Research Article

# Maps Completely Preserving Involutions and Maps Completely Preserving Drazin Inverse 

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#### Abstract

Let $X$ and $Y$ be infinite dimensional Banach spaces over the real or complex field $\mathbb{F}$, and let $\mathcal{A}$ and $B$ be standard operator algebras on $X$ and $Y$, respectively. In this paper, the structures of surjective maps from $\mathcal{A}$ onto $B$ that completely preserve involutions in both directions and that completely preserve Drazin inverse in both direction are determined, respectively. From the structures of these maps, it is shown that involutions and Drazin inverse are invariants of isomorphism in complete preserver problems.


## 1. Introduction

In the last decades, the study of preserver problems is an active topic in operator algebra or operator space theory (see [1]). In [2], the form of involutivity-preserving maps was given by using the known results of idempotence-preserving maps, and in [3], the authors gave the characterization of additive maps preserving Drazin inverse. These results showed that involutions and Drazin inverse are invariants of isomorphism in preserver problems. Since completely positive linear maps and completely bounded linear maps are very important in operator algebra or operator space theory [4], and the concept of completely rank nonincreasing linear maps was introduced by Hadwin and Larson in [5], many mathematicians began to focus on complete preserver problems, that is, characterizations of maps on operator spaces (subsets) that preserve some property (or invariant) completely [6]. Cui and Hou discussed the completely trace-rank-preserving linear maps and the completely invertibilitypreserving linear maps in $[7,8]$, respectively. Subsequently, in $[6,9]$, general surjective maps between standard operator algebras that completely preserve invertibility or spectrum and
that completely preserve spectral functions are studied, respectively, where a standard operator algebra is a norm closed subalgebra of some $B(X)$ over a Banach space $X$ containing the identity $I$ and all finite-rank operators. Recently, in [10], the authors discussed completely idempotents preserving surjective maps and completely square-zero operators preserving surjective maps. These results showed that idempotents and square-zero operators are invariants of isomorphism in complete preserver problems. Since involutions and Drazin inverse are closely related to idempotents, it is interesting to consider whether the involutions and Drazin inverse are still invariants of isomorphism in complete preserver problems.

Let $X$ and $Y$ be Banach spaces over the real or complex field $\mathbb{F}$, and let $B(X)$ be the Banach algebra of all bounded linear operators from $X$ to $X$. An operator $A \in B(X)$ is called an involution (idempotent) if $A^{2}=I\left(A^{2}=A\right)$, denoted by $\Gamma_{S}=\left\{A: A \in S\right.$ and $\left.A^{2}=I\right\}$ and $D_{S}=\left\{A: A \in S\right.$ and $\left.A^{2}=A\right\}$, where $S$ is an algebra and $I$ is an identity in $S$. An operator $A \in B(X)$ is said to have a Drazin inverse, or to be Drazin invertible if there exists $X \in B(X)$ such that

$$
\begin{equation*}
A^{k+1} X=A^{k}, \quad X A X=X, \quad A X=X A \tag{1.1}
\end{equation*}
$$

and $X$ is called the Drazin inverse of $A$, denoted by $A^{D}$. The concepts of involution and Drazin inverse are very useful in various applied mathematical areas. For example, in [11], the authors showed that involution has applications in Chi-square distribution, combinatorial problems, and so on. About Drazin inverse, it is helpful in singular differential and difference equations, Markov chain, multibody system dynamics, and so on [3].

Inspired by the above, the purpose of this paper is to consider the following two things:
(1) the characterization of surjective maps that completely preserve involutions between standard operator algebras on Banach spaces;
(2) the characterization of surjective maps that completely preserve Drazin inverse between standard operator algebras on Banach spaces.

Let $\mathcal{A}$ and $\mathbb{B}$ be standard operator algebras on $X$ and $Y$, respectively, and let $\Phi: \mathcal{A} \rightarrow \mathbb{B}$ be a surjective map. Define, for each $n \in \mathbb{N}$, a map $\Phi_{n}: \mathcal{A} \otimes M_{n}(\mathbb{F}) \rightarrow \mathcal{B} \otimes M_{n}(\mathbb{F})$ by

$$
\begin{equation*}
\Phi_{n}\left(\left(s_{i j}\right)_{n \times n}\right)=\left(\Phi\left(s_{i j}\right)\right)_{n \times n} \tag{1.2}
\end{equation*}
$$

Then $\Phi$ is called $n$-involutions preserving in both directions if $\Phi_{n}$ preserves involutions in both directions, that is, $\left(s_{i j}\right)_{n \times n}^{2}=I \Leftrightarrow \Phi_{n}^{2}\left(\left(s_{i j}\right)_{n \times n}\right)=I$; $\Phi$ is said to be completely involutionpreserving in both directions if $\Phi$ is $n$-involutions preserving in both directions for every positive integer $n$. Similarly, $\Phi$ is called $n$-Drazin inverse preserving in both directions if $\Phi_{n}$ preserves Drazin inverse in both directions, that is, $\left(s_{i j}\right)_{n \times n}^{D}=\left(t_{i j}\right)_{n \times n} \Leftrightarrow \Phi_{n}\left(\left(t_{i j}\right)_{n \times n}\right)=$ $\left(\Phi\left(s_{i j}\right)\right)_{n \times n^{n}}^{D}$; $\Phi$ is said to be completely Drazin inverse preserving in both directions if $\Phi$ is $n$-Drazin inverse preserving in both directions for every positive integer $n$.

We end this part by some notations. Let $X^{*}$ be the dual space of a Banach space $X$. For every nonzero $x \in X$ and $f \in X^{*}$, the symbol $x \otimes f$ standards for the rank one bounded linear operator on $X$ defined by $(x \otimes f) y=\langle y, f\rangle x$ for any $y \in X$. Given $P, Q \in P_{A}$, we say $P$ and $Q$ are orthogonal if $P Q=Q P=0$, where 0 is the zero operator in $\mathcal{A}$.

## 2. Maps Completely Preserving Involutions

Lemma 2.1 (see [10]). Let X, $Y$ be infinite dimensional Banach spaces over the real or complex field $\mathbb{F}$ and $P \subseteq B(X), Q \subseteq B(Y)$ be sets of idempotents which contain all rank one idempotents. Let $\Phi: P \rightarrow Q$ be a bijective map. If $\Phi$ preserves orthogonality in both directions, then there exists either a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P A^{-1}, \quad P \in P, \tag{2.1}
\end{equation*}
$$

or a bounded invertible linear or (in the complex case) conjugate linear operator $A: X^{*} \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P^{*} A^{-1}, \quad P \in p . \tag{2.2}
\end{equation*}
$$

In the second case, X and $\Upsilon$ must be reflexive.
Theorem 2.2. Let $X, Y$ be infinite-dimensional Banach spaces over the real or complex field $\mathbb{F}$ and $\mathcal{A}, B$ be standard operator algebras on $X$ and $\Upsilon$, respectively. Let $\Phi: \mathcal{A} \rightarrow B$ be a surjective map. Then the following statements are equivalent:
(1) $\Phi$ is completely involutions preserving in both directions.
(2) $\Phi$ is 2 -involution preserving in both directions.
(3) There exists a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(T)=\delta A T A^{-1}, \quad \forall T \in \mathcal{A}, \tag{2.3}
\end{equation*}
$$

where $\delta= \pm 1$.
Proof. Obviously, $(3) \Rightarrow(1) \Rightarrow(2)$. Then, $(2) \Rightarrow(3)$ is shown by proving the following claims. Assume that $\Phi$ is 2-involution preserving in both directions.

Claim 1. $\Phi(0)=0, \Phi(I)=\delta I$, where $\delta= \pm 1$ and $\Phi$ is injective.
For any $T \in \mathcal{A}$,

$$
\left(\begin{array}{cc}
I & T  \tag{2.4}\\
0 & -I
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)},
$$

where $\mathrm{X}^{2}=\mathrm{X} \times \mathrm{X}$ is a Banach space with a suitable norm, for example, $\left\|\left(x_{1}, x_{2}\right)\right\|=\left\|x_{1}\right\|+\left\|x_{2}\right\|$. Applying the assumption of $\Phi$, we get

$$
\left(\begin{array}{cc}
\Phi(I) & \Phi(T)  \tag{2.5}\\
\Phi(0) & \Phi(-I)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)} .
$$

Thus

$$
\begin{gather*}
\Phi^{2}(I)+\Phi(T) \Phi(0)=I,  \tag{2.6}\\
\Phi(I) \Phi(T)+\Phi(T) \Phi(-I)=0 . \tag{2.7}
\end{gather*}
$$

By the surjectivity of $\Phi$, we can find some $T_{0} \in \mathcal{A}$ such that $\Phi\left(T_{0}\right)=0$. Let $T=T_{0}$, (2.6) yields that $\Phi^{2}(I)=I$. Hence, $\Phi(T) \Phi(0)=0$ holds for all $T$; this entails that $\Phi(0)=0$, since $\Phi$ is surjective.

Taking $T=I$ in (2.7), and also by the invertibility of $\Phi(I)$, we have $\Phi(I)=-\Phi(-I)$. Then (2.7) yields that $\Phi(I) \Phi(T)=\Phi(T) \Phi(I)$ for all $T \in \mathcal{A}$. Because of the surjectivity of $\Phi$ and $\Phi^{2}(I)=I$, it is not difficult to get $\Phi(I)=\delta I$, where $\delta= \pm 1$.

If we replace $\Phi$ by $-\Phi$, it is still 2-involution preserving, then without loss of generality, we always assume that $\Phi(I)=I$ in the sequel. Next, we show that $\Phi$ is injective.

For any $T, S \in \mathcal{A}$ such that $\Phi(T)=\Phi(S)$, we have

$$
\begin{align*}
\left(\begin{array}{cc}
T & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} & \Longleftrightarrow\left(\begin{array}{cc}
\Phi(T) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & \Phi(-T)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)} \\
& \Longleftrightarrow\left(\begin{array}{cc}
\Phi(S) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & \Phi(-T)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)}  \tag{2.8}\\
& \Longleftrightarrow\left(\begin{array}{cc}
S & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)}
\end{align*}
$$

which imply that $T=S$. Therefore, $\Phi$ is injective.
Claim 2. Ф preserves idempotents in both directions.
For any $P \in D_{A}$, since

$$
\left(\begin{array}{cc}
I-P & P  \tag{2.9}\\
P & I-P
\end{array}\right) \in \Gamma_{B\left(X^{2}\right),} \quad\left(\begin{array}{cc}
I-P & I \\
P & P-I
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)}
$$

then using the assumption of $\Phi$, we have

$$
\begin{gather*}
\Phi^{2}(I-P)+\Phi^{2}(P)=I  \tag{2.10}\\
\Phi^{2}(I-P)+\Phi(P)=I  \tag{2.11}\\
\Phi(I-P)+\Phi(P-I)=0 \tag{2.12}
\end{gather*}
$$

From (2.10) and (2.11), it is derived that $\Phi(P) \in p_{\mathcal{B}}$ for any $P \in \rho_{\mathcal{A}}$. Applying (2.11) again, we see that

$$
\begin{equation*}
\Phi(I-P)=I-\Phi(P) \tag{2.13}
\end{equation*}
$$

Since $P \in D_{\AA}$ is arbitrary, then (2.12) yields that

$$
\begin{equation*}
\Phi(P)=-\Phi(-P), \quad \text { for any } P \in D_{\mathcal{A}} \tag{2.14}
\end{equation*}
$$

Thus, combining (2.13) and (2.14) with the bijectivity of $\Phi$, it is not difficult to get the result that if $\Phi(P) \in D_{\mathbb{B}}$, then $P \in D_{\mathcal{A}}$. Therefore, Claim 2 holds true.

Claim 3. There exists a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that $\Phi(P)=A P A^{-1}$ for every $P \in P_{\mathcal{A}}$.

For every $P, Q \in P_{\mathcal{A}}$,

$$
\begin{align*}
P Q=Q P=0 & \Longleftrightarrow\left(\begin{array}{cc}
I & P \\
Q & -I
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} \\
& \Longleftrightarrow\left(\begin{array}{cc}
\Phi(I) & \Phi(P) \\
\Phi(Q) & \Phi(-I)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)}  \tag{2.15}\\
& \Longleftrightarrow \Phi(P) \Phi(Q)=\Phi(Q) \Phi(P)=0,
\end{align*}
$$

that is, $\Phi$ preserves orthogonality in both directions from $p_{\mathcal{A}}$ to $p_{\mathcal{B}}$. From Lemma 2.1, we see that there exists either a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P A^{-1}, \quad P \in p_{A}, \tag{2.16}
\end{equation*}
$$

or a bounded invertible linear or (in the complex case) conjugate linear operator $A: X^{*} \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P^{*} A^{-1}, \quad P \in P_{\mathbb{A}} . \tag{2.17}
\end{equation*}
$$

Sequently, we show that the second case cannot occur. On the contrary, assume that $\Phi(P)=A P^{*} A^{-1}$ for all $P \in D_{\mathcal{A}}$. Similar to the proof of Theorem 3.2 in [10], for any linearly independent vectors $x, y \in X$, there exist $f, g \in X^{*}$ such that $\langle x, f\rangle=\langle x, g\rangle=\langle y, g\rangle=1$ and $\langle y, f\rangle=-1$. Then,

$$
M=\left(\begin{array}{cc}
I-x \otimes f & x \otimes g  \tag{2.18}\\
y \otimes f & I-y \otimes g
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} .
$$

By the assumption of $\Phi$ on $p_{\mathcal{A}}$ and (2.14), we see that

$$
\Phi_{2}(M)=\left(\begin{array}{cc}
A & 0  \tag{2.19}\\
0 & A
\end{array}\right)\left(\begin{array}{cc}
I-f \otimes x & g \otimes x \\
f \otimes y & I-g \otimes y
\end{array}\right)\left(\begin{array}{cc}
A^{-1} & 0 \\
0 & A^{-1}
\end{array}\right),
$$

but $\Phi_{2}(M)$ is not an involution, it is a contradiction. Therefore, Claim 3 holds true.
Let $\Psi(\cdot)=A^{-1} \Phi(\cdot) A$, then $\Psi$ is a bijective map preserving 2 involutions in both directions from $\mathcal{A}$ onto the standard operator algebra $A^{-1} \mathcal{A} A$. Furthermore, by Claim 3, $\Psi(P)=P$ for every $P \in P_{\mathcal{A}}$. Hence, without lose of generality, we suppose that

$$
\begin{equation*}
\Phi(P)=P, \tag{2.20}
\end{equation*}
$$

for all $P \in P_{\mathscr{A}}$.

Claim 4. $\Phi(x \otimes f)=x \otimes f$ for any rank one operator $x \otimes f$.
For any rank one operator $x \otimes f$, there exists $y \in X$ such that $y$ is linearly independent of $x$ and $\langle y, f\rangle=1$. Then there exist $g_{1}, g_{2} \in X^{*}$ such that $\left\langle x, g_{1}\right\rangle=\left\langle y, g_{2}\right\rangle=1,\left\langle y, g_{1}\right\rangle=$ $\left\langle x, g_{2}\right\rangle=0$. Let $g=g_{1}+g_{2}$, then we have $\langle x, g\rangle=1,\langle y, g\rangle=1$. Then using (2.20), we have

$$
\left(\begin{array}{cc}
I-x \otimes g & x \otimes f  \tag{2.21}\\
y \otimes g & I-y \otimes f
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} \Longleftrightarrow\left(\begin{array}{cc}
I-x \otimes g & \Phi(x \otimes f) \\
y \otimes g & I-y \otimes f
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)} .
$$

Hence,

$$
\begin{gather*}
\Phi(x \otimes f)(y \otimes g)=x \otimes g  \tag{2.22}\\
(y \otimes g) \Phi(x \otimes f)=y \otimes f  \tag{2.23}\\
2 \Phi(x \otimes f)=x \otimes g \Phi(x \otimes f)+\Phi(x \otimes f) y \otimes f \tag{2.24}
\end{gather*}
$$

Combining (2.22) and (2.23) with (2.24), we derive that $\Phi(x \otimes f)=\Phi(x \otimes f)(y \otimes g) \Phi(x \otimes f)$. Then using (2.22) and (2.23) again, we get

$$
\begin{align*}
& \Phi(x \otimes f)=x \otimes g \Phi(x \otimes f) \\
& \Phi(x \otimes f)=\Phi(x \otimes f) y \otimes f \tag{2.25}
\end{align*}
$$

From (2.25), it is easily seen that there exists $\mu_{x \otimes f} \in \mathbb{F} \backslash\{0\}$ such that

$$
\begin{equation*}
\Phi(x \otimes f)=\mu_{x \otimes f} x \otimes f . \tag{2.26}
\end{equation*}
$$

Taking (2.26) into (2.22), this yields that $\mu_{x \otimes f}=1$, Thus, Claim 4 holds true.
Claim 5. $\Phi(T)=T$ for all $T \in \mathcal{A}$.
For any $T \in \mathcal{A}$, since

$$
\left(\begin{array}{cc}
T & I  \tag{2.27}\\
I-T^{2} & -T
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} \Longleftrightarrow\left(\begin{array}{cc}
\Phi(T) & \Phi(I) \\
\Phi\left(I-T^{2}\right) & \Phi(-T)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)}
$$

By $\Phi(I)=I$, it follows that

$$
\begin{equation*}
\Phi(T)=-\Phi(-T), \quad \forall T \in \mathcal{A} \tag{2.28}
\end{equation*}
$$

For any $S \in \mathcal{A}$ and any invertible operator $T \in \mathcal{A}$,

$$
\left(\begin{array}{cc}
T S & T  \tag{2.29}\\
T^{-1}-S T S & -S T
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} \Longleftrightarrow\left(\begin{array}{cc}
\Phi(T S) & \Phi(T) \\
\Phi\left(T^{-1}-S T S\right) & \Phi(-S T)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)}
$$

Applying (2.28), we get

$$
\begin{equation*}
\Phi(T S) \Phi(T)=\Phi(T) \Phi(S T) \tag{2.30}
\end{equation*}
$$

For any rank one operator $x \otimes f$, let $S=x \otimes f$ in (2.30), and using Claim 4, we know that $T(x \otimes f) \Phi(T)=\Phi(T)(x \otimes f) T$. It follows that $T x \otimes \Phi(T)^{*} f=\Phi(T) x \otimes T^{*} f$. This yields that

$$
\begin{equation*}
\Phi(T)=T, \quad \text { for any invertible } T \in \mathcal{A} . \tag{2.31}
\end{equation*}
$$

For any rank one operator $T \in \mathcal{A}$, it is clearly that $I-T$ is either idempotent or invertible. Then using (2.20) or (2.31), we have

$$
\begin{equation*}
\Phi(I-T)=I-T, \quad \text { for any rank one operator } T \in \mathcal{A} . \tag{2.32}
\end{equation*}
$$

For any $T, S \in \mathcal{A}$,

$$
\left(\begin{array}{cc}
I-T S & -T  \tag{2.33}\\
-(2 I-S T) S & S T-I
\end{array}\right) \in \Gamma_{B\left(X^{2}\right)} \Longleftrightarrow\left(\begin{array}{cc}
\Phi(I-T S) & \Phi(-T) \\
\Phi(-(2 I-S T) S) & \Phi(S T-I)
\end{array}\right) \in \Gamma_{B\left(Y^{2}\right)} .
$$

By (2.28), we have

$$
\begin{equation*}
\Phi(I-T S) \Phi(T)=\Phi(T) \Phi(I-S T), \quad \text { for any } T, S \in \mathcal{A} . \tag{2.34}
\end{equation*}
$$

For any rank one operator $x \otimes f$, let $S=x \otimes f$ in (2.34), and using (2.32), we still get $T(x \otimes$ $f) \Phi(T)=\Phi(T)(x \otimes f) T$. Then similarly, we have

$$
\begin{equation*}
\Phi(T)=T, \quad \forall T \in \mathcal{A} . \tag{2.35}
\end{equation*}
$$

Therefore, the proof of this theorem is finished.
$\Phi$ is called $n$-identity product preserving in both directions if $\Phi_{n}$ preserves identity product in both directions, that is, $\left(s_{i j}\right)_{n \times n}\left(t_{i j}\right)_{n \times n}=I \Leftrightarrow \Phi_{n}\left(\left(s_{i j}\right)_{n \times n}\right) \Phi_{n}\left(\left(t_{i j}\right)_{n \times n}\right)=I$; $\Phi$ is said to be completely identity preserving product in both directions if $\Phi$ is $n$-identity product preserving in both directions for every positive integer $n$ and $\Phi$ is called $n$-identity Jordan product preserving in both directions if $\Phi_{n}$ preserves identity Jordan product in both directions, that is, $(1 / 2)\left(s_{i j}\right)_{n \times n}\left(t_{i j}\right)_{n \times n}+(1 / 2)\left(t_{i j}\right)_{n \times n}\left(s_{i j}\right)_{n \times n}=I \Leftrightarrow(1 / 2) \Phi_{n}\left(\left(s_{i j}\right)_{n \times n}\right) \Phi_{n}\left(\left(t_{i j}\right)_{n \times n}\right)+$ $(1 / 2) \Phi_{n}\left(\left(t_{i j}\right)_{n \times n}\right) \Phi_{n}\left(\left(s_{i j}\right)_{n \times n}\right)=I ; \Phi$ is said to be completely identity Jordan product preserving in both directions if $\Phi$ is $n$-identity Jordan product in both directions for every positive integer $n$.

Remark 2.3. Using the result of Theorem 2.2, it is not difficult to give the characterization of maps completely preserving identity product in both directions and maps completely preserving identity Jordan product preserving in both directions.

## 3. Maps Completely Preserving Drazin Inverse

Theorem 3.1. Let $X, Y$ be infinite-dimensional Banach spaces over the real or complex field $\mathbb{F}$ and $A, B$ be standard operator algebras on $X$ and $\Upsilon$, respectively. Let $\Phi: A \rightarrow B$ be a surjective map.

Then the following statements are equivalent:
(1) $\Phi$ is completely Drazin inverse preserving in both directions.
(2) $\Phi$ is 2-Drazin inverse preserving in both directions.
(3) There exists a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(T)=\delta A T A^{-1} \quad \forall T \in \mathscr{A} \tag{3.1}
\end{equation*}
$$

where $\delta= \pm 1$.
Proof. Clearly, we only need to prove that $(2) \Rightarrow(3)$. Assume that $\Phi$ is 2-Drazin inverse preserving in both directions.

Claim 1. $\Phi(0)=0, \Phi(I)=\delta I$, where $\delta= \pm 1$, and $\Phi$ is injective.
For any $T \in \mathcal{A}$, since

$$
\left(\begin{array}{ll}
I & T  \tag{3.2}\\
0 & 0
\end{array}\right)^{D}=\left(\begin{array}{ll}
I & T \\
0 & 0
\end{array}\right) \Longleftrightarrow\left(\begin{array}{cc}
\Phi(I) & \Phi(T) \\
\Phi(0) & \Phi(0)
\end{array}\right)^{D}=\left(\begin{array}{cc}
\Phi(I) & \Phi(T) \\
\Phi(0) & \Phi(0)
\end{array}\right)
$$

using (1.1), it entails that

$$
\begin{gather*}
\Phi^{2}(I) \Phi(T)+\Phi(T) \Phi(0) \Phi(T)+\Phi(I) \Phi(T) \Phi(0)+\Phi(T) \Phi^{2}(0)=\Phi(T)  \tag{3.3}\\
\Phi(0) \Phi^{2}(I)+\Phi^{2}(0) \Phi(I)+\Phi(0) \Phi(T) \Phi(0)+\Phi^{3}(0)=\Phi(0)  \tag{3.4}\\
\Phi(0) \Phi(I) \Phi(T)+\Phi^{2}(0) \Phi(T)+\Phi(0) \Phi(T) \Phi(0)+\Phi^{3}(0)=\Phi(0) \tag{3.5}
\end{gather*}
$$

As $\Phi$ is surjective, there exists some $T_{0} \in \mathscr{A}$ such that $\Phi\left(T_{0}\right)=0$. Taking $T=T_{0}$ in (3.4) and (3.5), respectively, we have

$$
\begin{gather*}
\Phi(0) \Phi^{2}(I)+\Phi^{2}(0) \Phi(I)=0  \tag{3.6}\\
\Phi^{3}(0)=\Phi(0) \tag{3.7}
\end{gather*}
$$

Taking (3.6) and (3.7) into (3.4) again, we see that

$$
\begin{equation*}
\Phi(0) \Phi(T) \Phi(0)=0 \tag{3.8}
\end{equation*}
$$

then let $T=0$ in (3.8) and use (3.7), we get

$$
\begin{equation*}
\Phi(0)=0 \tag{3.9}
\end{equation*}
$$

Taking (3.9) into (3.3), this yields that

$$
\begin{equation*}
\Phi^{2}(I) \Phi(T)=\Phi(T) \tag{3.10}
\end{equation*}
$$

by the surjectivity of $\Phi$, there exists a $T_{1} \in \mathcal{A}$ such that $\Phi\left(T_{1}\right)=I$; let $T=T_{1}$ in (3.10), we see that

$$
\begin{equation*}
\Phi^{2}(I)=I . \tag{3.11}
\end{equation*}
$$

Since

$$
\left(\begin{array}{ll}
I & T  \tag{3.12}\\
0 & I
\end{array}\right)^{D}=\left(\begin{array}{cc}
I & -T \\
0 & I
\end{array}\right), \quad \text { for any } T \in \mathcal{A}
$$

then by the assumption of $\Phi$ and applying (1.1) and (3.11), we see that

$$
\begin{equation*}
\Phi(I) \Phi(T) \Phi(I)=-\Phi(-T) \tag{3.13}
\end{equation*}
$$

Let $T=I$ in (3.13), we have

$$
\begin{equation*}
\Phi(I)=-\Phi(-I) \tag{3.14}
\end{equation*}
$$

For any $T \in \mathcal{A}$,

$$
\left(\begin{array}{cc}
I & T  \tag{3.15}\\
0 & -I
\end{array}\right)^{D}=\left(\begin{array}{cc}
I & T \\
0 & -I
\end{array}\right)
$$

then using (3.14) and (3.11), we have

$$
\begin{equation*}
\Phi(I) \Phi(T)=\Phi(T) \Phi(I), \quad \forall T \in \mathcal{A} \tag{3.16}
\end{equation*}
$$

Similar to the proof of Claim 1 in Theorem 2.2, we get $\Phi(I)=\delta I$, where $\delta= \pm 1$.
Without loss of generality, we always assume that $\Phi(I)=I$ in the sequel. Now, we show that $\Phi$ is injective.

Take $\Phi(I)=I$ into (3.13), it yields that

$$
\begin{equation*}
\Phi(T)=-\Phi(-T), \quad \text { for any } T \in \mathcal{A} \tag{3.17}
\end{equation*}
$$

For any $T, S \in \mathcal{A}$ such that $\Phi(T)=\Phi(S)$, we have

$$
\left(\begin{array}{cc}
T & -2 I  \tag{3.18}\\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right)^{D}=\left(\begin{array}{cc}
T & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right)
$$

then

$$
\left(\begin{array}{cc}
\Phi(T) & \Phi(-2 I)  \tag{3.19}\\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & \Phi(-T)
\end{array}\right)^{D}=\left(\begin{array}{cc}
\Phi(T) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & \Phi(-T)
\end{array}\right)
$$

therefore, by (3.17) and $\Phi(T)=\Phi(S)$, we see that

$$
\begin{align*}
& \left(\begin{array}{cc}
\Phi(S) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & -\Phi(S)
\end{array}\right)^{D}=\left(\begin{array}{cc}
\Phi(S) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & -\Phi(S)
\end{array}\right) \\
& \left(\begin{array}{cc}
\Phi(S) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & -\Phi(T)
\end{array}\right)^{D}=\left(\begin{array}{cc}
\Phi(S) & \Phi(-2 I) \\
\Phi\left(-\frac{1}{2}\left(I-T^{2}\right)\right) & -\Phi(T)
\end{array}\right) \tag{3.20}
\end{align*}
$$

then

$$
\begin{align*}
& \left(\begin{array}{cc}
S & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -S
\end{array}\right)^{D}=\left(\begin{array}{cc}
S & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -S
\end{array}\right)  \tag{3.21}\\
& \left(\begin{array}{cc}
S & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right)^{D}=\left(\begin{array}{cc}
S & -2 I \\
-\frac{1}{2}\left(I-T^{2}\right) & -T
\end{array}\right)
\end{align*}
$$

Applying (1.1), we derive that $S^{2}=S T, S-T=T^{2} S-T^{3}$ and $S^{2}=T^{2}$. By direct computation, it is easy to get $T=S$. Thus, $\Phi$ is an injective map, and Claim 1 holds true.

Claim 2. $\Phi$ preserves idempotents in both directions.
For any $T \in \mathcal{A}$,

$$
\left(\begin{array}{cc}
I-T & -T  \tag{3.22}\\
I & I
\end{array}\right)^{D}=\left(\begin{array}{cc}
I & T \\
-I & I-T
\end{array}\right),
$$

by the assumption of $\Phi$ and (1.1), we have

$$
\begin{equation*}
\Phi(I-T)=I-\Phi(T), \quad \text { for any } T \in \mathcal{A} \tag{3.23}
\end{equation*}
$$

For any $P \in P_{d}$,

$$
\left(\begin{array}{cc}
P & 0  \tag{3.24}\\
0 & I-P
\end{array}\right)^{D}=\left(\begin{array}{cc}
P & 0 \\
0 & I-P
\end{array}\right)
$$

it follows that $\Phi^{3}(P)=\Phi(P)$ and $\Phi^{3}(I-P)=\Phi(I-P)$. Then by (3.23), it derives that $\Phi(P) \in$ $p_{\mathcal{B}}$. Similarly, we get that if $\Phi(P) \in p_{\mathcal{B}}$, then $P \in p_{\mathcal{A}}$. Therefore, this claim is true.

Claim 3. There exists a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that $\Phi(P)=A P A^{-1}$ for every $P \in D_{\mathbb{A}}$.

For every $P, Q \in P_{d}$,

$$
\begin{align*}
P Q=Q P=0 & \Longleftrightarrow\left(\begin{array}{cc}
I & P \\
Q & -I
\end{array}\right)^{D}=\left(\begin{array}{cc}
I & P \\
Q & -I
\end{array}\right) \\
& \Longleftrightarrow\left(\begin{array}{cc}
\Phi(I) & \Phi(P) \\
\Phi(Q) & \Phi(-I)
\end{array}\right)^{D}=\left(\begin{array}{cc}
\Phi(I) & \Phi(P) \\
\Phi(Q) & \Phi(-I)
\end{array}\right)  \tag{3.25}\\
& \Longleftrightarrow \Phi(P) \Phi(Q)=\Phi(Q) \Phi(P)=0,
\end{align*}
$$

that is, $\Phi$ preserves orthogonality in both directions from $p_{\mathcal{A}}$ to $D_{\mathcal{B}}$. It follows from Lemma 2.1 that there exists either a bounded invertible linear or (in the complex case) conjugate linear operator $A: X \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P A^{-1}, \quad P \in D_{d}, \tag{3.26}
\end{equation*}
$$

or a bounded invertible linear or (in the complex case) conjugate linear operator $A: X^{*} \rightarrow Y$ such that

$$
\begin{equation*}
\Phi(P)=A P^{*} A^{-1}, \quad P \in P_{\mathbb{A}} \tag{3.27}
\end{equation*}
$$

We show that the second case cannot occur. On the contrary, assume that $\Phi(P)=$ $A P^{*} A^{-1}$ for all $P \in D_{\mathcal{A}}$. For any linearly independent vectors $x, y \in X$, similar to the proof of the Theorem 3.2 in [10], we can find $f, g \in X^{*}$ such that $\langle x, f\rangle=\langle x, g\rangle=\langle y, g\rangle=1$ and $\langle y, f\rangle=-1$. Then

$$
\left(\begin{array}{cc}
I-x \otimes f & x \otimes g  \tag{3.28}\\
y \otimes f & I-y \otimes g
\end{array}\right)^{D}=\left(\begin{array}{cc}
I-x \otimes f & x \otimes g \\
y \otimes f & I-y \otimes g
\end{array}\right),
$$

but by the assumption of $\Phi$ and (1.1), it is easy to check that

$$
\left(\begin{array}{cc}
\Phi(I-x \otimes f) & \Phi(x \otimes g)  \tag{3.29}\\
\Phi(y \otimes f) & \Phi(I-y \otimes g)
\end{array}\right)^{D} \neq\left(\begin{array}{cc}
\Phi(I-x \otimes f) & \Phi(x \otimes g) \\
\Phi(y \otimes f) & \Phi(I-y \otimes g)
\end{array}\right),
$$

which is a contradiction to the hypothesis that $\Phi_{2}$ is Drazin inverse preserving. Therefore, $\Phi(P)=A P A^{-1}$ holds for every $P \in P_{\mathbb{d}}$, and Claim 3 holds true.

In the sequel, without lose of generality, we suppose that

$$
\begin{equation*}
\Phi(P)=P, \quad \forall P \in p_{\mathcal{A}} . \tag{3.30}
\end{equation*}
$$

Claim 4. $\Phi(x \otimes f)=x \otimes f$ for any rank one operator $x \otimes f$.
Similar to the proof of Claim 4 in Theorem 2.1 in [10], for any rank one operator $x \otimes f$, we can find $g \in X^{*}$ such that $\langle x, g\rangle=1$ and $y \in X$ such that $\langle y, f\rangle=1$. Then

$$
\begin{gather*}
\left(\begin{array}{cc}
x \otimes g & x \otimes f \\
0 & 0
\end{array}\right)^{D}=\left(\begin{array}{cc}
x \otimes g & x \otimes f \\
0 & 0
\end{array}\right),  \tag{3.31}\\
\left(\begin{array}{ll}
0 & x \otimes f \\
0 & y \otimes f
\end{array}\right)^{D}=\left(\begin{array}{ll}
0 & x \otimes f \\
0 & y \otimes f
\end{array}\right) .
\end{gather*}
$$

Using (3.30), we have

$$
\begin{gather*}
\left(\begin{array}{cc}
\Phi(x \otimes g) & \Phi(x \otimes f) \\
0 & 0
\end{array}\right)^{D}=\left(\begin{array}{cc}
x \otimes g & \Phi(x \otimes f) \\
0 & 0
\end{array}\right) \\
\left(\begin{array}{ll}
0 & \Phi(x \otimes f) \\
0 & \Phi(y \otimes f)
\end{array}\right)^{D}=\left(\begin{array}{cc}
0 & \Phi(x \otimes f) \\
0 & y \otimes f
\end{array}\right) \tag{3.32}
\end{gather*}
$$

Therefore, we derive that $x \otimes g \Phi(x \otimes f)=\Phi(x \otimes f)$ and $\Phi(x \otimes f) y \otimes f=\Phi(x \otimes f)$. Then, it is easily seen that there exists $\mu_{x \otimes f} \in \mathbb{F} \backslash\{0\}$ such that

$$
\begin{equation*}
\Phi(x \otimes f)=\mu_{x \otimes f} x \otimes f \tag{3.33}
\end{equation*}
$$

Similar to the proof of Claim 4 in Theorem 2.2, for any rank one operator $x \otimes f$, we can find $y \in X$ such that $\langle y, f\rangle=1$ and $g \in X^{*}$ such that $\langle x, g\rangle=1,\langle y, g\rangle=1$. Then

$$
\left(\begin{array}{cc}
I-x \otimes g & x \otimes f  \tag{3.34}\\
y \otimes g & I-y \otimes f
\end{array}\right)^{D}=\left(\begin{array}{cc}
I-x \otimes g & x \otimes f \\
y \otimes g & I-y \otimes f
\end{array}\right)
$$

by (3.30) and the assumption of $\Phi$, we see that

$$
\left(\begin{array}{cc}
I-x \otimes g & \Phi(x \otimes f)  \tag{3.35}\\
y \otimes g & I-y \otimes f
\end{array}\right)^{D}=\left(\begin{array}{cc}
I-x \otimes g & \Phi(x \otimes f) \\
y \otimes g & I-y \otimes f
\end{array}\right)
$$

It entails that

$$
\begin{equation*}
y \otimes g \Phi(x \otimes f) y \otimes g=y \otimes g \tag{3.36}
\end{equation*}
$$

by (3.33), we know that $\mu_{x \otimes f}=1$. Therefore, Claim 4 holds true.
Claim 5. $\Phi(T)=T$ for all $T \in \mathcal{A}$.
For any $T, S \in \mathcal{A}$, since

$$
\begin{align*}
& \left(\begin{array}{cc}
I-T S & -T \\
S T S & I+S T
\end{array}\right)^{D}=\left(\begin{array}{cc}
I+T S & T \\
-S T S & I-S T
\end{array}\right),  \tag{3.37}\\
& \left(\begin{array}{cc}
I+T S & T \\
-S T S & I-S T
\end{array}\right)^{D}=\left(\begin{array}{cc}
I-T S & -T \\
S T S & I+S T
\end{array}\right),
\end{align*}
$$

then by the assumption of $\Phi$, using (3.17) and (3.23), we have

$$
\begin{align*}
& \left(\begin{array}{cc}
\Phi(I-T S) & \Phi(-T) \\
\Phi(S T S) & \Phi(I+S T)
\end{array}\right)^{D}=\left(\begin{array}{cc}
I+\Phi(T S) & \Phi(T) \\
-\Phi(S T S) & I-\Phi(S T)
\end{array}\right)  \tag{3.38}\\
& \left(\begin{array}{cc}
I+\Phi(T S) & \Phi(T) \\
-\Phi(S T S) & I-\Phi(S T)
\end{array}\right)^{D}=\left(\begin{array}{cc}
I-\Phi(T S) & -\Phi(T) \\
\Phi(S T S) & I+\Phi(S T)
\end{array}\right)
\end{align*}
$$

For simplification, let $N=\left(\begin{array}{cc}\Phi(T S) & \Phi(T) \\ -\Phi(S T S) & -\Phi(S T)\end{array}\right)$; then by (1.1), we derive that $(I+N)(I-N)(I+$ $N)=(I+N)$ and $(I-N)(I+N)(I-N)=(I-N)$. Therefore, we have $N^{2}=0$. Thus, by direct computation, we get

$$
\begin{equation*}
\Phi(T S) \Phi(T)=\Phi(T) \Phi(S T), \quad \text { for any } T, S \in \mathscr{A} \tag{3.39}
\end{equation*}
$$

For any rank one operator $x \otimes f$, let $S=x \otimes f$ in (3.39), and using Claim 4, we have $T(x \otimes$ $f) \Phi(T)=\Phi(T)(x \otimes f) T$. It follows that $T x \otimes \Phi(T)^{*} f=\Phi(T) x \otimes T^{*} f$. Then we get

$$
\begin{equation*}
\Phi(T)=T, \quad \forall T \in \mathcal{A} \tag{3.40}
\end{equation*}
$$

Therefore, the proof of this theorem is completed.

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