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Research Article

Some Classes of Pseudosymmetric Contact Metric 3-Manifolds

Evaggelia Moutafi

Department of Mathematics, University of Thessaloniki, Adendron 57007, Greece

Correspondence should be addressed to Evaggelia Moutafi, moutafi@sch.gr

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We study (a) the class of 3-dimensional pseudosymmetric contact metric manifolds with harmonic curvature and *Trl* constant along the direction of ξ and (b) the class of (κ, μ, ν)-contact metric pseudosymmetric 3-manifolds of type constant in the direction of ξ.

1. Introduction

A Riemannian manifold (M^m, g) is said to be pseudosymmetric according to Deszcz [1] if its curvature tensor R satisfies the condition $R(X,Y)\cdot R = L\{(X\wedge Y)\cdot R\}$, where the dot means that $R(X,Y)$ acts as a derivation on R, L is a smooth function and the endomorphism field $X\wedge Y$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \tag{1.1}$$

for all vectors fields X, Y, Z on M and it similarly acts as a derivation on R.

If L is constant, M is called a pseudosymmetric manifold of constant type and if particularly $L = 0$ then M is called a semisymmetric manifold first studied by E. Cartan. Semisymmetric spaces [2, 3] are a generalization of locally symmetric spaces ($\nabla R = 0$, [4]) while pseudosymmetric spaces are a natural generalization of semisymmetric spaces. There are many details and examples on pseudosymmetric manifolds in [1, 5]. We remark that in dimension three, the pseudosymmetry is equivalent to the condition: the eigenvalues $\rho_1,\ \rho_2,\ \rho_3$ of the Ricci tensor satisfy $\rho_1 = \rho_2$ (up to numeration) and the last one is constant [6, 7].

Kowalski and Sekizawa have studied [7–10] 3-dimensional pseudosymmetric spaces of constant type. Hashimoto and Sekizawa classified 3-dimensional conformally flat pseudosymmetric spaces of constant type [11] and finally Calvaruso [12] gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions >2. Cho and Inoguchi [13] studied pseudosymmetric contact homogeneous 3-manifolds while Cho et al. [14] give the conditions so as 3-dimensional trans-Sasakians, quasi-Sasakians, non-Sasakian generalized (κ, μ)-spaces to be pseudosymmetric. Belkhelfa et al. [15] studied pseudosymmetric Sasakian space forms of any dimension. Finally Gouli-Andreou and Moutafi in [16, 17] have studied some classes of pseudosymmetric contact metric 3-manifolds.

The aim of this paper is the study of the 3-dimension pseudosymmetric contact metric manifolds. The paper is organized in the following way: in Section 2 we will give some preliminaries on pseudosymmetric manifolds and contact manifolds as well and in the next sections we will study 3-dimensional manifolds which satisfy one of the following conditions.

(i) M is a pseudosymmetric contact metric manifold with harmonic curvature and Trl constant along the direction of ξ.

(ii) M is a (κ, μ, ν)-contact metric pseudosymmetric manifold of type constant in the direction of ξ.

2. Preliminaries

Let (M^m, g), $m \geq 3$ be a connected Riemannian smooth manifold. We denote by R its Riemannian curvature tensor given by the equation $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for any $X, Y, Z \in \mathfrak{X}(M)$ and where ∇ is the Levi-Civita connection of M^m.

Definition 2.1. A Riemannian manifold (M^m, g), $m \geq 3$, is called *pseudosymmetric* in the sense of Deszcz [1] if at every point of M the curvature tensor satisfies the condition:

$$(R(X,Y) \cdot R)(X_1, X_2, X_3) = L\{((X \wedge Y) \cdot R)(X_1, X_2, X_3)\} \tag{2.1}$$

or more explixitly:

$$\begin{aligned} R(X,Y)(R(X_1,X_2)X_3) &- R(R(X,Y)X_1,X_2)X_3 - R(X_1,R(X,Y)X_2)X_3 - R(X_1,X_2)(R(X,Y)X_3) \\ &= L\{(X \wedge Y)(R(X_1,X_2)X_3) - R((X \wedge Y)X_1,X_2)X_3 \\ &\quad -R(X_1,(X \wedge Y)X_2)X_3 - R(X_1,X_2)((X \wedge Y)X_3)\}, \end{aligned} \tag{2.2}$$

for any $X, Y, X_1, X_2, X_3 \in \mathfrak{X}(M)$, $X \wedge Y$ is given by (1.1) and L is a smooth function.

Definition 2.2. A differentiable manifold M^{2n+1} endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M is called a *contact* manifold.

Given a contact manifold (M,η), there is an underlying contact metric structure (η,ξ,ϕ,g) where g is a Riemannian metric (the *associated* metric), ϕ a global tensor of type (1,1), and ξ a unique global vector field (the *characteristic* or *Reeb vector field*). A differentiable $(2n+1)$-dimensional manifold endowed with a contact metric structure (η,ξ,ϕ,g) is called a *contact metric* (Riemannian) manifold denoted by $M(\eta,\xi,\phi,g)$. The structure tensors η, ξ, ϕ, and g satisfy the equations:

$$\begin{gathered}\phi^2 = -I + \eta\otimes\xi, \qquad \eta(X) = g(X,\xi), \qquad \eta(\xi) = 1, \\ d\eta(X,Y) = g(X,\phi Y), \qquad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y).\end{gathered} \tag{2.3}$$

The associated metrics can be constructed polarizing $d\eta$ on the contact subbundle D defined by $\eta = 0$. Denoting by L the Lie differentiation and R the curvature tensor, respectively, we define the tensor fields h, l, and τ by

$$hX = \frac{1}{2}(L_\xi\phi)X, \qquad lX = R(X,\xi)\xi, \qquad \tau(X,Y) = (L_\xi g)(X,Y). \tag{2.4}$$

These tensors also satisfy the following formulas:

$$\phi\xi = h\xi = l\xi = 0, \qquad \eta\circ\phi = \eta\circ h = 0, \qquad d\eta(\xi,X) = 0, \tag{2.5}$$

$$Trh = Trh\phi = 0, \qquad \nabla_X\xi = -\phi X - \phi hX, \qquad h\phi = -\phi h, \tag{2.6}$$

$$hX = \lambda X \implies h\phi X = -\lambda\phi X, \tag{2.7}$$

$$\nabla_\xi h = \phi - \phi l - \phi h^2, \qquad \phi l\phi - l = 2\left(\phi^2 + h^2\right), \tag{2.8}$$

$$\nabla_\xi\phi = 0, \qquad Trl = g(Q\xi,\xi) = 2n - Trh^2. \tag{2.9}$$

$h = 0$ (or equivalently $\tau = 0$) if and only if ξ is Killing and M is called *K-contact*. A contact structure on M implies an almost complex structure on the product manifold $M^{2n+1}\times\mathbb{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. More details on contact manifolds we can find in [18, 19].

Let (M,ϕ,ξ,η,g) be a 3-dimensional contact metric manifold and U the open subset of points $p\in M$ where $h\neq 0$ in a neighborhood of p and U_0 the open subset of points $p\in M$ such that $h = 0$ in a neighborhood of p. Because h is a smooth function on M then $U\cup U_0$ is an open and dense subset of M so if a property is satisfied in $U_0\cup U$ then this property will be satisfied in M. For any point $p\in U\cup U_0$ there exists a local orthonormal basis $\{e,\phi e,\xi\}$ of smooth eigenvectors of h in a neighborhood of p (a ϕ-basis). On U, we put $he = \lambda e$, where λ is a non vanishing smooth function which is supposed positive. From (2.7), we have $h\phi e = -\lambda\phi e$. We recall the following.

Lemma 2.3 (see [20]). *On U, one has*

$$\begin{gathered}
\nabla_\xi e = a\phi e, \qquad \nabla_e e = b\phi e, \qquad \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi, \\
\nabla_\xi \phi e = -ae, \qquad \nabla_e \phi e = -be + (1+\lambda)\xi, \qquad \nabla_{\phi e}\phi e = ce, \\
\nabla_\xi \xi = 0, \qquad \nabla_e \xi = -(1+\lambda)\phi e, \qquad \nabla_{\phi e}\xi = (1-\lambda)e,
\end{gathered} \tag{2.10}$$

where a is a smooth function and

$$\begin{aligned}
b &= \frac{1}{2\lambda}\left[(\phi e \cdot \lambda) + A\right] \quad \textit{with } A = S(\xi, e), \\
c &= \frac{1}{2\lambda}\left[(e \cdot \lambda) + B\right] \quad \textit{with } B = S(\xi, \phi e).
\end{aligned} \tag{2.11}$$

From Lemma 2.3 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ we can prove that

$$\begin{aligned}
[e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\
[e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a + \lambda + 1)\phi e, \\
[\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a - \lambda + 1)e,
\end{aligned} \tag{2.12}$$

and from (1.1) we estimate

$$\begin{gathered}
(e \wedge \phi e)e = -\phi e, \qquad (e \wedge \xi)e = -\xi, \qquad (\phi e \wedge \xi)\xi = \phi e, \\
(e \wedge \phi e)\phi e = e, \qquad (e \wedge \xi)\xi = e, \qquad (\phi e \wedge \xi)\phi e = -\xi,
\end{gathered} \tag{2.13}$$

while $(X \wedge Y)Z = 0$, whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations we calculate the non vanishing independent components of the Riemannian curvature tensor field R (1,3):

$$\begin{gathered}
R(\xi, e)\xi = -Ie - Z\phi e, \qquad R(e, \phi e)e = -C\phi e - B\xi, \\
R(\xi, \phi e)\xi = -Ze - D\phi e, \qquad R(\xi, e)\phi e = -Ke + Z\xi, \\
R(e, \phi e)\xi = Be - A\phi e, \qquad R(\xi, \phi e)\phi e = He + D\xi, \\
R(\xi, e)e = K\phi e + I\xi, \qquad R(e, \phi e)\phi e = Ce + A\xi, \\
R(\xi, \phi e)e = -H\phi e + Z\xi,
\end{gathered} \tag{2.14}$$

where

$$\begin{aligned}
C &= -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b), \\
H &= b(\lambda - a - 1) + (\xi \cdot c) + (\phi e \cdot a), \\
K &= c(\lambda + a + 1) + (\xi \cdot b) - (e \cdot a), \\
I &= -2a\lambda - \lambda^2 + 1, \\
D &= 2a\lambda - \lambda^2 + 1, \\
Z &= \xi \cdot \lambda.
\end{aligned} \tag{2.15}$$

Setting $X = e$, $Y = \phi e$, $Z = \xi$ in the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ and using (2.12), we get

$$\begin{aligned}
b(a + \lambda + 1) - (\xi \cdot c) - (\phi e \cdot \lambda) - (\phi e \cdot a) &= 0, \\
c(a - \lambda + 1) + (\xi \cdot b) + (e \cdot \lambda) - (e \cdot a) &= 0
\end{aligned} \tag{2.16}$$

or equivalently: $A = H$ and $B = K$.

We give the components of the Ricci operator Q with respect to a ϕ-basis:

$$\begin{aligned}
Qe &= \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right)e + Z\phi e + A\xi, \\
Q\phi e &= Ze + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right)\phi e + B\xi, \\
Q\xi &= Ae + B\phi e + 2\left(1 - \lambda^2\right)\xi,
\end{aligned} \tag{2.17}$$

where

$$r = TrQ = 2\left[1 - \lambda^2 - b^2 - c^2 + 2a + (e \cdot c) + (\phi e \cdot b)\right] \tag{2.18}$$

is the scalar curvature. The relations (2.15) and (2.18) yield

$$C = -b^2 - c^2 + \lambda^2 - 1 + 2a + (e \cdot c) + (\phi e \cdot b) = 2\lambda^2 - 2 + \frac{r}{2}, \tag{2.19}$$

and the relation (2.9):

$$Trl = 2\left(1 - \lambda^2\right). \tag{2.20}$$

Remark 2.4. If $M^3 = U_0$ (see [21]), Lemma 2.3 is expressed in a similar form with $\lambda = 0$, e is a unit vector field belonging to the contact distribution and for the functions A, B, D, H, I, K and Z we have: $A = B = Z = H = K = 0$, $I = D = 1$ and $C = r/2 - 2$.

Definition 2.5. An M^m Riemannian manifold has harmonic curvature if the Ricci operator Q satisfies the condition:

$$(\nabla_X Q)Y = (\nabla_Y Q)X, \quad \forall X, Y \in \mathfrak{X}(M). \tag{2.21}$$

From now on we shall work on a $(M^3, \phi, \xi, \eta, g)$ contact metric 3-manifold concerning a ϕ-basis $\{e, \phi e, \xi\}$ at any point $p \in M$. First from the equation $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y)$, Lemma 2.3 and the relations (2.17), we get the equations:

$$\begin{aligned}
(\nabla_e Q)\phi e = {}& \left[(e \cdot \xi \cdot \lambda) - 2b\lambda(2a + \lambda + 1) + (1 + \lambda)(\phi e \cdot \lambda)\right]e \\
&+ \left[\frac{(e \cdot r)}{2} + 2b(\xi \cdot \lambda) + 2\lambda(e \cdot a) + (4\lambda + 2a + 2)(e \cdot \lambda) - 4c\lambda(1 + \lambda)\right]\phi e \\
&+ \Big[\frac{r}{2}(\lambda + 1) + 2\lambda(e \cdot c) + 2c(e \cdot \lambda) - (e \cdot e \cdot \lambda) + 3\left(\lambda^2 - 1\right)(\lambda + 1) \\
&\quad - b(\phi e \cdot \lambda) + 2\lambda\left(b^2 + a\lambda + a\right)\Big]\xi, \\
(\nabla_e Q)\xi = {}& \left[3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) - 2bc\lambda + (1 + \lambda)(\xi \cdot \lambda)\right]e \\
&+ \Big[\frac{r}{2}(\lambda + 1) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) - (e \cdot e \cdot \lambda) + 3\left(\lambda^2 - 1\right)(\lambda + 1) \\
&\quad - b(\phi e \cdot \lambda) + 2\lambda\left(b^2 + a\lambda + a\right)\Big]\phi e \\
&+ \left[4c\lambda(1 + \lambda) - (2 + 6\lambda)(e \cdot \lambda)\right]\xi, \\
(\nabla_{\phi e} Q)e = {}& \left[\frac{(\phi e \cdot r)}{2} + (4\lambda - 2a - 2)(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot a) + 4b\lambda(1 - \lambda) + 2c(\xi \cdot \lambda)\right]e \\
&+ \left[4ac\lambda + (\phi e \cdot \xi \cdot \lambda) + 2c\lambda(1 - \lambda) + (\lambda - 1)(e \cdot \lambda)\right]\phi e \\
&+ \Big[\frac{r}{2}(\lambda - 1) + 2\lambda(\phi e \cdot b) - c(e \cdot \lambda) - (\phi e \cdot \phi e \cdot \lambda) + 3\left(\lambda^2 - 1\right)(\lambda - 1) \\
&\quad + 2b(\phi e \cdot \lambda) + 2\lambda\left(c^2 - a\lambda + a\right)\Big]\xi, \\
(\nabla_{\phi e} Q)\xi = {}& \Big[\frac{r}{2}(\lambda - 1) + 2\lambda(\phi e \cdot b) - (\phi e \cdot \phi e \cdot \lambda) - c(e \cdot \lambda) + 3\left(\lambda^2 - 1\right)(\lambda - 1) \\
&\quad + 2b(\phi e \cdot \lambda) + 2\lambda\left(c^2 - a\lambda + a\right)\Big]e \\
&+ \left[-2bc\lambda + 3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) + (\lambda - 1)(\xi \cdot \lambda)\right]\phi e \\
&+ \left[4b\lambda(\lambda - 1) - (6\lambda - 2)(\phi e \cdot \lambda)\right]\xi,
\end{aligned}$$

$$
\begin{aligned}
(\nabla_{\xi} Q)e &= \left[\frac{(\xi \cdot r)}{2} + (2\lambda - 4a)(\xi \cdot \lambda) - 2\lambda(\xi \cdot a)\right]e \\
&\quad + \left[-4a^2\lambda + (\xi \cdot \xi \cdot \lambda)\right]\phi e \\
&\quad + \left[-2ac\lambda + 2b(\xi \cdot \lambda) + 2\lambda(\xi \cdot b) - (\xi \cdot \phi e \cdot \lambda) + a(e \cdot \lambda)\right], \\
(\nabla_{\xi} Q)\phi e &= \left[(\xi \cdot \xi \cdot \lambda) - 4a^2\lambda\right]e \\
&\quad + \left[\frac{(\xi \cdot r)}{2} + (2\lambda + 4a)(\xi \cdot \lambda) + 2\lambda(\xi \cdot a)\right]\phi e \\
&\quad + \left[2ab\lambda + 2c(\xi \cdot \lambda) + 2\lambda(\xi \cdot c) - (\xi \cdot e \cdot \lambda) - a(\phi e \cdot \lambda)\right]\xi.
\end{aligned}
\tag{2.22}
$$

Applying (2.21) to the vectors fields of the ϕ-basis of the contact metric manifold M^3 we have: $(\nabla_e Q)\phi e = (\nabla_{\phi e} Q)e$, $(\nabla_e Q)\xi = (\nabla_{\xi} Q)e$ and $(\nabla_{\phi e} Q)\xi = (\nabla_{\xi} Q)\phi e$. We use the previous relations and we get the following nine (9) conditions for a contact metric 3-manifold to have harmonic curvature:

$$
\begin{aligned}
&(e \cdot \xi \cdot \lambda) + (3 - 3\lambda + 2a)(\phi e \cdot \lambda) - \frac{(\phi e \cdot r)}{2} - 2c(\xi \cdot \lambda) \\
&\qquad + 2\lambda(\phi e \cdot a) - 2b\lambda(2a + 3 - \lambda) = 0, \\
&2b(\xi \cdot \lambda) + 2\lambda(e \cdot a) + (3\lambda + 2a + 3)(e \cdot \lambda) + \frac{(e \cdot r)}{2} \\
&\qquad - (\phi e \cdot \xi \cdot \lambda) - 2c\lambda(3 + 2a + \lambda) = 0, \\
&r + 2\lambda(e \cdot c) - 2\lambda(\phi e \cdot b) + 3c(e \cdot \lambda) - 3b(\phi e \cdot \lambda) - (e \cdot e \cdot \lambda) \\
&\qquad + 6\left(\lambda^2 - 1\right) + (\phi e \cdot \phi e \cdot \lambda) + 2\lambda\left(b^2 - c^2 + 2a\lambda\right) = 0, \\
&3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) + 2\lambda(\xi \cdot a) - \frac{(\xi \cdot r)}{2} - 2bc\lambda + (4a + 1 - \lambda)(\xi \cdot \lambda) = 0, \\
&-(2 + 6\lambda + a)(e \cdot \lambda) - 2b(\xi \cdot \lambda) - 2\lambda(\xi \cdot b) + 2\lambda c(a + 2 + 2\lambda) + (\xi \cdot \phi e \cdot \lambda) = 0, \\
&(a - 6\lambda + 2)(\phi e \cdot \lambda) - 2c(\xi \cdot \lambda) - 2\lambda(\xi \cdot c) + 2b\lambda(2\lambda - 2 - a) + (\xi \cdot e \cdot \lambda) = 0, \\
&3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) - 2\lambda(\xi \cdot a) - \frac{(\xi \cdot r)}{2} - 2bc\lambda - (\lambda + 1 + 4a)(\xi \cdot \lambda) = 0, \\
&\frac{r}{2}(\lambda + 1) - b(\phi e \cdot \lambda) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) - (e \cdot e \cdot \lambda) \\
&\qquad - (\xi \cdot \xi \cdot \lambda) + 3\left(\lambda^2 - 1\right)(\lambda + 1) + 2\lambda\left(2a^2 + b^2 + a\lambda + a\right) = 0,
\end{aligned}
$$

$$\frac{r}{2}(\lambda-1)+2b(\phi e\cdot\lambda)+2\lambda(\phi e\cdot b)-c(e\cdot\lambda)-(\phi e\cdot\phi e\cdot\lambda)-(\xi\cdot\xi\cdot\lambda)$$
$$+3\left(\lambda^2-1\right)(\lambda-1)+2\lambda\left(2a^2+c^2-a\lambda+a\right)=0. \tag{2.23}$$

Remark 2.6. From these nine conditions, we can derive some useful results: (a) by subtracting the ninth equation from the first and using (2.16), we get $\phi e\cdot r=0$, (b) by adding the equations two and six and using similarly (2.16) we have $e\cdot r=0$. From the relations $\phi e\cdot r=0$, $e\cdot r=0$, and (2.12), we can conclude $\xi\cdot r=0$ and hence we are led to the known result that the scalar curvature r is constant in a contact metric 3-manifold with harmonic curvature. Later for our study, we will use the r as a constant and we will give to these equations a more convenient form.

Definition 2.7 (see [22]). Let M^3 be a 3-dimensional contact metric manifold and $h=\lambda h^+-\lambda h^-$ the spectral decomposition of h on U. If

$$\nabla_{h^-X}h^-X=[\xi,h^+X], \tag{2.24}$$

for all vector fields X on M^3 and all points of an open subset W of U and $h=0$ on the points of M^3 which do not belong to W, then the manifold is said to be semi-K *contact* manifold. From Lemma 2.3 and the relations (2.12), the above condition for $X=e$ leads to $[\xi,e]=0$ and for $X=\phi e$ to $\nabla_{\phi e}\phi e=0$. Hence on a semi-K contact manifold we have $a+\lambda+1=c=0$. If we apply the deformation $e\to\phi e$, $\phi e\to e$, $\xi\to-\xi$, $\lambda\to-\lambda$, $b\to c$, and $c\to b$ then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a-\lambda+1=b=0$.

Definition 2.8. A (κ,μ,ν)-contact metric manifold is defined in [23] as a contact metric manifold $(M^{2n+1},\eta,\xi,\phi,g)$ on which the curvature tensor satisfies for every $X,Y\in X(M)$ the condition:

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY)$$
$$+\nu(\eta(Y)\phi hX-\eta(X)\phi hY), \tag{2.25}$$

where κ,μ,ν are smooth functions on M. If $\nu=0$ we have a generalized (κ,μ)-contact metric manifold [24] and if additionally κ,μ are constants then the manifold is a contact metric (κ,μ)-space [25, 26]. Moreover, in [23] and [24] it is proved, respectively, that for a (κ,μ,ν) or a generalized (κ,μ)-contact metric manifold M^{2n+1} of dimension greater than 3 the functions κ,μ are constants and ν is the zero function.

Now, we will give some known results concerning contact metric 3-manifolds and pseudosymmetric contact metric 3-manifolds.

Proposition 2.9 (see [16]). *In a 3-dimensional contact metric manifold one has*

$$Q\phi=\phi Q\Longleftrightarrow(\xi\cdot\lambda=2b\lambda-(\phi e\cdot\lambda)=2c\lambda-(e\cdot\lambda)=a\lambda=0). \tag{2.26}$$

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In case $M = U_0$, that is, (ξ, η, ϕ, g) is a Sasakian structure, then M is a pseudosymmetric space of constant type [13]. Next, assume that U is not empty and let $\{e, \phi e, \xi\}$ be a ϕ-basis as in Lemma 2.3. We have the following.

Lemma 2.10 (see [16]). *Let (M, η, g, ϕ, ξ) be a contact metric three manifold. Then M is pseudosymmetric if and only if*

$$\begin{gathered} B(\xi \cdot \lambda) + \left(-2a\lambda - \lambda^2 + 1\right) A = LA, \\ A(\xi \cdot \lambda) + \left(2a\lambda - \lambda^2 + 1\right) B = LB, \\ (\xi \cdot \lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right) + AB = L(\xi \cdot \lambda), \\ A^2 - |(\xi \cdot \lambda)|^2 + \left(2a\lambda - \lambda^2 + 1\right)\left(-2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right) = L\left(-2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right), \\ B^2 - |(\xi \cdot \lambda)|^2 + \left(-2a\lambda - \lambda^2 + 1\right)\left(2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right) = L\left(2a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right), \end{gathered} \tag{2.27}$$

where L is the function in the pseudosymmetry definition (2.2).

Using (2.15), (2.19), the system (2.27) takes a more convenient form:

$$\begin{gathered} ZB + IA = LA, \\ ZA + DB = LB, \\ ZC + AB = LZ, \\ A^2 - Z^2 + D(I - C) = L(I - C), \\ B^2 - Z^2 + I(D - C) = L(D - C). \end{gathered} \tag{2.28}$$

Remark 2.11. If $L = 0$, the manifold is semisymmetric and the above system (2.28) is in accordance with equations (3.1)–(3.5) in [27].

Proposition 2.12 (see [16]). *Let M^3 be a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$. Then, M^3 is a pseudosymmetric space of constant type.*

3. Pseudosymmetric Contact Metric 3-Manifolds with Harmonic Curvature and Trl Constant in the Direction of ξ

Theorem 3.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold with harmonic curvature and Trl constant in the direction of ξ. Then, there are at most eight open subsets of M^3 for which their union is an open and dense subset of M^3 and each of them as an open submanifold of M^3 is either: (a) Sasakian or (b) flat or (c) locally isometric to the Lie groups $SU(2)$, $SL(2, R)$ equipped with a left invariant metric or (d) pseudosymmetric of constant type and with scalar curvature $r = 2(1 - \lambda^2 + 2a)$ or (e) semi-K contact with $L = -3a^2 - 4a$ $(a \neq 0)$ or (f) semi-K contact with $L = a^2$ $(a \neq 0)$ or (g) semi-K contact of type constant along ξ and ϕe or (h) semi-K contact of type constant along ξ and e.*

Proof. We consider the next open subsets of M:

$$U_0 = \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\},$$
$$U = \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\}, \tag{3.1}$$

where $U_0 \cup U$ is open and dense subset of M.

In case $M = U_0$, M is a pseudosymmetric space of constant type [13] and we get the (a) case of present Theorem 3.1. Next, assume that U is not empty and let $\{e, \phi e, \xi\}$ be a ϕ-basis. First we note that in the neighborhood U where $\lambda \neq 0$ we have from (2.20)

$$\xi \cdot Trl = 0 \Longleftrightarrow \xi \cdot \lambda = 0. \tag{3.2}$$

Equations (2.23) because of (3.2) and the fact that r is constant become, respectively,

$$(3 - 3\lambda + 2a)(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot a) - 2b\lambda(2a + 3 - \lambda) = 0, \tag{3.3}$$

$$2\lambda(e \cdot a) + (3\lambda + 2a + 3)(e \cdot \lambda) - 2c\lambda(3 + 2a + \lambda) = 0, \tag{3.4}$$

$$\begin{aligned} r + 2\lambda(e \cdot c) - 2\lambda(\phi e \cdot b) + 3c(e \cdot \lambda) - 3b(\phi e \cdot \lambda) - (e \cdot e \cdot \lambda) + 6(\lambda^2 - 1) \\ + (\phi e \cdot \phi e \cdot \lambda) + 2\lambda(b^2 - c^2 + 2a\lambda) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 3b(e \cdot \lambda) + 2\lambda(e \cdot b) - (e \cdot \phi e \cdot \lambda) + 2\lambda(\xi \cdot a) - 2bc\lambda = 0, \\ \text{or} \quad 3b(e \cdot \lambda) + (e \cdot A) - 6bc\lambda + 2\lambda(\xi \cdot a) = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{r}{2}(\lambda + 1) - b(\phi e \cdot \lambda) + 2c(e \cdot \lambda) + 2\lambda(e \cdot c) + 2\lambda(2a^2 + b^2 + a\lambda + a) \\ - (e \cdot e \cdot \lambda) + 3(\lambda^2 - 1)(\lambda + 1) = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} -(2 + 6\lambda + a)(e \cdot \lambda) - 2\lambda(\xi \cdot b) + 2\lambda c(a + 2 + 2\lambda) + (\xi \cdot \phi e \cdot \lambda) = 0, \\ \text{or} \quad -(2 + 6\lambda + a)(e \cdot \lambda) + 2\lambda c(a + 2 + 2\lambda) - (\xi \cdot A) = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \frac{r}{2}(\lambda - 1) + 2b(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot b) - c(e \cdot \lambda) - (\phi e \cdot \phi e \cdot \lambda) \\ + 3(\lambda^2 - 1)(\lambda - 1) + 2\lambda(2a^2 + c^2 - a\lambda + a) = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} 3c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c) - (\phi e \cdot e \cdot \lambda) - 2\lambda(\xi \cdot a) - 2bc\lambda = 0, \\ \text{or} \quad 3c(\phi e \cdot \lambda) + (\phi e \cdot B) - 2\lambda(\xi \cdot a) - 6bc\lambda = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} (a - 6\lambda + 2)(\phi e \cdot \lambda) - 2\lambda(\xi \cdot c) + 2b\lambda(2\lambda - 2 - a) + (\xi \cdot e \cdot \lambda) = 0, \\ \text{or} \quad (a - 6\lambda + 2)(\phi e \cdot \lambda) - (\xi \cdot B) + 2b\lambda(2\lambda - 2 - a) = 0, \end{aligned} \tag{3.11}$$

where for the second form of (3.6), (3.8), we also used $A = 2b\lambda - (\phi e \cdot \lambda)$ and $B = 2c\lambda - (e \cdot \lambda)$ in (3.10), (3.11).

In the neighborhood U, the system (2.28) for the pseudosymmetric contact metric 3-manifolds of Lemma 2.10 because of (3.2) becomes

$$\begin{gathered}(I-L)A=0,\\ (D-L)B=0,\\ AB=0,\\ A^2+(D-L)(I-C)=0,\\ B^2+(I-L)(D-C)=0,\end{gathered}\tag{3.12}$$

where A, B, C, D, I are given by (2.15), (2.19).

Studying the third equation, we regard the following open subsets of U:

$$\begin{gathered}W=\{p\in U : A=2b\lambda-(\phi e\cdot\lambda)=0 \text{ in a neighborhood of } p\},\\ W_3=\{p\in U : A=2b\lambda-(\phi e\cdot\lambda)\neq 0 \text{ in a neighborhood of } p\},\end{gathered}\tag{3.13}$$

where $W\cup W_3$ is open and dense in the closure of U.

In W we have

$$\begin{gathered}(D-L)B=0,\\ (D-L)(I-C)=0,\\ B^2+(I-L)(D-C)=0,\end{gathered}\tag{3.14}$$

hence, we regard the subsets of W:

$$\begin{gathered}W_1=\{p\in W : B=2c\lambda-(e\cdot\lambda)=0 \text{ in a neighborhood of } p\},\\ W_2=\{p\in W : B=2c\lambda-(e\cdot\lambda)\neq 0 \text{ in a neighborhood of } p\},\end{gathered}\tag{3.15}$$

where $W_1\cup W_2$ is open and dense in the closure of W and $W_1\cup W_2\cup W_3$ is open and dense in the closure of U. We study the initial system at each W_i for $i=1,2,3$.

In W_1 the initial system (2.28) becomes

$$\begin{gathered}(D-L)(I-C)=0,\\ (I-L)(D-C)=0\end{gathered}\tag{3.16}$$

or more explicitly

$$\begin{gathered}(\phi e\cdot\lambda)=2b\lambda,\\ (e\cdot\lambda)=2c\lambda,\\ \xi\cdot\lambda=0,\\ \left[2a\lambda-2\lambda^2+2+b^2+c^2-2a-(e\cdot c)-(\phi e\cdot b)\right]\times\left(-2a\lambda-\lambda^2+1-L\right)=0,\\ \left[-2a\lambda-2\lambda^2+2+b^2+c^2-2a-(e\cdot c)-(\phi e\cdot b)\right]\times\left(2a\lambda-\lambda^2+1-L\right)=0.\end{gathered}\tag{3.17}$$

We have studied this system in [17] (Theorem 4.1) and we get the cases (b), (c), (d), (e), and (f) of the present Theorem 3.1.

In W_2 the initial system (2.28) becomes

$$\begin{gathered}D-L=0,\\ B^2+(I-L)(D-C)=0.\end{gathered}\tag{3.18}$$

Apart from (3.2), we also have the following equations:

$$(\phi e\cdot\lambda)=2b\lambda,\tag{3.19}$$

$$B=2c\lambda-(e\cdot\lambda)\neq 0,\tag{3.20}$$

$$2a\lambda-\lambda^2+1-L=0,\tag{3.21}$$

$$B^2=4a\lambda\left(2a\lambda-3\lambda^2+3-\frac{r}{2}\right)=4a\lambda\left(L-2\lambda^2+2-\frac{r}{2}\right)\tag{3.22}$$

while we will also use (3.3), (3.4), (3.6), and (3.11).

Differentiating (3.21) with respect to ξ, ϕe we get, respectively,

$$2\lambda(\xi\cdot a)=\xi\cdot L,\tag{3.23}$$

$$2\lambda(\phi e\cdot a)+4ab\lambda-4b\lambda^2=\phi e\cdot L\tag{3.24}$$

(the derivative $e\cdot L=2\lambda(e\cdot a)+2(a-\lambda)(e\cdot\lambda)$ can not be estimated any further). Equations (3.3), (3.11) because of (3.19) yield, respectively,

$$\phi e\cdot a=\phi e\cdot\lambda=2b\lambda,\tag{3.25}$$

$$\xi\cdot B=-8b\lambda^2.\tag{3.26}$$

Differentiating (3.22) with respect to ξ and using (3.2), (3.26) and the fact that r is constant, we get

$$-4b\lambda B = \left(4a\lambda - 3\lambda^2 + 3 - \frac{r}{2}\right)(\xi \cdot a). \tag{3.27}$$

The first form of (3.6) and (3.19) give

$$bB = 2\lambda(\xi \cdot a), \tag{3.28}$$

hence (3.27), (3.28) yield

$$\left(4a\lambda + 5\lambda^2 + 3 - \frac{r}{2}\right)(\xi \cdot a) = 0. \tag{3.29}$$

We suppose that there is a point $p \in W_2$ where $\xi \cdot a \neq 0$. Because of the continuity of the function $\xi \cdot a$, there is a neighborhood of this point $S \subset W_2 \subset U$: $S = \{q \in W_2 : \xi \cdot a \neq 0\}$. In S, we have $4a\lambda + 5\lambda^2 + 3 - r/2 = 0$. Differentiating this equation with respect to ξ and using (3.2), the constancy of r and the fact that we work in U where $\lambda \neq 0$ we conclude that $\xi \cdot a = 0$ in S, which is a contradiction. Hence

$$\xi \cdot a = 0, \tag{3.30}$$

everywhere in W_2. Because of (3.30) the equations (3.20), (3.27) give

$$b = 0. \tag{3.31}$$

Differentiating (3.2) with respect to ϕe, (3.19) with respect to ξ, subtracting and using (2.12), we get

$$(a - \lambda + 1)(e \cdot \lambda) = 0. \tag{3.32}$$

Let's suppose that there is a point p in W_2 where $a-\lambda+1 \neq 0$. The function $a-\lambda+1$ is continuous, hence there is an open neighborhood V of p, $V \subset W_2$, where $a - \lambda + 1 \neq 0$ everywhere in V, hence

$$e \cdot \lambda = 0. \tag{3.33}$$

From (3.20) and (3.33) we have in $V \subset W_2 \subset U$:

$$c \neq 0. \tag{3.34}$$

Equation (3.4) because of (3.33) gives $e \cdot a = c(2a + \lambda + 3)$. From the second of (2.16) and because of (3.31), (3.33), (3.34), we get in V: $e \cdot a = c(a - \lambda + 1) \neq 0$. By equalizing these two results and because of (3.34), we get: $a+2\lambda+2=0$. We differentiate this equation with respect to e and because of (3.33), we get $e\cdot a=0$, which is a contradiction in V. Hence $a-\lambda+1=0$ everywhere in W_2 and because of (3.31), we can conclude according to Definition 2.7 that the structure is semi-K contact and pseudosymmetric with L constant along the directions of ξ and ϕe because of (3.23), (3.24) and (3.25), (3.30), (3.31).

In W_3 the initial system (2.28) becomes

$$\begin{gathered} I-L=0,\\ A^2+(D-L)(I-C)=0. \end{gathered} \tag{3.35}$$

We have the following equations and (3.2):

$$(e\cdot\lambda)=2c\lambda, \tag{3.36}$$

$$A=2b\lambda-(\phi e\cdot\lambda)\neq 0, \tag{3.37}$$

$$-2a\lambda-\lambda^2+1-L=0, \tag{3.38}$$

$$A^2=-4a\lambda\left(-2a\lambda-3\lambda^2+3-\frac{r}{2}\right)=-4a\lambda\left(L-2\lambda^2+2-\frac{r}{2}\right) \tag{3.39}$$

while we will also use (3.3), (3.4), (3.6), (3.8).

Differentiating (3.38) with respect to ξ, e we get, respectively,

$$\begin{gathered} -2\lambda(\xi\cdot a)=\xi\cdot L,\\ -2\lambda(e\cdot a)-4ac\lambda-4c\lambda^2=e\cdot L\ , \end{gathered} \tag{3.40}$$

(we neglect the derivative $\phi e\cdot L$ because we can not estimate it). Equations (3.4), (3.8) because of (3.36) yield, respectively,

$$e\cdot a=-e\cdot\lambda=-2c\lambda, \tag{3.41}$$

$$\xi\cdot A=-8c\lambda^2. \tag{3.42}$$

Differentiating (3.39) with respect to ξ and using (3.2), (3.42) and the fact that r is constant, we get

$$-4c\lambda A=\left(4a\lambda+3\lambda^2-3+\frac{r}{2}\right)(\xi\cdot a). \tag{3.43}$$

The first form of (3.10) and (3.36) give

$$-cA=2\lambda(\xi\cdot a), \tag{3.44}$$

hence (3.43), (3.44) yield

$$\left(4a\lambda - 5\lambda^2 - 3 + \frac{r}{2}\right)(\xi \cdot a) = 0. \tag{3.45}$$

We suppose that there is a point $p \in W_3$ where $\xi \cdot a \neq 0$. Because of the continuity of this function, there is a neighborhood of p $S \subset W_3 \subset U$: $S = \{q \in W_3 : \xi \cdot a \neq 0\}$. In S, we have $4a\lambda - 5\lambda^2 - 3 + r/2 = 0$. Differentiating this equation with respect to ξ and using (3.2), the constancy of r and the fact that we work in U where $\lambda \neq 0$ we conclude that $\xi \cdot a = 0$ in S, which is a contradiction. Hence

$$\xi \cdot a = 0, \tag{3.46}$$

everywhere in W_3. Because of (3.46) the equations (3.37), (3.43) give

$$c = 0. \tag{3.47}$$

Differentiating (3.2) with respect to e, (3.36) with respect to ξ, subtracting and using (2.12), we get

$$-(a + \lambda + 1)(\phi e \cdot \lambda) = 0. \tag{3.48}$$

Let's suppose that there is a point p in W_3 where $a + \lambda + 1 \neq 0$. This function is smooth, then because of its continuity, there is an open neighborhood V of p, $V \subset W_3$, where $a + \lambda + 1 \neq 0$ everywhere in V, hence

$$\phi e \cdot \lambda = 0. \tag{3.49}$$

From (3.37) and (3.49) we have in $V \subset W_3 \subset U$:

$$b \neq 0. \tag{3.50}$$

From (3.3) and (3.49), we get $\phi e \cdot a = b(2a - \lambda + 3)$. From the first of (2.16) and because of (3.47), (3.49), (3.50), we get in V: $\phi e \cdot a = b(a + \lambda + 1) \neq 0$. By equalizing these two results and because of (3.50), we get: $a - 2\lambda + 2 = 0$. We differentiate this equation with respect to ϕe and because of (3.49), we get $\phi e \cdot a = 0$, which is a contradiction in V. Hence $a + \lambda + 1 = 0$ everywhere in W_3 and because of (3.47), we can conclude according to Definition 2.7 that the structure is semi-K contact and pseudosymmetric with L constant along the directions of ξ and e because of (3.40) and (3.41), (3.46), (3.47).

Finally, we remark that the cases (g) and (h) of the present Theorem 3.1 that result from the structures studied in the sets W_2 and W_3, respectively. □

Remark 3.2. (i) The conditions of harmonic curvature help us to the systems in the neighborhoods W_2 and W_3 where we had equations of the type $A^2 = -4a\lambda(-2a\lambda - 3\lambda^2 + 3 - r/2)$ and which we could not handle in our previous articles [16, 17].

(ii) In case (d) where L is constant, we can also use the classification of [11] to improve our results as the manifolds with harmonic curvature are a special case of conformally flat manifolds in dimension 3.

4. Pseudosymmetric (κ,μ,ν)-Contact Metric 3-Manifolds of Type Constant in the Direction of ξ

Theorem 4.1. *Let M^3 be a pseudosymmetric (κ,μ,ν)-contact metric 3-manifold of type constant along the direction ξ. Then, there are at most five open subsets of M^3 for which their union is an open and dense subset of M^3 and each of them as an open submanifold of M^3 is either (a) Sasakian or (b) flat or (c) pseudosymmetric of constant type $L=\kappa=1/2(Trl)$, $\mu=\nu=0$ and of constant scalar curvature $r=2\kappa$ or (d) pseudosymmetric generalized (κ,μ)-contact metric manifold of type $L=\kappa-\mu\lambda$, of scalar curvature $r=2(3\kappa-\mu\lambda)$ and $\xi\cdot\mu=\xi\cdot\kappa=0$ or (e) pseudosymmetric generalized (κ,μ)-contact metric manifold of type $L=\kappa+\mu\lambda$, of scalar curvature $r=2(3\kappa+\mu\lambda)$ and $\xi\cdot\mu=\xi\cdot\kappa=0$.*

Proof. We study pseudosymmetric (κ,μ,ν)-contact metric 3-manifolds with

$$\xi\cdot L=0, \tag{4.1}$$

where L is the function in (2.2). We consider the next open subsets of M,

$$\begin{aligned} U_0&=\{p\in M:\lambda=0 \text{ in a neighborhood of } p\},\\ U&=\{p\in M:\lambda\neq 0 \text{ in a neighborhood of } p\}, \end{aligned} \tag{4.2}$$

where $U_0\cup U$ is open and dense subset of M.

In case $M=U_0$, (M,ξ,η,ϕ,g) is a Sasakian structure which is a pseudosymmetric space of constant type [13] with $\kappa=1$, $\mu,\nu\in\mathbb{R}$ and $h=0$ and we get the (a) case of present Theorem 4.1. Next, assume that U is not empty and let $\{e,\phi e,\xi\}$ be a ϕ-basis. From (2.25), we can calculate the following components of the Riemannian curvature tensor:

$$\begin{gathered} R(\xi,e)\xi=-(\kappa+\lambda\mu)e-\lambda\nu\phi e,\qquad R(e,\phi e)\xi=0,\\ R(\xi,\phi e)\xi=-\lambda\nu e-(\kappa-\lambda\mu)\phi e. \end{gathered} \tag{4.3}$$

By virtue of (2.14) we can conclude that

$$\begin{gathered} A=2b\lambda-(\phi e\cdot\lambda)=0,\qquad B=2c\lambda-(e\cdot\lambda)=0,\qquad Z=\xi\cdot\lambda=\lambda\nu,\\ D=2a\lambda-\lambda^2+1=\kappa-\lambda\mu,\qquad I=-2a\lambda-\lambda^2+1=\kappa+\lambda\mu, \end{gathered} \tag{4.4}$$

and hence the system (2.28) becomes

$$Z(C-L)=0,$$
$$-Z^2+(D-L)(I-C)=0, \qquad (*)$$
$$-Z^2+(I-L)(D-C)=0,$$

where A, B, C, D, I, Z are given by (2.19) and (4.4). Substituting from (4.4) $(\phi e\cdot\lambda)$, $(e\cdot\lambda)$ in (2.16) we also have

$$\xi\cdot c=-(\phi e\cdot a)+b(a-\lambda+1), \tag{4.5}$$

$$\xi\cdot b=(e\cdot a)-c(\lambda+a+1). \tag{4.6}$$

First we will prove that $Z=\xi\cdot\lambda=0$ (equivalently $\nu=0$ as we work in U where $\lambda\neq 0$). We suppose that there is a point $p\in U$ where $\xi\cdot\lambda\neq 0$. By the continuity of this function, we can consider that there is a neighborhood V of p, where $\xi\cdot\lambda\neq 0$ everywhere in $V\subset U$. We work in V. Then the first equation of $(*)$ becomes $C-L=0$ or equivalently:

$$(e\cdot c)+(\phi e\cdot b)=L+b^2+c^2-\lambda^2+1-2a. \tag{4.7}$$

We differentiate this equation with respect to ξ and by virtue of (4.1) we get

$$\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(\xi\cdot b)+2c(\xi\cdot c)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a), \tag{4.8}$$

which because of (4.5), (4.6) becomes

$$\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(e\cdot a)-2c(\phi e\cdot a)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a)-4bc\lambda. \tag{4.9}$$

Next, we differentiate (4.5) and (4.6) with respect to e and ϕe, respectively, and adding we have

$$\begin{aligned} e\cdot\xi\cdot c+\phi e\cdot\xi\cdot b={}&-[e,\phi e]a-(a+\lambda+1)(\phi e\cdot c)+(a-\lambda+1)(e\cdot b)-c(\phi e\cdot a)\\ &+b(e\cdot a)-4bc\lambda. \end{aligned} \tag{4.10}$$

We subtract this last equation from (4.9) and we get

$$\begin{aligned} [\xi,e]c+[\xi,\phi e]b={}&b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)+[e,\phi e]a\\ &+(a+\lambda+1)(\phi e\cdot c)-(a-\lambda+1)(e\cdot b) \end{aligned} \tag{4.11}$$

or because of (2.12)

$$
\begin{aligned}
&(a+\lambda+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b)\\
&\quad=b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)\\
&\qquad-b(e\cdot a)+c(\phi e\cdot a)+2(\xi\cdot a)+(\lambda+a+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b)
\end{aligned}
\tag{4.12}
$$

or equivalently: $\lambda(\xi\cdot\lambda)=0$ and because we work in $V\subset U$, we have $\xi\cdot\lambda=0$, which is a contradiction. Hence, we can deduce everywhere in U:

$$
\xi\cdot\lambda=0\Longleftrightarrow\nu=0.
\tag{4.13}
$$

Next we will derive some useful relations. From (4.4) we have:

$$
\begin{gathered}
\phi e\cdot\lambda=2b\lambda,\\
e\cdot\lambda=2c\lambda.
\end{gathered}
\tag{4.14}
$$

We differentiate these equations with respect to e and ϕe, respectively, we subtract, we use the relations (2.12), (4.4) and we get

$$
\xi\cdot\lambda=\lambda\left[(e\cdot b)-(\phi e\cdot c)\right]
\tag{4.15}
$$

or because of (4.13)

$$
e\cdot b=\phi e\cdot c.
\tag{4.16}
$$

We differentiate the relations $\phi e\cdot\lambda=2b\lambda$ and (4.13) with respect to ξ and ϕe, respectively, and subtracting we obtain: $[\xi,\phi e]\lambda=2\lambda(\xi\cdot b)$ or because of (2.12), (4.4), (4.6)

$$
e\cdot a=2c\lambda.
\tag{4.17}
$$

We differentiate the relations $e\cdot\lambda=2c\lambda$ and (4.13) with respect to ξ and e, respectively, and subtracting we obtain: $[\xi,e]\lambda=2\lambda(\xi\cdot c)$ or because of (2.12), (4.4), (4.5)

$$
\phi e\cdot a=-2b\lambda.
\tag{4.18}
$$

Finally, after substituting D,I,Z from (4.4), (4.13) the final form of the system $(*)$ is

$$
\begin{gathered}
\mu(C-L)=0,\\
(\kappa-\lambda\mu-L)(\kappa+\lambda\mu-C)=0.
\end{gathered}
\tag{4.19}
$$

In order to study this system we regard the following open subsets of U:

$$V_1 = \{p \in U : C - L \neq 0 \text{ in a neighborhood of } p\},$$
$$V_2 = \{p \in U : C - L = 0 \text{ in a neighborhood of } p\}, \tag{4.20}$$

where $V_1 \cup V_2$ is open and dense in the closure of U.

In V_1, we have $\mu = 0$ and hence from (4.4): $I = D = \kappa$ or $2a\lambda - \lambda^2 + 1 = -2a\lambda - \lambda^2 + 1$ or finally $a = 0$ and $\kappa = 1 - \lambda^2$. From (4.17), (4.18) we deduce that $b = c = 0$. Having also the second equation of (4.19), we regard the open subsets of V_1

$$Y_1 = \{p \in V_1 : \kappa - C = 0 \text{ in a neighborhood of } p\},$$
$$Y_2 = \{p \in V_1 : \kappa - C \neq 0 \text{ in a neighborhood of } p\}, \tag{4.21}$$

where $Y_1 \cup Y_2$ is open and dense in the closure of V_1.

In Y_1 substituting in $\kappa - C = 0$, C from (2.15), $a = b = c = 0$, we get $\kappa = 1 - \lambda^2 = 0$ and hence the structure is flat with $\kappa = \mu = \nu$.

In Y_2 from $\mu = 0$ we have again $I = D = \kappa$, $a = 0$ and from (4.17), (4.18) $b = c = 0$ while we must also have $\kappa = L$. Hence, $L = \kappa = (1/2)Trl$ and from (2.18) of constant scalar curvature $r = 2(1 - \lambda^2)$.

In V_2 having $C = L$, the second equation of (4.19) becomes $(2a\lambda - \lambda^2 + 1 - L)(-2a\lambda - \lambda^2 + 1 - L) = 0$. Hence, we regard the open subsets of V_2

$$W_1 = \left\{p \in V_2 : -2a\lambda - \lambda^2 + 1 - L \neq 0 \text{ in a neighborhood of } p\right\},$$
$$W_2 = \left\{p \in V_2 : -2a\lambda - \lambda^2 + 1 - L = 0 \text{ in a neighborhood of } p\right\}, \tag{4.22}$$

where $W_1 \cup W_2$ is open and dense in the closure of V_2.

In W_1 we must have $2a\lambda - \lambda^2 + 1 - L = 0$ while in W_2 we have $-2a\lambda - \lambda^2 + 1 - L = 0$. We differentiate these equations with respect to ξ and because of (4.13) we get

$$\xi \cdot a = 0. \tag{4.23}$$

By virtue of I and D in (4.4) we deduce $\mu = -2a$ and hence

$$\xi \cdot \mu = 0. \tag{4.24}$$

In W_2 we differentiate $-2a\lambda - \lambda^2 + 1 - L = 0$ with respect to ξ and similarly we also obtain (4.24). Each of W_1 and W_2 is a generalized (κ, μ)-contact metric 3-manifold with $\xi \cdot \mu = 0$ and scalar curvature $r = 2(2a\lambda - 3\lambda^2 + 3) = 2(3\kappa - \mu\lambda)$ or $r = 2(-2a\lambda - 3\lambda^2 + 3) = 2(3\kappa + \mu\lambda)$ respectively and from (4.1), (4.13) and (4.23) or (4.24) $\xi \cdot \kappa = 0$ and $\xi \cdot r = 0$.

Concluding: the structure in U_0 gives the Sasakian case, the structures in Y_1 and Y_2 give the (b) and (c) cases of the present Theorem 4.1 and the structures in W_1 and W_2 give (d) and (e) respectively. □

Remark 4.2. The generalized (κ, μ)-contact metric manifolds in dimension 3 with $\kappa < 1$ (equivalently $\lambda \neq 0$) and $\xi \cdot \mu = 0$ have been studied by Koufogiorgos and Tsichlias [28]. They proved in their Theorem 4.1 of [28] that at any point of $P \in M$, precisely one of the following relations is valid: $\mu = 2(1+\sqrt{1-\kappa})$, or $\mu = 2(1-\sqrt{1-\kappa})$, while there exists a chart $(U, (x, y, z))$ with $P \in U \subseteq M$ such that the functions κ, μ depend only on z and the tensors fields η, ξ, ϕ, g take a suitable form. Each of our submanifolds W_1 and W_2 is such a generalized (κ, μ)-contact metric 3-manifold.

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References

[1] R. Deszcz, "On pseudo-symmetric spaces," *Bulletin de la Société Mathématique de Belgique. Série A*, vol. 44, no. 1, pp. 1–34, 1992.
[2] Z. I. Szabó, "Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version," *Journal of Differential Geometry*, vol. 17, no. 4, pp. 531–582, 1982.
[3] Z. I. Szabó, "Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$ II. Global versions," *Geometriae Dedicata*, vol. 19, no. 1, pp. 65–108, 1985.
[4] H. Takagi, "An example of Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$ but not $\nabla R = 0$," *The Tohoku Mathematical Journal*, vol. 24, pp. 105–108, 1972.
[5] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk, and L. Verstraelen, "On some type of curvature conditions," in *PDEs, Submanifolds and Affine Differential Geometry*, vol. 57 of *Banach Center Publications*, pp. 179–194, Institute of Mathematics of the. Polish Academy of Sciences, Warsaw, Poland, 2002.
[6] J. Deprez, R. Deszcz, and L. Verstraelen, "Examples of pseudo-symmetric conformally flat warped products," *Chinese Journal of Mathematics*, vol. 17, no. 1, pp. 51–65, 1989.
[7] O. Kowalski and M. Sekizawa, *Three—Dimensional Riemannian Manifolds of c—Conullity Two*, Riemannian Manifolds of Conullity Two, chapter 11, World Scientific, Singapore, 1996.
[8] O. Kowalski and M. Sekizawa, "Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues $\rho_1 = \rho 2 \neq \rho 3$," *Archivum Mathematicum*, vol. 32, no. 2, pp. 137–145, 1996.
[9] O. Kowalski and M. Sekizawa, "Pseudo-symmetric spaces of constant type in dimension three—elliptic spaces," *Rendiconti di Matematica e Delle sue Applicazioni. Serie VII*, vol. 17, no. 3, pp. 477–512, 1997.
[10] O. Kowalski and M. Sekizawa, "Pseudo-symmetric spaces of constant type in dimension three—non-elliptic spaces," *Bulletin of Tokyo Gakugei University*, vol. 50, pp. 1–28, 1998.
[11] N. Hashimoto and M. Sekizawa, "Three-dimensional conformally flat pseudo-symmetric spaces of constant type," *Archivum Mathematicum*, vol. 36, no. 4, pp. 279–286, 2000.
[12] G. Calvaruso, "Conformally flat pseudo-symmetric spaces of constant type," *Czechoslovak Mathematical Journal*, vol. 56, no. 2, pp. 649–657, 2006.
[13] J. T. Cho and J.-I. Inoguchi, "Pseudo-symmetric contact 3-manifolds," *Journal of the Korean Mathematical Society*, vol. 42, no. 5, pp. 913–932, 2005.
[14] J. T. Cho, J.-I. Inoguchi, and J.-E. Lee, "Pseudo-symmetric contact 3-manifolds. III," *Colloquium Mathematicum*, vol. 114, no. 1, pp. 77–98, 2009.
[15] M. Belkhelfa, R. Deszcz, and L. Verstraelen, "Symmetry properties of Sasakian space forms," *Soochow Journal of Mathematics*, vol. 31, no. 4, pp. 611–616, 2005.
[16] F. Gouli-Andreou and E. Moutafi, "Two classes of pseudosymmetric contact metric 3-manifolds," *Pacific Journal of Mathematics*, vol. 239, no. 1, pp. 17–37, 2009.
[17] F. Gouli-Andreou and E. Moutafi, "Three classes of pseudosymmetric contact metric 3-manifolds," *Pacific Journal of Mathematics*, vol. 245, no. 1, pp. 57–77, 2010.
[18] D. E. Blair, *Contact Manifolds in Riemannian geometry*, vol. 509 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

[19] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, vol. 203 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 2002.
[20] F. Gouli-Andreou and Ph. J. Xenos, "On 3-dimensional contact metric manifolds with $\nabla_{\xi}\tau = 0$," *Journal of Geometry*, vol. 62, no. 1-2, pp. 154–165, 1998.
[21] F. Gouli-Andreou and Ph. J. Xenos, "On a class of 3-dimensional contact metric manifolds," *Journal of Geometry*, vol. 63, no. 1-2, pp. 64–75, 1998.
[22] F. Gouli-Andreou, J. Karatsobanis, and Ph. Xenos, "Conformally flat $3 - \tau - a$ manifolds," *Differential Geometry—Dynamical Systems*, vol. 10, pp. 107–131, 2008.
[23] T. Koufogiorgos, M. Markellos, and V. J. Papantoniou, "The harmonicity of the Reeb vector field on contact metric 3-manifolds," *Pacific Journal of Mathematics*, vol. 234, no. 2, pp. 325–344, 2008.
[24] T. Koufogiorgos and C. Tsichlias, "On the existence of a new class of contact metric manifolds," *Canadian Mathematical Bulletin*, vol. 43, no. 4, pp. 440–447, 2000.
[25] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, "Contact metric manifolds satisfying a nullity condition," *Israel Journal of Mathematics*, vol. 91, no. 1–3, pp. 189–214, 1995.
[26] E. Boeckx, "A full classification of contact metric (κ, μ)-spaces," *Illinois Journal of Mathematics*, vol. 44, no. 1, pp. 212–219, 2000.
[27] G. Calvaruso and D. Perrone, "Semi-symmetric contact metric three-manifolds," *Yokohama Mathematical Journal*, vol. 49, no. 2, pp. 149–161, 2002.
[28] T. Koufogiorgos and C. Tsichlias, "Generalized (κ, μ)-contact metric manifolds with $\xi\mu = 0$," *Tokyo Journal of Mathematics*, vol. 31, no. 1, pp. 39–57, 2008.


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