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Research Article

On Pre-Hilbert Noncommutative Jordan Algebras Satisfying $\|x^2\| = \|x\|^2$

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Let A be a real or complex algebra. Assuming that a vector space A is endowed with a pre-Hilbert norm $\|\cdot\|$ satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. We prove that A is finite dimensional in the following cases. (1) A is a real weakly alternative algebra without divisors of zero. (2) A is a complex powers associative algebra. (3) A is a complex flexible algebraic algebra. (4) A is a complex Jordan algebra. In the first case A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, and A is isomorphic to $\mathbb{C}$ in the last three cases. These last cases permit us to show that if A is a complex pre-Hilbert noncommutative Jordan algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$. Moreover, we give an example of an infinite-dimensional real pre-Hilbert Jordan algebra with divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

1. Introduction

Let A be a real or complex algebra not necessarily associative or finite dimensional. Assuming that a vector space A is endowed with a pre-Hilbert norm $\|\cdot\|$ satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Zalar (1995, [1]) proved that.

(1) If A is a real alternative algebra containing a unit element e such that $\|e\| = 1$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.

(2) If A is a real associative algebra satisfying $\|x^2\| = \|x\|^2$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

(3) If A is a complex normed algebra containing a unit element e such that $\|e\| = 1$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.

These results were extended, respectively, to the following cases.

(1) If A is a real alternative algebra containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ (2008, [2]).

(2) If A is a real alternative algebra satisfying $\|x^2\| = \|x\|^2$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ (2008, [2]).

(3) If A is a complex normed algebra without divisors of zero and containing an invertible element v such that $\|vx\| = \|xv\| = \|v\|\|x\|$, then A is finite dimensional and is isomorphic to $\mathbb{C}$ (2010, [3]).

In the present paper, we extend the above results to more general situation. Indeed, we prove that, if A is a real or complex pre-Hilbert algebra satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$, then A is finite dimensional in the following cases.

(1) A is a real weakly alternative algebra without divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$ (Theorem 3.5).

(2) A is a real weakly alternative algebra without divisors of zero and containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$ (Theorem 3.7).

(3) A is a complex powers associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$ (Theorem 4.8).

In the first two cases A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and A is isomorphic to $\mathbb{C}$ in the last two cases. This last allows us to show that if A is a complex pre-Hilbert noncommutative Jordan algebra (resp., flexible algebraic algebra or Jordan algebra) satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$ (Theorems 4.9 and 4.10). Moreover, we give an example of an infinite-dimensional real pre-Hilbert Jordan algebra (weakly alternative algebra) with divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

2. Notation and Preliminary Results

Throughout the paper, the word algebra refers to a nonnecessarily associative algebra over $\mathbb{R}$ or $\mathbb{C}$.

Definitions 1. Let B be an arbitrary algebra and K is a field of characteristic not 2.

(1)

(i) B is called alternative if it is satisfied the identities $(y, x, x) = 0$ and $(x, x, y) = 0$ (where $(\cdot, \cdot, \cdot)$ means associator), for all $x, y \in B$ (1966, [4]).

(ii) B is called a powers associative if, for every x in B, the subalgebra $B(x)$ generated by x is associative.

(iii) B is called flexible if $(x, y, x) = 0$ for all $x, y \in B$.

(iv) B is called a Jordan algebra if it is commutative and satisfied the Jordan identity: (J) $(x^2, y, x) = 0$ for all $x, y \in B$.

(v) B is called a noncommutative Jordan algebra if it is flexible and satisfied the Jordan identity (J).

(vi) B is called weakly alternative if it is a noncommutative Jordan algebra and satisfied the identity $(x, x, [x, y]) = 0$ (where $[\cdot, \cdot]$ means commutator). An alternative algebra or Jordan algebra is evidently weakly alternative.

(vii) B is called quadratic if it has an identity element e and satisfied the identity $x^2 = \alpha e + \beta x$ for all $x \in B$ and $\alpha, \beta \in \mathbb{K}$.

(2)

(viii) We say that B is algebraic if, for every x in B, the subalgebra $B(x)$ of B generated by x is finite dimensional (1947, [5]).

(ix) A symmetric bilinear form $(\cdot, \cdot)$ over B is called a trace form if $(xy, z) = (x, yz)$ for all $x, y, z \in B$.

(x) B is termed normed (resp., absolute valued) if it is endowed with a space norm $\| \cdot \|$ such that $\|xy\| \leq \|x\|\|y\|$ (resp., $\|xy\| = \|x\|\|y\|$), for all $x, y \in B$.

(xi) B is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product $(\cdot \mid \cdot)$.

(xii) We mean by a nonzero central element in B, a nonzero element which commute with all elements of the algebra B.

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}, \overset{*}{\mathbb{C}}, \mathbb{H}$ (the algebra of Hamilton quaternion) and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values (1991, [6]) and (2004, [7]). The algebra $\overset{*}{\mathbb{C}}$ (1949, [8]) was obtained by replacing the product of $\mathbb{C}$ with the one defined by $x \circ y = x^* y^*$, where $*$ means the standard involution of $\mathbb{C}$.

We have the following very known results.

Lemma 2.1 (see [4]). *Let A be a powers associative algebra over K and without divisors of zero. If e is a nonzero idempotent in A, then A has an identity element e.*

Proposition 2.2 (see [9]). *If $\{x_i\}$ is a set of commuting elements in a flexible algebra A over K, then the subalgebra generated by the $\{x_i\}$ is commutative.*

Proposition 2.3 (see [10]). *Let A be a noncommutative Jordan algebra over K, then A is a powers associative algebra.*

Lemma 2.4 (see [11]). *Let $A = (V, (\cdot, \cdot), \times)$ be a quadratic algebra over K. Then A flexible if and only if $(\cdot, \cdot)$ is symmetric and the following equivalent statements hold.*

(1) *$(\cdot, \cdot)$ is a trace form over A.*

(2) *$(\cdot, \cdot)$ is a trace over V.*

(3) *$(u \times v, u) = 0$ for every $u, v \in V$.*

Theorem 2.5 (see [4]). *The subalgebra generated by any two elements of an alternative algebra A is associative.*

We need the following results.

Theorem 2.6 (see [1]). *Let A be a real pre-Hilbert associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.*

Theorem 2.7 (see [2]). *Let A be a real pre-Hilbert commutative algebra without divisors of zero and satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Suppose that A containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$. Then A is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\overset{*}{\mathbb{C}}$.*

Theorem 2.8 (see [1]). *Let A be a real pre-Hilbert alternative algebra with identity e. Suppose that $\|x^2\| \leq \|x\|^2$ for all $x \in A$ and $\|e\| = 1$. Then A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

3. Real Pre-Hilbert Weakly Alternative Algebras

In this subparagraph, we prove that, if A is a real pre-Hilbert algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional in the following cases.

(1) A is a real weakly alternative algebra without divisors of zero.

(2) A is a real Jordan algebra without divisors of zero.

In the first case A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, and A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ in the last case. Moreover, we give an example of an infinite-dimensional real pre-Hilbert Jordan algebra with divisors of zero and satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$.

Lemma 3.1 (see [12]). *Let A be a real pre-Hilbert algebra with identity e such that $\|a^2\| = \|a\|^2$ for all $a \in A$ and let $V = \{x \in A / (x \mid e) = 0\}$ then.*

(1) $V = \{x \in A / x^2 = -\|x\|^2 e\}$.

(2) $xy + yx = -2(x \mid y)e$ *for all* $x, y \in V$.

Remark 3.2. (i) The product $x \wedge y = xy - (xy \mid e)e$, for all $x, y \in V$, provides V the structure of an anticommutative algebra.

(ii) If A is flexible, then $(xy \mid e) = -(x \mid y)$ for all $x, y \in V$.

Proof. (i) Let $x, y \in V$, we have

$$\begin{aligned} x \wedge y + y \wedge x &= xy - (xy \mid e)e + yx - (yx \mid e)e \\ &= xy + yx - (xy + yx \mid e)e \\ &= -2(x \mid y)e + 2(x \mid y)e \quad \text{(Lemma 3.1)} \\ &= 0. \end{aligned} \tag{3.1}$$

(ii) As A is a flexible algebra, then

$$\begin{aligned}0 &= (xy)x - x(yx)\\ &= (x\wedge y + (xy \mid e)e)x - x(y\wedge x + (xy \mid e)e)\\ &= (x\wedge y)x + x(x\wedge y) + ((xy \mid e) - (yx \mid e))x\\ &= -2(x \mid x\wedge y)e + ((xy \mid e) - (yx \mid e))x \quad \text{(Lemma 3.1)}\\ &= ((xy \mid e) - (yx \mid e))x.\end{aligned} \tag{3.2}$$

This implies that $(xy \mid e) = (yx \mid e)$ for all $x, y \in V$, and by Lemma 3.1, we have $(xy + yx \mid e) = -2(x \mid y)$. Thus, $(xy \mid e) = -(x \mid y)$. □

Theorem 3.3. *Let A be a real pre-Hilbert weakly alternative algebra with identity e and without divisors of zero. Suppose that $\|x^2\| \leq \|x\|^2$ for all $x \in A$ and $\|e\| = 1$. Then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof. It is sufficient to prove that A is an alternative algebra.

Let $x, y \in \{e\}^\perp$ such that $(x \mid y) = 0$, according to Lemma 3.1 we have

$$xy + yx = 0. \tag{3.3}$$

This implies that

$$0 = (x, x, [x, y]) = (x, x, xy). \tag{3.4}$$

So

$$x[x(xy)] = x^2(xy) = -\|x\|^2 xy. \tag{3.5}$$

As A has nonzero divisors, then

$$x(xy) = -\|x\|^2 y = x^2 y. \tag{3.6}$$

Therefore, $(x, x, y) = 0$. Now we take two arbitrary elements $x, y \in \{e\}^\perp$, and let $z = y - \|x\|^{-2}(x \mid y)x \in \{e\}^\perp$. Or $(x \mid z) = 0$, then

$$(x, x, y) = \left(x, x, z + \|x\|^{-2}(x \mid y)x\right) = (x, x, z) = 0. \tag{3.7}$$

Let $a = \alpha e + x$ and $b = \beta e + y$ two elements in A, with $x, y \in \{e\}^\perp$ and $\alpha, \beta \in \mathbb{R}$, we have $(a - \alpha e), (b - \beta e) \in \{e\}^\perp$. Therefore $(a - \alpha e, a - \alpha e, b - \beta e) = 0$, thus $(a, a, b) = 0$. So A is a left alternative algebra. Now we show that A is a right alternative algebra, if $x, y \in \{e\}^{\perp}$ are two orthogonal elements. Then

$$(xy \mid x) = -(yx \mid x) = -\left(y \mid x^2\right) = (y \mid e) = 0 \quad \text{(Lemma 2.4)}. \tag{3.8}$$

And $(xy \mid e) = -(x \mid y) = 0$ (Remark 3.2), thus,

$$\begin{aligned} (y, x, x) &= (yx)x - yx^2 \\ &= -x(yx) + \|x\|^2 y \quad \text{(Lemma 3.1)} \\ &= x(xy) - x^2 y \\ &= -(x, x, y) \\ &= 0. \end{aligned} \tag{3.9}$$

Similarly, we prove that $(b, a, a) = 0$ for all $a, b \in A$, then A is a right alternative algebra. Thus, A is an alternative algebra, the result ensues then of Theorem 2.8. □

Corollary 3.4. *Let A be a real pre-Hilbert Jordan algebra with identity e and without divisors of zero. Suppose that $\|x^2\| \leq \|x\|^2$ for all $x \in A$ and $\|e\| = 1$, then A is finite dimensional and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Theorem 3.5. *Let A be a real pre-Hilbert weakly alternative algebra without divisors of zero. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof. A is a powers associative algebra (Proposition 2.3) then the subalgebra $A(x)$ of A, generated by $x \in A$, is associative and verifying the conditions of Theorem 2.6. Therefore, $A(x)$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, thus there is a nonzero idempotent $e \in A$ such that $xe = ex = x$; that is, A is a unital algebra of unit e (Lemma 2.1). So the result is a consequence of Theorem 3.3. □

Corollary 3.6. *Let A be a real pre-Hilbert Jordan algebra without divisors of zero. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

We give an extension of Theorem 3.3.

Theorem 3.7. *Let A be a real pre-Hilbert weakly alternative algebra without divisors of zero and satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Suppose that A containing a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof. Let $x \in A$, the subalgebra $A(a, x)$ of A generated by $\{x, a\}$ is commutative. Theorem 2.7 implies that $\|x^2\| = \|x\|^2$, thus the result is a consequence of Theorem 3.5. □

Corollary 3.8. *Let A be a real pre-Hilbert Jordan algebra without divisors of zero and satisfying $\|x^2\| \leq \|x\|^2$ for all $x \in A$. Suppose that A contains a nonzero central element a such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Remark 3.9. In the previous results the hypothesis without divisors of zero is necessary. The following example proves it.

Let H be an infinite-dimensional real Hilbert space, we define the multiplication on the vector space $A = \mathbb{R} \oplus H$ by:

$$(\alpha + x)(\beta + y) = (\alpha\beta - (x \mid y)) + (\alpha y + \beta x). \tag{3.10}$$

And the scalar product by

$$((\alpha + x) \mid (\beta + y)) = \alpha\beta + (x \mid y). \tag{3.11}$$

So A is a commutative algebra satisfying $\|a^2\| = \|a\|^2$ and $(a^2, b, a) = 0$ for all $a, b \in A$. Indeed, we put $a = \alpha + x$ and $b = \beta + y$. We have

$$\begin{aligned}\left\|(\alpha + x)^2\right\|^2 &= \left\|\left(\alpha^2 - \|x\|^2\right) + 2\alpha x\right\|^2 \\ &= \left(\alpha^2 - \|x\|^2\right)^2 + 4\alpha^2\|x\|^2 \\ &= \left(\alpha^2 + \|x\|^2\right)^2 \\ &= \|\alpha + x\|^4.\end{aligned} \tag{3.12}$$

Then $\|a^2\| = \|a\|^2$, moreover,

$$\begin{aligned}\left(a^2 b\right)a &= \left[(\alpha + x)^2(\beta + y)\right](\alpha + x) \\ &= \left[\left(\left(\alpha^2 - \|x\|^2\right) + 2\alpha x\right)(\beta + y)\right](\alpha + x) \\ &= \left[\left(\left(\alpha^2 - \|x\|^2\right)\beta - 2\alpha(xy)\right) + \left(2\alpha\beta x + \left(\alpha^2 - \|x\|^2\right)y\right)\right](\alpha + x).\end{aligned} \tag{3.13}$$

Then

$$\begin{aligned}\left(a^2 b\right)a = {}& \alpha\left[\left(\alpha^2 - \|x\|^2\right)\beta - 2\alpha(x \mid y)\right] - \left[2\alpha\beta\|x\|^2 + \left(\alpha^2 - \|x\|^2\right)y\right] \\ &+ \left[\left(\alpha^2 - \|x\|^2\right)\beta - 2\alpha(xy)\right]x + \alpha\left[2\alpha\beta x + \left(\alpha^2 - \|x\|^2\right)y\right].\end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned}\left(a^2 b\right)a = {}& \left[\left(\alpha^2 - \|x\|^2\right)(\alpha\beta - (x \mid y)) - 2\alpha^2(x \mid y) - 2\alpha\beta\|x\|^2\right] \\ &+ \left[\left(\alpha^2 - \|x\|^2\right)(\alpha y + \beta x) - 2\alpha(\alpha\beta - (x \mid y))x\right].\end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned} a^2(ba) &= (\alpha+x)^2\left[(\beta+y)(\alpha+x)\right] \\ &= \left[\left(\alpha^2-\|x\|^2\right)+2\alpha x\right]\left[(\alpha\beta-(x\mid y))+(\alpha y+\beta x)\right]. \end{aligned} \tag{3.16}$$

Thus,

$$\begin{aligned} \left(a^2b\right)a &= \left[\left(\alpha^2-\|x\|^2\right)(\alpha\beta-(x\mid y))-2\alpha^2(x\mid y)-2\alpha\beta\|x\|^2\right] \\ &\quad+\left[\left(\alpha^2-\|x\|^2\right)(\alpha y+\beta x)-2\alpha(\alpha\beta-(x\mid y))x\right]. \end{aligned} \tag{3.17}$$

From the two equalities (3.15) and (3.17), we conclude that $(a^2b)a = a^2(ba)$; that is, $(a^2, b, a) = 0$ for all $a, b \in A$. This implies that A is an infinite-dimensional real pre-Hilbert Jordan (weakly alternative) algebra with identity satisfying $\|a^2\| = \|a\|^2$ and has a zero divisors. Indeed, let x and y be two orthogonal nonzero elements in H, as defined multiplication of A, we have $xy = -(x \mid y) = 0$. Hence, A is an algebra with zero divisors.

4. Complex Pre-Hilbert Noncommutative Jordan Algebras Satisfying $\|x^2\| = \|x\|^2$

We show that if A is a noncommutative Jordan complex pre-Hilbert algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.

4.1. Complex Pre-Hilbert Alternative Algebras Satisfying $\|x^2\| = \|x\|^2$

We need the following results.

Proposition 4.1 (see [3]). *Let A be a complex pre-Hilbert commutative associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Theorem 4.2 (see [3]). *Let A be a complex pre-Hilbert algebra with identity e. Suppose that $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Lemma 4.3 (see [3]). *Let A be a complex pre-Hilbert commutative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A has nonzero divisors.*

Theorem 4.4 (see [3]). *Let A be a complex pre-Hilbert commutative algebraic algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Lemma 4.5. *Let A be a complex pre-Hilbert alternative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A has nonzero divisors.*

Proof. Let a be a nonzero element in A and let b an element in A such that $ab = 0$. The subalgebra $A(a, b)$ of A generated by $\{a, b\}$ is associative (Theorem 2.5). We have $\|ba\|^2 = \|(ba)^2\| = \|baba\| = 0$ then $ba = ab = 0$. Thus, $A(a, b)$ is a commutative and associative, therefore, the Proposition 4.1 complete the demonstration. □

Theorem 4.6. *Let A be a complex pre-Hilbert alternative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$, the subalgebra $A(a)$ of A generated by a is commutative and associative (Theorem 2.5). Proposition 4.1 proves that $A(a)$ is isomorphic to $\mathbb{C}$, then there exists a nonzero idempotent $f \in A$. According to Theorem 4.2 it is sufficient to prove that f is a unit element of A. Let $b \in A$, we have $f(b - fb) = 0$ and $(b - bf)f = 0$. As A is without divisors of zero (Lemma 4.5), then $fb = bf = b$. Thus, A is finite dimensional and is isomorphic to $\mathbb{C}$. □

4.2. Complexes Pre-Hilbert Powers Associative Algebras Satisfying $\|x^2\| = \|x\|^2$

In this subparagraph we show that if $(A, \|\cdot\|)$ is a complex pre-Hilbert powers associative algebra (resp., flexible algebraic algebra, noncommutative Jordan algebra, or weakly alternative algebra) satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A is finite dimensional and is isomorphic to $\mathbb{C}$.

We have the following importing result.

Lemma 4.7. *Let A be a complex pre-Hilbert powers associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$. Then A has nonzero divisors.*

Proof. Let a be a nonzero element in A, the subalgebra $A(a)$ of A is associative. According to Theorem 4.6, $A(a)$ is isomorphic to $\mathbb{C}$. Therefore, there exist a nonzero idempotent $e \in A$ and $\alpha \in \mathbb{R} - \{0\}$ such that $a = \alpha e$. Suppose there is a nonzero element $b \in \{a\}^{\perp}$, as $A(b)$ is isomorphic to $\mathbb{C}$ (Theorem 4.6), then there exist a nonzero idempotent $f \in A$ and $\beta \in \mathbb{R} - \{0\}$ such that $b = \beta f$. We have $(e + f)^2 = e + f + ef + fe$, and

$$(e - f)^2 = e + f - ef - fe = 2(e + f) - (e + f)^2. \tag{4.1}$$

This implies that $(e - f)^2 \in A(e + f) \cap A(e - f) = \{0\}$, because $(e + f \mid e - f) = (e \mid f) = 0$. Thus, $(e - f)^2 = 0$ or

$$0 = \left\|(e - f)^2\right\| = \|e - f\|^2 = 2. \tag{4.2}$$

This is absurd and hence, A has nonzero divisors. □

Theorem 4.8. *Let A be a complex pre-Hilbert powers associative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. According to Lemma 4.7, A has a nonzero divisors. Let a be a nonzero element in A, then the subalgebra $A(a)$ of A is associative. Theorem 4.6 implies that $A(a)$ is isomorphic to $\mathbb{C}$. Hence, A containing a nonzero idempotent, this gives that A has a unit element (Lemma 2.1). The result is a consequence of Theorem 4.2. □

Theorem 4.9. *Let A be a complex pre-Hilbert flexible algebraic algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$ be a nonzero element, according to Proposition 2.2 and Lemma 4.3, the subalgebra $A(a)$ of A is commutative, algebraic, and without divisors of zero. Thus $A(a)$ is isomorphic to $\mathbb{C}$ (Theorem 4.4). This implies that A is a powers associative algebra, then the result is a consequence of Theorem 4.8. □

We state the main theorem.

Theorem 4.10. *Let A be a complex pre-Hilbert noncommutative Jordan algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. Proposition 2.3 implies that A is a powers associative algebra, and hence, A is isomorphic to $\mathbb{C}$ (Theorem 4.8). □

Corollary 4.11. *Let A be a complex pre-Hilbert weakly alternative algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. A is a noncommutative Jordan algebra. By Theorem 4.10, A is finite dimensional and is isomorphic to $\mathbb{C}$. □

Corollary 4.12. *Let A be a complex pre-Hilbert Jordan algebra satisfying $\|x^2\| = \|x\|^2$ for all $x \in A$, then A is finite dimensional and is isomorphic to $\mathbb{C}$.*

Proof. A is a weakly alternative algebra. By Corollary 4.11, A is finite dimensional and is isomorphic to $\mathbb{C}$. □

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