parameters of Burr distribution type III based on a trimmed samples and prior information will be considered. In this paper, we study the estimation of unknown parameters based on doubly censored type II. The problem discussed using maximum likelihood method and Bayesian approach to estimate the shape parameters of Burr type III distribution. The numerical illustration requires solving nonlinear equations, therefore, MathCAD (2001) statistical package used to asses these effects numerically.

1. Introduction

Burr [1] introduced a family of twelve cumulative distribution functions for modeling life-time data. The two important members of the family are Burr types III and XII. The two important distributions, Burr type III and Burr type XII, are interrelated through simple transformation. Burr type III distribution allows for a wider region for the skewness and kurtosis plane, which covers several distributions including the log-logistic, and the Weibull and Burr type XII distributions.

However, outliers may occur in the data set. Trimmed samples are widely employed in several areas of statistical practice, especially when some sample values at either or both extremes might have been contaminated. The problem of estimating the parameters of Burr distribution type III based on a trimmed sample and prior information is considered.

Many authors discuss different methods of estimation for Burr type XII distribution. Al-Hussaini and Jaheen [2] and Al-Hussaini et al. [3] used different techniques for obtaining

Bayes estimates of the shape parameters c and k , reliability and failure rate functions based on type II censored samples. Al-Hussaini et al. [4] obtained the maximum likelihood, uniformly minimum variance unbiased, Bayes and empirical Bayes estimators for the parameter k and reliability function when c is known. Wingo [5] developing the theory for the ML point estimation of the parameters of the Burr distribution when Type II singly censored sample is at hand. Ali-Mousa and Jaheen [6] obtained interval estimates of the parameter k and reliability function when c is known, using a Bayesian approach based on Type II censored data. Ali-Mousa [7] obtained empirical Bayes estimation of the parameter k and the reliability function based on accelerated Type II censored data. Gupta et al. [8] discusses analysis of failure time data by Burr distribution. Wang et al. [9] obtained the maximum likelihood estimation of the Burr XII distributions parameter with censored and uncensored data. Hossain and Nath [10] deal with unweighted least squares estimation of the parameters. They compared the results with the maximum likelihood and maximum product of spacing methods. Ali-Mousa and Jaheen [11] obtained the maximum likelihood and Bayes estimates for two parameters c , k and the reliability function $r(t)$ of the Burr Type XII distribution based on progressive type II censored samples.

The objective of this paper is to obtain the estimators of the unknown shape parameters of Burr type III based on doubly censored type II. The problem discussed using maximum likelihood method and Bayesian approach to estimate the shape parameters of Burr type III distribution requires solving nonlinear equations, therefore, numerical study is carried out to asses these effects using MathCAD (2001) statistical package.

2. Burr Type III Distribution

The Burr family of distributions has, in recent years, assumed an important position in the field of life testing because of its uses to fit business failure data, also it includes the well-known exponential and Weibull distributions as special cases. Burr [1] has suggested this family of distributions by solving the following differential equation:

$$\frac{dy}{dx} = y(1-y)g(x,y), \quad y = F(x), \quad (2.1)$$

where $g(x,y) > 0$, $0 \leq y \leq 1$, x in the range over which the solution is being satisfied. By using different forms of $g(x,y)$ and Pearson systems, Burr obtained twelve distribution functions which listed in Burr [1].

Burr family of distributions includes twelve types of cumulative distribution functions, which yield a variety of density shapes. Many standard theoretical distributions, including the Weibull, exponential logistic, generalized logistic, Gompertz, normal, extreme value, and uniform distributions are special cases or limiting cases of Burr family of distributions. The simple closed form of these distributions has been applied to studies in simulation. Burr [1] developed the family of Burr distributions as an outgrowth into methods for fitting cumulative frequency functions rather than probability density functions to frequency data. Some of the forms of Burr distributions are related by simple transformation. For example, the Burr type III distribution can be obtained from Burr type II distribution by replacing X with $\ln(X)$, see Johnson et al. [12]. Similarly, Burr type XII distribution can be derived from Burr type III distribution by replacing X with $(1/X)$, see Burr and Cislak [13]. The Burr type I family is more commonly known as the uniform distribution.

The Burr III, IV, V, IX, and XII families have a variety of density shapes. Types III and XII are the simplest functionally and therefore, the two distributions are the most desirable for statistical modeling. The Burr type III distribution is much richer than Burr type XII distribution.

The twelve distributions of Burr are listed in Burr [1] and Johnson et al. [12].

Properties of Type III Burr (c, k)

The distribution of Burr (c, k) type III is

$$F(x) = (1 + x^{-c})^{-k}, \quad x > 0, \quad (2.2)$$

where the parameters $c > 0$ and $k > 0$ are the shape parameters of the distribution. Its density function is

$$f(x) = ckx^{-(c+1)}(1 + x^{-c})^{-(k+1)}. \quad (2.3)$$

The r th moment

$$E(x^r) = \mu'_r = ck \int_0^\infty \frac{x^r x^{-(c+1)}}{(1 + x^{-c})^{k+1}} dx = k\beta\left(1 - \frac{r}{c}, k + \frac{r}{c}\right), \quad r < c, \quad (2.4)$$

where $\beta(a, b)$ is the standard beta function.

The expectation of the distribution is obtained as follows.

If $r = 1$ in (2.4) we have:

$$\mu'_1 = k\beta\left(1 - \frac{1}{c}, k + \frac{1}{c}\right) = \frac{\lambda_1}{\Gamma(k)}, \quad c > 1, \quad (2.5)$$

where $\lambda_i = \Gamma(1 - i/c)\Gamma(k + i/c)$, $i = 1, 2, \dots, r$.

The variance

$$\text{var}(x) = \frac{\Gamma(k)\lambda_2 - \lambda_1^2}{\Gamma^2(k)}, \quad c > 2. \quad (2.6)$$

The mode

$$x = \left\{ \frac{c(k+1)}{c+1} - 1 \right\}^{1/c}, \quad ck \geq 1, \quad c > 0, \quad k > 0. \quad (2.7)$$

The median

$$x = \frac{1}{(2^{1/k} - 1)^{1/c}}. \quad (2.8)$$

3. Bayesian and Non-Bayesian Estimation Methods

In this section we will obtain the estimation of Burr parameters based on trimmed samples using Bayesian and non Bayesian methods of estimation.

3.1. Non-Bayesian Estimation for Burr Parameters Based on Trimmed Sample

In this section, the estimation problem of Burr type III with two parameters (c, k) under type II double censored data (trimmed samples) are obtained. Some data may not be observed, a known number of observation in an ordered sample are missing at both ends in failure censored experiments, the observations are the smallest r and the largest r_s are random then the data collected will be $x_{(r+1)} \leq x_{(r+2)} \leq \dots \leq x_{(n-s)}$ and the likelihood function in double censored type II takes the following form:

$$L(x, \theta) = \frac{n!}{(r-1)!(n-s)!} \prod_{i=r}^s f(x_{(i)}, \theta) [F(x_{(r)}, \theta)]^{r-1} [1 - F(x_{(s)}, \theta)]^{n-s},$$

$$L(x, c, k) = \frac{n!}{(r-1)!(n-s)!} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \quad (3.1)$$

$$* \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right].$$

It is usually easier to use maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is

$$\ln L = \ln w + (r-1) \ln \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right] - (n-s)k \ln \left(1 + x_{(s)}^{-c} \right) \quad (3.2)$$

$$- \sum_{i=r}^s \left[(c+1) \ln x_{(i)} + (k+1) \ln \left(1 + x_{(i)}^{-c} \right) \right] + (s-r+1) \ln(ck),$$

where $w = n! / ((r-1)!(n-s)!)$. The maximum likelihood estimators $\hat{c}, \hat{k}$ of c, k are the solutions of the system of equations obtained by letting the first partial derivatives of the total log likelihood with respect to c, k be zero. The systems of equations are as follows:

$$\frac{\partial \ln L}{\partial k} = \frac{(r-1) \ln \left(1 + x_{(r)}^{-c} \right) \left[1 + x_{(r)}^{-c} \right]^{-k}}{1 - \left(1 + x_{(r)}^{-c} \right)^{-k}} - (n-s) \ln \left(1 + x_{(s)}^{-c} \right) \quad (3.3)$$

$$+ \frac{(s-r+1)}{k} - \sum_{i=r}^s \ln \left(1 + x_{(i)}^{-c} \right),$$

$$\begin{aligned} \frac{\partial \ln L}{\partial c} = & - \frac{(r-1)k \left(1 + x_{(r)}^{-c}\right)^{-k-1} x_{(r)}^{-c} \ln(x_{(r)})}{1 - \left(1 + x_{(r)}^{-c}\right)^{-k}} + \frac{(n-s)k x_{(s)}^{-c} \ln(x_{(s)})}{1 + x_{(s)}^{-c}} \\ & + \frac{(s-r+1)}{c} - \sum_{i=r}^s \ln(x_{(i)}) - (k+1) \sum_{i=r}^s \frac{x_{(i)}^{-c} \ln(x_{(i)})}{1 + x_{(i)}^{-c}}. \end{aligned} \quad (3.4)$$

From (3.3) the maximum likelihood estimator of k is expressed by

$$(n-s) \ln\left(1 + x_{(s)}^{-\hat{c}}\right) + \sum_{i=1}^s \ln\left(1 + x_{(i)}^{-\hat{c}}\right) = \frac{(s-r+1)}{\hat{k}} + \frac{(r-1) \ln\left(1 + x_{(r)}^{-\hat{c}}\right)}{\left[1 + x_{(r)}^{-\hat{c}}\right]^{\hat{k}} - 1}. \quad (3.5)$$

From (3.4) the maximum likelihood estimator of c is expressed by

$$\begin{aligned} & \frac{(n-s)k x_{(s)}^{-\hat{c}} \ln(x_{(s)})}{1 + x_{(s)}^{-\hat{c}}} + (k+1) \sum_{i=r}^s \frac{x_{(i)}^{-\hat{c}} \ln(x_{(i)})}{1 + x_{(i)}^{-\hat{c}}} \\ & = \frac{(r-1)k \left(1 + x_{(r)}^{-\hat{c}}\right)^{-k-1} x_{(r)}^{-\hat{c}} \ln(x_{(r)})}{1 - \left(1 + x_{(r)}^{-\hat{c}}\right)^{-k}} + \frac{(s-r+1)}{\hat{c}} + \sum_{i=r}^s \ln(x_{(i)}). \end{aligned} \quad (3.6)$$

Since the closed form solution to nonlinear equations (3.5) and (3.6) is very hard to obtain, Newton-Raphson method is applied for solving the nonlinear equations simultaneously to obtain $\hat{c}$, $\hat{k}$.

3.1.1. Asymptotic Variances Covariance Matrix

The asymptotic variances covariance matrix of the parameters (c and k) is obtained by inverting the Fisher information matrix

$$I_{ij} = E \left[-\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \right], \quad i, j = 1, 2, \quad (3.7)$$

where $\theta_1, \theta_2 = c$ or k . Hence the approximate variance-covariance matrix is given by

$$\begin{bmatrix} \left. \frac{\partial^2 \ln L}{\partial c^2} \right|_{\hat{c}, \hat{k}} & \left. \frac{\partial^2 \ln L}{\partial c \partial k} \right|_{\hat{c}, \hat{k}} \\ \left. \frac{\partial^2 \ln L}{\partial k \partial c} \right|_{\hat{c}, \hat{k}} & \left. \frac{\partial^2 \ln L}{\partial k^2} \right|_{\hat{c}, \hat{k}} \end{bmatrix}^{-1} = \begin{bmatrix} \text{Var}(\hat{c}) & \text{Cov}(\hat{c}, \hat{k}) \\ \text{Cov}(\hat{k}, \hat{c}) & \text{Var}(\hat{k}) \end{bmatrix}. \quad (3.8)$$

The elements of matrix I are the negative of second derivatives of the natural logarithm of likelihood function defined in (3.2).

The elements of matrix I are given as follows:

$$\frac{\partial^2 \ln L}{\partial k^2} = \frac{-(s-r+1)}{k^2} - \frac{(r-1) \left[\ln(1+x_{(r)}^{-c}) \right]^2 (1+x_{(r)}^{-c})^{-k}}{\left[1 - (1+x_{(r)}^{-c})^{-k} \right]^2}, \quad (3.9)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial c^2} &= \frac{(r-1)k \ln(x_{(r)}) \left[\mathcal{A} x_{(r)}^c \ln(x_{(r)}) - x_{(r)}^{-c} (k+1) (1+x_{(r)}^{-c})^k x_{(r)}^{-c} \ln(x_{(r)}) - x_{(r)}^{-c} \ln(x_{(r)}) \right]}{\left[(1+x_{(r)}^c)^{k+1} - (1+x_{(r)}^c) \right]^2} \\ &\quad - (n-s)k \ln x_{(s)} \frac{x_{(s)}^{-c} \ln x_{(s)}}{(1+x_{(s)}^c)^2} - \frac{(s-r+1)}{c^2} + (k+1) \sum_{i=r}^s \frac{\ln(x_{(i)})^2}{(1+x_{(i)}^{-c})^2}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial c \partial k} &= - \sum_{i=r}^s \frac{x_{(i)}^{-c} \ln(x_{(i)})}{1+x_{(i)}^{-c}} + \frac{(n-s)x_{(s)}^{-c} \ln(x_{(s)})}{1+x_{(s)}^{-c}} - \frac{(r-1)x_{(r)}^{-c} \ln x_{(r)}}{1+x_{(r)}^{-c}} \\ &\quad + \frac{(r-1) \left(1 - (1+x_{(r)}^{-c})^{-k} \right) x_{(r)}^{-c} \ln(x_{(r)})}{(1+x_{(r)}^{-c}) \left(\left(1 - (1+x_{(r)}^{-c})^{-k} \right) \right)^2} - \left[\ln(1+x_{(r)}^{-c}) \right] k (1+x_{(r)}^{-c})^{-k-1} \\ &\quad + x_{(r)}^{-c} \ln x_{(r)}, \end{aligned} \quad (3.11)$$

where $\mathcal{A}$ denotes $[(1+x_{(r)}^{-c})^{k+1} - (1+x_{(r)}^{-c})]$. The maximum likelihood estimators $\hat{c}, \hat{k}$ have asymptotic variance-covariance matrix defined by inverting the Fisher information matrix.

3.2. Numerical Illustration

In estimation problem, it is required to study the properties of the derived expressions for the estimators theoretically or analytically. Sometimes it seems very difficult to study the properties of the estimators theoretically because of the complicated formula of the estimators. Consequently, a simulation study will be set up, treating separately the sampling distribution of the estimators. Simulation studies have been performed using MathCAD (2001) for illustrating the theoretical results of the estimation problem. The simulation procedures will be described below.

Step 1. Generate a random sample of size 10, 20, 40, 60, 80, and 100 from Burr type III distribution. The generation of Burr type III distribution is very simple if U has a uniform (0,1) random number, then $Y = [U^{(-1/k)} - 1]^{(-1/c)}$ follows a Burr type III distribution. In double censored type II, the true parameters selected values are $(c = 1.2, k = 1.5)$, $(c = 1.2, k = 2)$, $(c = 2.5, k = 1.5)$, and $(c = 2.5, k = 2)$.

Step 2. Choose censored failure ($r = 0.6, s = 6$).

Step 3. For each sample and for the four sets of parameters distribution were estimated under doubly censored type II.

Step 4. Newton-Raphson method was used for solving the nonlinear equations for c and k defined in (3.4) and (3.6).

Step 5. Equations (3.9) and (3.10) were used to obtain the variance covariance matrix of $(\hat{c}, \hat{k})$.

Results are tabulated in Table 1.

From Table 1, we note that the standard deviation decreases when n is increasing. Also, we note that, when c and k increase the standard deviation of the estimators decrease. Similarly, the maximum likelihood estimator of the parameters has the same behaviors when the sample size becomes large and the properties of two parameters c and k at (1.2 and 1.5), respectively is better than the other values.

3.3. Bayesian Analysis for Burr Distribution Type III Based on Trimmed Samples

The Bayesian approach allows both sample and prior information to be incorporated into analysis, which will improve the quality of the inferences. In this section, Bayesian estimators and posterior variance of the shapes parameters are obtained in the case of double censored type II. The prior distribution could be “none informative” such as a flat, uniform distribution, assuming equal probability for the parameter value within a realistic range, and when the prior could be more informative such as a normal distribution or other possible distribution that represents an initial assessment of what is known about the parameter before the collection of data and the analysis. Moreover, a numerical examples are given for illustration study.

3.3.1. Bayesian Estimation for Burr Distribution Type III Parameters Based on Trimmed Sample in Case of Noninformative Prior

The likelihood function takes the form in (3.1). Assumed that the parameters c, k have independent prior distribution and let the noninformative prior (NIP) for c and k are, respectively, given by:

$$\begin{aligned} u_1(c) &= \alpha c^{-1} \quad c > 0, \\ u_2(k) &= \alpha k^{-1} \quad k > 0. \end{aligned} \tag{3.12}$$

Consequently, the joint (NIP) will be defined as follows:

$$u(c, k) = (\alpha c k)^{-1} \quad c, k > 0. \tag{3.13}$$

Table 1: The maximum likelihood estimator, the standard deviation, and covariance of the Burr distribution with two parameter under doubly censored sample when (a) ($c = 1.2, k = 1.5$), (b) ($c = 1.2, k = 2$), (c) ($c = 2.5, k = 1.5$), (d) ($c = 2.5, k = 2$).

(a)							
n	s	r	$\hat{c}$		$\hat{k}$		$\text{Co var}(\hat{c}, \hat{k})$
			MLE	Standard deviation	MLE	Standard deviation	
10	5	2	4.619	3.689	7.841	4.341	58.814
	7		4.534	3.524	4.672	4.172	50.042
	9		4.338	3.308	4.722	3.202	22.784
20	10	2	6.874	3.874	6.654	4.154	66.589
	15		6.483	2.483	6.439	3.939	64.752
	18		6.147	3.147	6.307	3.807	41.058
40	20	3	6.498	3.498	6.802	4.302	74.941
	30		5.384	3.384	6.471	3.971	62.603
	38		5.287	2.287	5.629	3.129	39.913
60	30	3	5.692	2.692	6.794	4.294	62.054
	45		5.542	2.542	6.678	4.178	47.829
	58		5.201	2.201	5.489	2.989	16.475
80	40	5	5.448	2.448	6.824	4.324	78.475
	65		5.707	2.707	6.671	4.171	66.942
	78		5.334	2.334	5.391	2.891	11.313
100	50	5	6.689	3.689	6.841	4.341	58.814
	75		6.524	3.524	6.672	4.172	50.042
	98		6.308	3.308	5.702	3.202	30.784
(b)							
n	s	r	$\hat{c}$		$\hat{k}$		$\text{Co var}(\hat{c}, \hat{k})$
			MLE	Standard deviation	MLE	Standard deviation	
10	5	2	6.689	3.689	6.841	4.341	58.814
	7		6.524	3.524	6.672	4.172	50.042
	9		6.308	3.308	5.702	3.202	30.784
20	10	2	6.874	3.874	6.654	4.154	66.589
	15		6.483	3.483	6.439	3.939	64.752
	18		6.147	3.147	6.307	3.807	41.058
40	20	3	6.498	3.498	6.802	4.302	74.941
	30		6.384	3.384	6.471	3.971	62.603
	38		5.287	2.287	5.629	3.129	39.913
60	30	3	5.692	2.692	6.794	4.294	62.054
	45		5.542	2.542	6.678	4.178	47.829
	58		5.201	2.201	5.489	2.989	16.475
80	40	5	5.448	2.448	6.824	4.324	78.475
	65		5.707	2.707	6.671	4.171	66.942
	78		5.334	2.334	5.391	2.891	11.313
100	50	5	6.689	3.689	6.841	4.341	58.814
	75		6.524	3.524	6.672	4.172	50.042
	98		6.308	3.308	5.702	3.202	30.784
(c)							
n	s	r	$\hat{c}$		$\hat{k}$		$\text{Co var}(\hat{c}, \hat{k})$
			MLE	Standard deviation	MLE	Standard deviation	
10	5	2	5.952	2.952	5.159	3.159	69.043
	7		5.712	2.712	4.914	2.914	55.876
	9		5.819	2.819	4.249	2.249	38.874

(c) Continued.

n	s	r	$\hat{c}$		$\hat{k}$		Co var($\hat{c}, \hat{k}$)
			MLE	Standard deviation	MLE	Standard deviation	
20	10	2	5.984	2.984	5.147	3.147	68.497
	15		5.147	2.147	4.271	2.271	24.984
	18		5.078	2.078	4.014	2.014	14.872
40	20	3	5.618	2.618	5.081	3.081	69.187
	30		5.491	2.491	4.941	2.941	57.078
	38		5.107	2.107	4.028	2.028	23.174
60	30	3	5.347	2.347	4.627	2.627	70.842
	45		5.148	2.148	4.489	2.489	58.247
	58		5.004	2.004	4.309	2.309	41.872
80	40	5	5.334	2.334	4.658	2.658	74.194
	65		5.238	2.238	4.748	2.748	60.827
	78		4.997	1.997	4.102	2.102	47.145
100	50	5	0.447	0.553	1.39	0.19	1.987
	75		0.474	0.526	1.317	0.117	1.854
	98		1.218	0.218	1.234	0.034	1.823

(d)

n	s	r	$\hat{c}$		$\hat{k}$		Co var($\hat{c}, \hat{k}$)
			MLE	Standard deviation	MLE	Standard deviation	
10	5	2	1.395	0.395	2.717	0.717	42.175
	7		1.501	0.501	2.585	0.585	35.214
	9		1.014	0.014	2.461	0.461	18.258
20	10	2	1.284	0.284	2.648	0.648	40.954
	15		1.481	0.481	2.441	0.441	33.254
	18		1.008	0.008	2.314	0.314	16.418
40	20	3	1.658	0.658	2.995	0.995	68.284
	30		1.018	0.018	2.147	0.147	50.258
	38		1.001	0.001	1.654	0.346	52.547
60	30	3	1.228	0.228	2.954	0.954	68.952
	45		1.109	0.109	2.756	0.756	53.218
	58		1.019	0.019	2.341	0.341	40.021
80	40	5	1.721	0.721	2.481	0.481	77.248
	65		1.856	0.856	2.441	0.441	80.514
	78		1.007	0.007	1.761	0.239	54.951
100	50	5	1.395	0.395	2.717	0.717	42.175
	75		1.501	0.501	2.585	0.585	35.214
	98		1.014	0.014	2.461	0.461	18.258

The joint posterior density functions of c, k under double censored sample type II will be

$$g(c, k | x) = \frac{u(c, k) L(x | c, k)}{\lambda} c > 0, \quad k > 0,$$

$$g(c, k | x) = \frac{(ck)^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ck x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right]}{\lambda}, \quad (3.14)$$

where λ is the normalized constant defined as follows:

$$\iint_0^\infty (ck)^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk. \quad (3.15)$$

Now, the marginal posterior of one parameter is obtained by integrating the joint posterior distribution with respect to the other parameter, hence the marginal posterior probability density function of c will be

$$g_1(c \mid k, x)$$

$$= \frac{c^{-1} \int_0^\infty k^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dk}{\lambda}. \quad (3.16)$$

Similarly integrating the joint posterior (3.14) with respect to k . Then, the marginal posterior of k is

$$g_2(k \mid c, x)$$

$$= \frac{k^{-1} \int_0^\infty c^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc}{\lambda}. \quad (3.17)$$

It is well known that under a squared error loss function, the Bayes estimator of the parameter will be its posterior expectation. To obtain the posterior mean and posterior variance, a numerical integration is required. Then, the posterior mean and posterior variance of the shape parameters (c, k) are expressed as follows:

$$E(c \mid k, x) = \tilde{c},$$

$$\tilde{c} = \iint_0^\infty k^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk,$$

$$\text{var}(\tilde{c} \mid k, x) = E(\tilde{c} - c)^2,$$

$$\begin{aligned}
\text{var}(\tilde{c} | k, x) &= \int \int_0^\infty (\tilde{c} - c)^2 \times (ck)^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\
&\quad \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk, \\
E(k | c, x) &= \tilde{k}, \\
\tilde{k} &= \int \int_0^\infty c^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk, \\
\text{var}(\tilde{k} | c, x) &= E(\tilde{k} - k)^2, \\
\text{var}(\tilde{k} | c, x) &= \int \int_0^\infty (\tilde{k} - k)^2 \times (ck)^{-1} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\
&\quad \times \prod_{i=r}^s \left[ckx_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk.
\end{aligned} \tag{3.18}$$

Equations (3.18) are very difficult to be solved exactly so that an iterative procedure is needed to solve these equations numerically. Using the statistical package, MathCAD (2001), the posterior mean and posterior variance of the shapes parameter (c, k) will be obtained.

The posterior mean and the posterior variance of c, k are obtained numerically as described below.

Repeat Steps 1 and 2 in Section 3.2.

Step 3. Solve the nonlinear equations to obtain posterior variance of the shape parameter (c, k) using (3.18).

Step 4. The posterior mean and the posterior variance of the estimators for the shape parameters (c, k) for all sample size and for sets of parameter values were obtained.

Numerical results are summarized in Table 2.

3.3.2. Bayes Estimator under LINEX Loss Function

Under the assumption that minimal loss function occurs $\tilde{\theta} = \theta$, the LINEX loss function for $\theta = \theta(c, k)$ can be expressed as defined by Zellner [14], the Bayes estimator $\tilde{\theta}$ of θ under LINEX loss function is

$$\begin{aligned}
\tilde{\theta} &= -\frac{1}{a} \log(E_\theta(\exp(-a\theta))), \\
\tilde{\theta} - \frac{1}{a} \ln \int e^{-a\theta} f(\theta | x) d\theta.
\end{aligned} \tag{3.19}$$

Table 2: The posterior mean and posterior variance of the Burr distribution with two parameter distribution under doubly censored sample when ($c = 1.2, k = 1.5$) in case of noninformative prior.

n	r	S	c		k	
			Posterior mean	Posterior variance	Posterior mean	Posterior variance
10	2	5	1.017	0.685	0.671	0.765
		7	0.701	0.514	0.569	0.648
		9	0.608	0.338	0.408	0.491
20	2	10	1.010	0.742	0.948	0.785
		15	0.871	0.537	0.653	0.676
		18	0.691	0.338	0.527	0.527
40	3	20	0.437	0.674	0.741	0.576
		30	0.288	0.582	0.687	0.477
		38	0.119	0.449	0.354	0.345
60	3	30	0.609	0.762	0.671	0.758
		45	0.547	0.684	0.507	0.862
		58	0.489	0.449	0.407	0.647
80	5	40	0.827	0.761	0.718	0.901
		65	0.667	0.895	0.651	0.763
		78	0.524	0.612	0.581	0.522
100	5	50	0.718	0.761	0.718	0.901
		75	0.651	0.895	0.651	0.763
		98	0.581	0.612	0.581	0.522

Now, in (3.19) put $\theta = c$, then the Bayes estimate $\tilde{c}$ of parameter c relative to the LINEX loss function is

$$\tilde{c} = -\frac{1}{a} \ln \int e^{-ac} g_1(c | k, x) dc. \quad (3.20)$$

Set $\theta = k$ in (3.19), then the Bayes estimate $\tilde{k}$ of parameter k relative to the LINEX loss function is

$$\tilde{k} = -\frac{1}{a} \ln \int e^{-ak} g_2(k | c, x) dk. \quad (3.21)$$

The equations (3.20) to (3.21) are very difficult to obtain their solutions exactly, an iterative procedure is needed to solve these equations numerically using MathCAD (2001) statistical package to obtain posterior variance of shapes parameters (c, k). The numerical procedures will describe as follow.

Repeat Steps 1 and 2 in Section 3.2.

Step 3. Solve the nonlinear equations to obtain the posterior variance of the shape parameter (c, k) in (3.20) and (3.21).

Step 4. The posterior mean and the posterior variance of the estimators for the shape parameter (c, k) for all sample size and for sets of parameters were obtained.

Numerical results are summarized in Table 3.

Table 3: The expectation mean and variance of the Burr distribution with two parameter distribution under doubly censored sample when $(c = 1.2, k = 1.5)$ under LINEX loss function.

n	r	S	c		k	
			Expectation mean	Variance	Expectation mean	Variance
10	2	5	1.698	0.396	2.596	1.631
		7	1.353	0.205	1.700	0.810
		9	1.189	0.292	1.663	0.726
20	2	10	1.468	0.267	2.552	1.819
		15	1.238	0.166	2.032	1.619
		18	1.237	0.178	1.773	1.152
40	3	20	1.382	0.294	2.343	1.198
		30	1.092	0.200	1.411	0.848
		38	1.022	0.196	0.936	0.847
60	3	30	1.075	0.197	1.543	1.237
		45	1.013	0.174	1.018	1.062
		58	0.988	0.170	0.935	1.138
80	5	40	1.275	0.162	1.007	1.188
		65	0.986	0.182	0.340	1.193
		78	0.906	0.151	1.762	2.269
100	5	50	0.992	0.761	0.718	0.901
		75	0.745	0.895	0.651	0.763
		98	0.604	0.612	0.581	0.522

3.3.3. Bayes Estimator under General Entropy (GE) Loss Function

The Bayes estimator $\hat{\theta}$ under GE loss function is

$$\hat{\theta} = (E_{\theta}(\theta^{-v}))^{-1/v},$$

$$\hat{\theta} = \left[\int \theta^{-v} f(\theta | x) d\theta \right]^{-1/v} \quad v > 0. \quad (3.22)$$

Now, in (3.22) put $\hat{\theta} = c$, then the Bayes estimate $\hat{c}$ of parameter c relative to the GE loss function is

$$\hat{c} = \left[\int c^{-v} g_1(c | k, x) dc \right]^{-1/v}. \quad (3.23)$$

In (3.22), set $\hat{\theta} = k$, then the Bayes estimate $\hat{k}$ of parameter k relative to the GE loss function is

$$\hat{k} = \left[\int k^{-v} g_2(k | c, x) dk \right]^{-1/v}. \quad (3.24)$$

Table 4: The expectation mean and variance of the Burr distribution with two parameter distribution under doubly censored sample when ($c = 1.2, k = 1.5$) general entropy loss function.

n	r	S	c		k	
			Expectation mean	Variance	Expectation mean	Variance
10	2	5	1.641	0.685	3.825	0.765
		7	1.375	0.514	2.614	0.648
		9	1.255	0.338	1.661	0.491
20	2	10	1.474	0.742	3.423	0.785
		15	1.299	0.537	2.490	0.676
		18	1.147	0.338	1.550	0.527
40	3	20	1.294	0.674	3.076	0.576
		30	1.087	0.582	1.378	0.477
		38	1.016	0.449	1.263	0.345
60	3	30	1.032	0.762	1.672	0.758
		45	0.977	0.684	1.173	0.862
		58	0.924	0.449	1.038	0.647
80	5	40	1.162	0.761	0.860	0.901
		65	1.034	0.895	0.396	0.763
		78	0.888	0.612	0.185	0.522
100	5	50	0.992	0.761	0.718	0.901
		75	0.745	0.895	0.651	0.763
		98	0.604	0.612	0.581	0.522

The equations (3.23) to (3.24) cannot have exact solutions, an iterative procedure is needed to solve these equations numerically using MathCAD (2001) statistical package to obtain posterior variance of shapes parameter. The numerical procedures will be described as follow.

Repeat Steps 1 and 2 in Section 3.2.

Step 3. Solving the nonlinear Bayesian for posterior variance of the shape parameter (c, k) in (3.23) and (3.24).

Step 4. The posterior mean and the posterior variance of the estimators for the shape parameter (c, k) for all sample size and for sets of parameters were obtained.

Numerical results are summarized in Table 4.

3.4. Bayesian Estimation for Burr Distribution Type III Parameters Based on Trimmed Sample in Case of Informative Prior (IP)

Assume that the parameters c, k have independent prior distribution and let the informative prior (IP) for the parameter k is given by the gamma density as follow:

$$\pi_1(k | c) = \frac{c^{\alpha+1}}{\Gamma(\alpha+1)} k^{\alpha} e^{-kc} \quad k > 0, \quad c > 0, \quad \alpha + 1 > 0, \quad (3.25)$$

and the prior density function of c is the exponential density as follow:

$$\pi_2(c) = \frac{1}{\beta} e^{-c/\beta} \quad c > 0, \beta > 0. \quad (3.26)$$

Then the joint prior density of c and k , given by

$$\begin{aligned} \pi(c, k) &= \pi_1(k | c) \pi_2(c), \\ \pi(c, k) &= \frac{c^{\alpha+1}}{\beta \Gamma \alpha + 1} k^\alpha e^{-(k+1/\beta)c} \quad c > 0, k > 0. \end{aligned} \quad (3.27)$$

The joint posterior density function of c, k using double censored sample type II will be

$$\begin{aligned} f(c, k | x) &= \frac{\pi(c, k) L(x | c, k)}{E} \quad c > 0, k > 0, \\ &= \frac{c^{\alpha+1} k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right]}{E \beta \Gamma \alpha + 1}, \end{aligned} \quad (3.28)$$

where E is the normalized constant equal to

$$\begin{aligned} E &= \int_0^\infty \int_0^\infty \frac{c^{\alpha+1}}{\beta \Gamma \alpha + 1} k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ &\quad \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk. \end{aligned} \quad (3.29)$$

Now, the marginal posterior of one parameter is obtained by integrating the joint posterior distribution with respect to the other parameter, hence the marginal posterior probability density function of c is

$$\begin{aligned} f_1(c | k, x) &= \frac{c^{\alpha+1}}{\beta \Gamma \alpha + 1} \int_0^\infty k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ &\quad \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] \frac{dk}{\lambda}. \end{aligned} \quad (3.30)$$

Similarly integrating the joint posterior (3.28) with respect to k the marginal posterior of k is

$$f_2(k | c, x) = \frac{k^\alpha}{\beta\Gamma\alpha + 1} \int_0^\infty c^{\alpha+1} e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] \frac{dc}{\lambda}. \quad (3.31)$$

Hence, under a squared error loss function, the Bayes estimator of the parameter will be its posterior expectation. To obtain the posterior mean and posterior variance a numerical integration is required. Then, the posterior mean and posterior variance of the shape parameters (c, k) are expressed as follows:

$$E(c | k, x) = \check{c},$$

$$\check{c} = \frac{1}{\beta\Gamma\alpha + 1} \iint_0^\infty c^{\alpha+2} k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk,$$

$$\text{var}(\check{c} | k, x) = E(\check{c} - c)^2,$$

$$\text{var}(\check{c} | k, x)$$

$$= \frac{1}{\beta\Gamma\alpha + 1} \iint_0^\infty (\check{c} - c)^2 \iint_0^\infty c^{\alpha+1} k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk,$$

$$E(k | c, x) = \check{k},$$

$$\check{k} = \frac{1}{\beta\Gamma\alpha + 1} \iint_0^\infty c^{\alpha+1} k^{\alpha+1} e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\ \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk,$$

$$\text{var}(\check{k} | c, x) = E(\check{k} - k)^2,$$

Table 5: The posterior mean and posterior variance of the Burr distribution with two parameter under doubly censored sample when ($c = 1.2, k = 1.5$) in case of informative prior.

n	r	s	c		k	
			Posterior mean	Posterior variance	Posterior mean	Posterior variance
10	2	5	0.495	0.204	1.017	0.719
		7	0.384	0.214	0.701	0.645
		9	0.258	0.217	0.608	0.201
20	2	10	0.234	0.391	1.010	0.604
		15	0.128	8.215	0.871	0.556
		18	0.087	0.319	0.691	0.493
40	3	20	0.482	0.339	0.437	0.911
		30	0.352	0.337	0.288	0.425
		38	0.241	0.190	0.119	0.307
60	3	30	0.852	0.199	0.609	0.827
		45	0.729	0.160	0.547	0.647
		58	0.635	0.139	0.489	0.558
80	5	40	0.957	1.171	0.827	0.728
		65	0.721	0.206	0.667	0.661
		78	0.408	0.078	0.524	0.337
100	5	50	0.992	0.761	0.718	0.901
		75	0.745	0.895	0.651	0.763
		98	0.604	0.612	0.581	0.522

$$\begin{aligned}
& \text{var}(\check{K} | c, x) \\
&= \frac{1}{\beta\Gamma\alpha + 1} \iint_0^\infty (\check{k} - k)^2 \iint_0^\infty c^{\alpha+1} k^\alpha e^{-(k+1/\beta)c} \left[1 - \left(1 + x_{(r)}^{-c} \right)^{-k} \right]^{r-1} \left[\left(1 + x_{(s)}^{-c} \right)^{-k} \right]^{n-s} \\
&\quad \times \prod_{i=r}^s \left[c k x_{(i)}^{-(c+1)} \left(1 + x_{(i)}^{-c} \right)^{-(k+1)} \right] dc dk.
\end{aligned} \tag{3.32}$$

Nonlinear equations (3.32) are very difficult to be solved exactly an iterative procedure is needed to solve these equations numerically. By using the statistical package, MathCAD (2001), the posterior mean and posterior variance of the shape parameters (c, k) will be obtained.

The posterior mean and the posterior variance of c, k are obtained numerically in the following procedures as follow.

Repeat Steps 1 and 2 in Section 3.2.

Step 3. Solve the nonlinear Bayesian for posterior variance of the shape parameter (c, k) in (3.32).

Step 4. The posterior mean and the posterior variance of the estimators for the shape parameter (c, k) for all sample size and for sets of parameters were obtained.

Numerical results are summarized in Table 5.

It is noted that the posterior mean decreases when n is increasing. Similarly the posterior variance of the parameters has the same behaviors when the sample size becomes large.

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