Research Article

Sobolev Regularity in Neutron Transport Theory

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The main purpose of this paper is to extend the $W^{1,p}$ regularity results in neutron transport theory, with respect to the Lebesgue measure due to Mokhtar-Kharroubi, (1991), and to abstract measures covering, in particular, the continuous models or multigroup models. The results are obtained for vacuum boundary conditions as well as periodic boundary conditions. H^2 regularity results are derived when the velocity space is endowed with an appropriate class of measures (signed in the multidimensional case).

1. Introduction

Let D be an open bounded convex subset of $\mathbb{R}^N$. Let $d\mu$ be a positive bounded measure on $\mathbb{R}^N$ supported by $V = \{v \in \mathbb{R}^N; |v| < R\}$, $0 < R < +\infty$. We denote

$$X_p = \left\{ \psi \in L^p(dx \otimes d\mu(v)); v \cdot \frac{\partial \psi}{\partial x} \in L^p(dx \otimes d\mu(v)) \right\}, \quad (1.1)$$

where $L^p(dx \otimes d\mu(v)) = L^p(D \times V; dx \otimes d\mu(v))$, $1 \leq p < +\infty$, and dx is the Lebesgue measure on D .

The streaming operator T is defined

$$\begin{aligned} T\psi &= -v \cdot \frac{\partial \psi}{\partial x} - \sigma(v)\psi, \\ D(T) &= \{ \psi \in X_p; \psi|_{\Gamma_-} = 0 \}, \end{aligned} \quad (1.2)$$

where

$$\begin{aligned} \sigma(\cdot) &\in L^\infty(V; d\mu(v)), \\ \Gamma_- &= \{(x, v) \in \partial D \times V; v \cdot n(x) < 0\}. \end{aligned} \quad (1.3)$$

Here, $n(x)$ is the unit outward normal at $x \in \partial D$. It is well known that under the hypothesis $d\mu(\{0\}) = 0$, any function $\psi \in X_p$ possesses traces on Γ_- (see [1–4]).

Let $k_i(\cdot, \cdot)$ ($i = 1, 2$) be two measurable functions on $V \times V$ such that

$$\begin{aligned} K_i &\in \mathcal{L}(L^p(V; d\mu(v))), \text{ where } K_i \text{ is the operator with kernel } k_i, \\ \text{i.e. } K_i : \varphi &\in L^p(V; d\mu(v)) \longrightarrow \int_V k_i(\cdot, v') \varphi(v') d\mu(v'). \end{aligned} \quad (1.4)$$

Note that, for all $\lambda > -\lambda^* = -d\mu \text{ ess-inf } \sigma(\cdot)$, the operators $\mathcal{N}_\lambda = K_1(\lambda - T)^{-1}K_2$ and $\mathcal{M}_\lambda = K_1(\lambda - T)^{-1}$ map continuously $L^p(dx \otimes d\mu(v))$ into itself. In [5], several $W^{1,p}$ regularity results of the operators $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$ were established when the velocity space V is endowed with the Lebesgue measure. These results play a cornerstone role in the proof of the neutron transport approximations [6–8].

The main purpose of this paper is to extend the results of [5] in two directions. First, we deal with a general class of abstract measures covering, in particular, the continuous models or multigroup models. Secondly, we extend the results to periodic boundary conditions on the torus.

We also give a proof of the $H^{1/2}$ Sobolev regularity result, due to Agoshkov, for a general class of measures considered in [9]. We use analogous arguments as in [9] but the result is not a consequence of [9, Theorem 4], (see the commentary on the beginning of Section 4).

Finally, we prove how suitable assumptions on the abstract velocity measure (signed for $N > 1$) can be useful to derive the H^2 Sobolev regularity for the operator $\mathcal{M}_\lambda$.

Before closing the introduction, let us recall that the smoothing effect of velocity averages ($\mathcal{M}_0$, with $k_1 = 1$), in terms of the $H^{1/2}$ Sobolev regularity, is given in [10] when the velocity space is endowed with the Lebesgue measure, while a systematic analysis of the fractional Sobolev regularity, for general velocity measures in L^p space ($p > 1$), is given in [9] (see also [11–16]). These results play a cornerstone role in the analysis of kinetic models (see, e.g., [17, 18]). Finally, velocity averages turn out to play an important role in the context of inverse problems (see [19, 20]).

2. $W_x^{1,p}$ Regularity of the Operator $\mathcal{N}_\lambda = K_1(\lambda - T)^{-1}K_2$

Let $d\mu(v) = d\alpha(\rho) \otimes dS(w)$, where dS is the Lebesgue measure on $\mathbb{S}^{N-1}$ (the unit sphere of $\mathbb{R}^N$) and $d\alpha$ is a positive bounded measure on $[0, 1)$ such that (for simplicity we assume that $R = 1$)

$$\int_0^1 \frac{d\alpha(\rho)}{\rho} < \infty. \quad (2.1)$$

For $(z, v, v') \in V$, we set

$$\begin{aligned} G(z, v, v') &:= k_1(v, z)k_2(z, v'), \\ \lambda^* &:= d\mu \text{ ess-inf } \sigma(\cdot). \end{aligned} \quad (2.2)$$

Let $\psi \in L^p(dx \otimes d\mu(v))$. It is easy to show that for any $\lambda > -\lambda^*$

$$\begin{aligned} \mathcal{N}_\lambda \psi(x, v) &= K_1(\lambda - T)^{-1} K_2 \psi(x, v) \\ &= \int_V d\mu(v') \int_V d\mu(v'') \int_0^{t(x, v'')} G(v'', v, v') e^{-t(\lambda + \sigma(v''))} \psi(x - tv'', v') dt, \end{aligned} \quad (2.3)$$

where

$$t(x, v) = \inf\{t > 0, x - tv \notin D\}. \quad (2.4)$$

To prove that $\mathcal{N}_\lambda$ is an integral operator, we set

$$\begin{aligned} I &= \int_V d\mu(v'') \int_0^{t(x, v'')} G(v'', v, v') e^{-t(\lambda + \sigma(v''))} \psi(x - tv'', v') dt \\ &= \int_0^1 d\alpha(\rho) \int_{S^{N-1}} dS(w) \int_0^{t(x, w)/\rho} G(\rho w, v, v') e^{-t(\lambda + \sigma(\rho w))} \psi(x - t\rho w, v') dt \\ &= \int_0^1 \frac{d\alpha(\rho)}{\rho} \int_{S^{N-1}} dS(w) \int_0^{t(x, w)} G(\rho w, v, v') e^{-(t/\rho)(\lambda + \sigma(\rho w))} \psi(x - tw, v') dt. \end{aligned} \quad (2.5)$$

Since D is convex, the change of variables $x' = x - tw$ ($dx' = t^{N-1} dt dS(w)$) leads to

$$I = \int_0^1 \frac{d\alpha(\rho)}{\rho} \int_D \frac{G(\rho((x - x')/|x - x'|), v, v') e^{-(|x - x'|/\rho)(\lambda + \sigma(\rho((x - x')/|x - x'|)))}}{|x - x'|^{N-1}} \psi(x', v') dx'. \quad (2.6)$$

Thus, $\mathcal{N}_\lambda$ is an integral operator with kernel

$$N_\lambda(x - x', v, v') = \int_0^1 \frac{G(\rho((x - x')/|x - x'|), v, v') e^{-(|x - x'|/\rho)(\lambda + \sigma(\rho((x - x')/|x - x'|)))}}{|x - x'|^{N-1}} \frac{d\alpha(\rho)}{\rho}. \quad (2.7)$$

Let us now introduce the following hypotheses:

(A1)

$$\begin{aligned} \sigma(z) &= \sigma(-z), \quad d\mu(v) \text{ a.e.}, \\ G(z, v, v') &= G(-z, v, v'), \quad d\mu(v) \text{ a.e.}, \end{aligned} \quad (2.8)$$

(A2)

$G^\infty \in \mathcal{L}(L^p(V; d\mu(v)))$, where

G^∞ is the integral operator with kernel $G^\infty(v, v') := \sup_{z \in V} |G(z, v, v')|$, (2.9)

(A3)

$\bar{G} \in \mathcal{L}(L^p(V; d\mu(v)))$, where

$\bar{G}$ is the integral operator with kernel $\bar{G}(v, v') := \sup_{z \in V} \frac{|G(z, v, v')|}{|z|}$, (2.10)

(A4)

$G_i \in \mathcal{L}(L^p(V; d\mu(v)))$, where

G_i is the integral operator with kernel $G_i(v, v') := \sup_{z \in V} \left| \frac{\partial G}{\partial z_i}(z, v, v') \right|$, $1 \leq i \leq N$, (2.11)

(A5)

$$\sigma(\cdot) \in W^{1,\infty}(V; d\mu(v)). \quad (2.12)$$

Remark 2.1. (1) (A1) is the key assumption.

(2) Assumption (A2) is a consequence of (A3).

(3) Assumption (A3) is not necessary if $d\mu$ is the Lebesgue measure (see [5]).

Throughout this paper we set

$$\bar{\sigma}(\cdot) = \lambda + \sigma(\cdot). \quad (2.13)$$

Now, we are ready to state our first main result.

Theorem 2.2. *Let assumptions (A1), (A3)–(A5) be satisfied. Then, for any $\lambda > -\lambda^*$ and $p > 1$, the operator $\mathcal{N}_\lambda$ maps continuously $L^p(dx \otimes d\mu(v))$ into $W_x^{1,p}(dx \otimes d\mu(v))$, where*

$$W_x^{1,p}(dx \otimes d\mu(v)) := \left\{ \psi \in L^p(dx \otimes d\mu(v)) \text{ such that } \frac{\partial \psi}{\partial x_i} \in L^p(dx \otimes d\mu(v)) \right\}. \quad (2.14)$$

Moreover, $\|\mathcal{N}_\lambda\|_{\mathcal{L}(L^p(dx \otimes d\mu(v)); W_x^{1,p})}$ is locally bounded with respect to $\lambda > -\lambda^*$.

Proof. Let us first compute $\partial \mathcal{N}_\lambda \varphi / \partial x_i$ in the distributional sense for $\varphi \in \mathfrak{D}(D \times V)$. To this end, let $\varphi \in \mathfrak{D}(D \times V)$, and extend $k_i(\cdot, \cdot)$, $i = 1, 2$, by zero outside of V ; then

$$\begin{aligned} \left\langle \frac{\partial \mathcal{N}_\lambda \varphi}{\partial x_i}, \varphi \right\rangle_{\mathfrak{D}', \mathfrak{D}} &= - \int_{D \times V} \frac{\partial \varphi}{\partial x_i}(x, v) dx d\mu(v) \int_{D \times V} N_\lambda(x - x', v, v') \varphi(x', v') dx' d\mu(v') \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{V \times V} d\mu(v) d\mu(v') \int_{\mathbb{R}^N} \varphi(x', v') dx' \\ &\quad \times \int_{|x-x'| \geq \varepsilon} N_\lambda(x - x', v, v') \frac{\partial \varphi}{\partial x_i}(x, v) dx. \end{aligned} \quad (2.15)$$

The use of the Green formula leads to

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[\int_{V \times V} d\mu(v) d\mu(v') \int_{\mathbb{R}^N} \varphi(x', v') dx' \int_{|x-x'| \geq \varepsilon} \frac{\partial N_\lambda}{\partial x_i}(x - x', v, v') \varphi(x, v) dx \right. \\ \left. - \int_{V \times V} d\mu(v) d\mu(v') \int_{\mathbb{R}^N} \varphi(x', v') dx' \int_{|x-x'| = \varepsilon} N_\lambda(x - x', v, v') \varphi(x, v) \nu_i d\sigma(x) \right]. \end{aligned} \quad (2.16)$$

We claim that the second part of (2.16) goes to zero as $\varepsilon \rightarrow 0$.

Indeed,

$$\begin{aligned} &\int_{|x-x'| = \varepsilon} N_\lambda(x - x', v, v') \varphi(x, v) \nu_i d\sigma(x) \\ &= \varphi(x', v) \int_{|x-x'| = \varepsilon} N_\lambda(x - x', v, v') \nu_i d\sigma(x) \\ &\quad + \int_{|x-x'| = \varepsilon} N_\lambda(x - x', v, v') |x - x'| O(|x - x'|) \nu_i d\sigma(x) \\ &= I_1 + I_2. \end{aligned} \quad (2.17)$$

Clearly $I_1 = 0$ (because $N_\lambda(y, v, v') = N_\lambda(-y, v, v')$). Thus, using the dominated convergence theorem of Lebesgue, it is enough to prove that $I_2 \rightarrow 0$ as ε goes to zero. This follows from the estimate

$$|I_2| \leq \left(\int_0^1 \frac{d\alpha(\rho)}{\rho} \right) G^\infty(v, v') \int_{|x-x'| = \varepsilon} \frac{d\sigma(x)}{\varepsilon^{N-2}} = \left(\int_0^1 \frac{d\alpha(\rho)}{\rho} \right) G^\infty(v, v') \left| \mathbb{S}^{N-1} \right| \varepsilon, \quad (2.18)$$

which proves the claim.

Next, we compute the derivative of the kernel N_λ .

Indeed, by straightforward calculations

$$\begin{aligned}
\frac{\partial N_\lambda}{\partial x_i}(x, v, v') = & - \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+2}} G_j\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} d\alpha(\rho) \\
& + \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+2}} G_i\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} d\alpha(\rho) \\
& + \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+1}} \sigma_j\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+1}} \sigma_i\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{x_i}{|x|^N} \bar{\sigma}\left(\rho \frac{x}{|x|}\right) \frac{G(\rho(x/|x|), v, v')}{\rho} e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - (N-1) \int_0^1 \frac{x_i}{|x|^{N+1}} G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho},
\end{aligned} \tag{2.19}$$

where

$$\sigma_j(\cdot) := \frac{\partial \sigma}{\partial z_j}(\cdot), \quad G_j(\cdot, v, v') := \frac{\partial G}{\partial z_j}(\cdot, v, v'). \tag{2.20}$$

It is easily seen that $e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} = 1 - (|x|/\rho) \bar{\sigma}(\rho(x/|x|)) B(x, \rho)$, where $B(\cdot, \cdot)$ is a bounded function.

Therefore, $\partial N_\lambda / \partial x_i$ may be decomposed as

$$\begin{aligned}
\frac{\partial N_\lambda}{\partial x_i}(x, v, v') = & - \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+2}} G_j\left(\rho \frac{x}{|x|}, v, v'\right) d\alpha(\rho) + \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+2}} G_i\left(\rho \frac{x}{|x|}, v, v'\right) d\alpha(\rho) \\
& + \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+1}} G_j\left(\rho \frac{x}{|x|}, v, v'\right) \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+1}} G_i\left(\rho \frac{x}{|x|}, v, v'\right) \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho} \\
& + \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+1}} \sigma_j\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+1}} \sigma_i\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho}
\end{aligned}$$

$$\begin{aligned}
& - \int_0^1 \frac{x_i}{|x|^N} \bar{\sigma}\left(\rho \frac{x}{|x|}\right) \frac{G(\rho(x/|x|), v, v')}{\rho} e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - (N-1) \int_0^1 \frac{x_i}{|x|^{N+1}} G\left(\rho \frac{x}{|x|}, v, v'\right) \frac{d\alpha(\rho)}{\rho} \\
& + (N-1) \int_0^1 \frac{x_i}{|x|^N} \frac{G(\rho(x/|x|), v, v')}{\rho} \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho}.
\end{aligned} \tag{2.21}$$

Set

$$\begin{aligned}
S_i(x, v, v') &:= - \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^2} G_j\left(\rho \frac{x}{|x|}, v, v'\right) d\alpha(\rho) + \int_0^1 \frac{|x|^2 - x_i^2}{|x|^2} G_i\left(\rho \frac{x}{|x|}, v, v'\right) d\alpha(\rho) \\
& - (N-1) \int_0^1 \frac{x_i}{|x|} G\left(\rho \frac{x}{|x|}, v, v'\right) \frac{d\alpha(\rho)}{\rho}, \\
R_i(x, v, v') &:= \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+1}} G_j\left(\rho \frac{x}{|x|}, v, v'\right) \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+1}} G_i\left(\rho \frac{x}{|x|}, v, v'\right) \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho} \\
& + \sum_{j \neq i} \int_0^1 \frac{x_i x_j}{|x|^{N+1}} \sigma_j\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{|x|^2 - x_i^2}{|x|^{N+1}} \sigma_i\left(\rho \frac{x}{|x|}\right) G\left(\rho \frac{x}{|x|}, v, v'\right) e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& - \int_0^1 \frac{x_i}{|x|^N} \bar{\sigma}\left(\rho \frac{x}{|x|}\right) \frac{G(\rho(x/|x|), v, v')}{\rho} e^{-(|x|/\rho) \bar{\sigma}(\rho(x/|x|))} \frac{d\alpha(\rho)}{\rho} \\
& + (N-1) \int_0^1 \frac{x_i}{|x|^N} \frac{G(\rho(x/|x|), v, v')}{\rho} \bar{\sigma}\left(\rho \frac{x}{|x|}\right) B(x, \rho) \frac{d\alpha(\rho)}{\rho}.
\end{aligned} \tag{2.22}$$

Now, (2.16) becomes

$$\begin{aligned}
\left\langle \frac{\partial N_\lambda \varphi}{\partial x_i}, \varphi \right\rangle_{\mathfrak{D}, \mathfrak{D}} &= \lim_{\varepsilon \rightarrow 0} \left[\int_{V \times V} d\mu(v) d\mu(v') \int_{\mathbb{R}^N} \varphi(x, v) dx \int_{|x-x'| \geq \varepsilon} R_i(x-x', v, v') \varphi(x', v') dx' \right. \\
& \quad \left. + \int_{V \times V} d\mu(v) d\mu(v') \int_{\mathbb{R}^N} \varphi(x, v) dx \int_{|x-x'| \geq \varepsilon} \frac{S_i(x-x', v, v') \varphi(x', v')}{|x-x'|^N} dx' \right] \\
&= J_1 + J_2.
\end{aligned} \tag{2.23}$$

Clearly,

$$|R_i(x - x', v, v')| \leq \frac{C \left(G^\infty(v, v') + \overline{G}(v, v') + \sum_{j=1}^N G_j(v, v') \right)}{|x - x'|^{N-1}}, \quad (2.24)$$

where

$$C = \left(\int_0^1 \frac{d\alpha(\rho)}{\rho} \right) \max \left(\sum_{j=1}^N \|\sigma_j\|_\infty, (N\|B\|_\infty + 1)(\lambda + \|\sigma\|_\infty) \right). \quad (2.25)$$

Hence, according to (A2)–(A4) and the dominated convergence theorem of Lebesgue

$$J_2 = \int_{D \times V} \psi(x, v) dx d\mu(v) \int_{D \times V} R_i(x - x', v, v') \varphi(x', v') dx' d\mu(v'). \quad (2.26)$$

Therefore, using the Hölder inequality and the boundedness of D , the mapping

$$\psi \longrightarrow \int_{D \times V} R_i(x - x', v, v') \varphi(x', v') dx' d\mu(v') \quad (2.27)$$

defines a bounded operator $\mathcal{R}_i \in \mathcal{L}(L^p(dx \otimes d\mu))$.

Note, parenthetically, that from (2.25) we deduce the second part of the theorem.

Next, to deal with J_1 , we first consider the truncated operator

$$\mathcal{S}_{i,\varepsilon} : \varphi \longrightarrow \int_{|x-x'| \geq \varepsilon} \frac{S_i(x - x', v, v') \varphi(x', v')}{|x - x'|^N} dx' \quad \text{for fixed } (v, v'). \quad (2.28)$$

It is easily seen that $S_i(\cdot, v, v')$ is positively homogeneous of degree 0 and even and satisfies

$$|S_i(x, v, v')| \leq \left(\int_0^1 \frac{d\alpha(\rho)}{\rho} \right) \left((N-1)G^\infty(v, v') + \sum_{j=1}^N G_j(v, v') \right). \quad (2.29)$$

So, $\mathcal{S}_{i,\varepsilon}$ is a Caldéron-Zygmund operator which converges in L^p , as $\varepsilon \rightarrow 0$ to $\mathcal{S}_i$ (see [3, 21]), where

$$\mathcal{S}_i \varphi := p \cdot v \cdot \int_{\mathbb{R}^N} \frac{S_i(x - x', v, v') \varphi(x', v')}{|x - x'|^N} dx'. \quad (2.30)$$

Furthermore, the truncated maximal operator $\mathcal{S}_i^*$ defined by (see [3, 21])

$$\mathcal{S}_i^* := \sup_{\varepsilon > 0} |\mathcal{S}_{i,\varepsilon}^* \varphi| \quad (2.31)$$

satisfies

$$\|\mathcal{S}_i^*\|_{L_x^p(\mathbb{R}^N)} \leq C' \|S_i(\cdot, v, v')\|_{L^\infty(S^{N-1})} \|\varphi(\cdot, v')\|_{L_x^p(\mathbb{R}^N)}. \quad (2.32)$$

By the Hölder inequality and (2.29)

$$\begin{aligned} & \left| \int \varphi(x, v) \mathcal{S}_{i,\varepsilon} \varphi(x, v) dx \right| \\ & \leq C'' \left[(N-1)G^\infty(v, v') + \sum_{j=1}^N G_j(v, v') \right] \left(\int |\varphi(x, v)|^q \right)^{1/q} \|\varphi(\cdot, v')\|_{L_x^p(\mathbb{R}^N)}, \end{aligned} \quad (2.33)$$

where q is the conjugate of p , that is, $q = p/(p-1)$. Finally, by the dominated convergence theorem of Lebesgue

$$J_2 = (\mathcal{S}_i \varphi, \varphi)_{L^p, L^q}, \quad \mathcal{S}_i \in \mathcal{L}\left(L^p(\mathbb{R}^N \times V; dx \otimes d\mu(v))\right), \quad (2.34)$$

which amounts to

$$\left\langle \frac{\partial \mathcal{N}_\lambda \varphi}{\partial x_i}, \varphi \right\rangle_{\mathfrak{D}', \mathfrak{D}} = (\mathcal{R}_i \varphi, \varphi)_{L^p, L^q} + (\mathcal{S}_i \varphi, \varphi)_{L^p, L^q}. \quad (2.35)$$

The density of $\mathfrak{D}$ in $L^p(dx \otimes d\mu)$ concludes the proof. $\square$

Remark 2.3. We note that Theorem 2.2 has already been obtained by [5, Theorem 1] of Mokhtar-Kharroubi when the velocity space is endowed with the Lebesgue measure. The choice of an abstract measure is motivated by our desire to give a unified treatment which covers either continuous models (the Lebesgue measures on open subsets of $\mathbb{R}^N$) or multigroup models (the Lebesgue measure on spheres).

3. H_x^1 Regularity of the Operator $\mathcal{N}_\lambda = K_1(\lambda - T_p)^{-1}K_2$ in Periodic Transport

Let us first precise the functional setting of our problem. Let μ be a positive bounded measure on V satisfying the following:

$$d\mu \text{ is invariant by symmetry with respect to the origin} \quad (3.1)$$

$$\text{ess-} \sup_{w \in \mathbb{S}^{N-1}} \mu\{v \in V; |v \cdot w| \leq \varepsilon\} \leq C\varepsilon, \quad (3.2)$$

where C is a positive constant.

We define the streaming operator T_p on $L^2(dx \otimes d\mu(v))$ by

$$\begin{aligned} T_p \psi &= -v \cdot \frac{\partial \psi}{\partial x} - \sigma(v) \psi, \\ D(T_p) &= \{ \psi \in X_2; \psi|_{x_i=0} = \psi|_{x_i=2\pi} \ (1 \leq i \leq N) \}, \end{aligned} \quad (3.3)$$

where $D = (0, 2\pi)^N$. We expand $\psi \in L^2(dx \otimes d\mu(v))$ into the Fourier series with respect to x :

$$\psi(x, v) = \sum_{k \in \mathbb{Z}^N} f_k(v) e^{i(x \cdot k)}, \quad (3.4)$$

where

$$f_k(v) = \frac{1}{(2\pi)^{n/2}} \int_D \psi(x, v) e^{-i(x \cdot k)} dx \in L^2(V; d\mu(v)), \quad k \in \mathbb{Z}^n, \quad (3.5)$$

and by the Parseval formula

$$\|\psi\|_{L^2(dx \otimes d\mu(v))}^2 = \sum_{k \in \mathbb{Z}^N} \int_V |f_k(v)|^2 d\mu(v) < \infty. \quad (3.6)$$

By simple computations we have

$$\mathcal{N}_\lambda \psi(x, v) = \sum_{k \in \mathbb{Z}^N} \left[\int_V f_k(v') d\mu(v') \int_V \frac{G(v'', v, v')}{\overline{\sigma(v'')} + i(v'' \cdot k)} d\mu(v'') \right] e^{i(x \cdot k)}. \quad (3.7)$$

The regularity result of $\mathcal{N}_\lambda$ is obtained under the following two weaker assumptions:

(A7)

$$\sigma(v) = \sigma(-v), \quad d\mu(v) \text{ a.e.}, \quad \sigma \in L^\infty(V; d\mu(v)), \quad (3.8)$$

(A8)

$$\begin{aligned} G(z, v, v') &= G(-z, v, v'), \quad d\mu(v) \text{ a.e.}, \\ G^\infty(v, v') &:= \operatorname{ess-sup}_{z \in V} |G(z, v, v')| \in L^2(V \times V; d\mu \otimes d\mu), \end{aligned} \quad (3.9)$$

where G is defined by (2.2). The following lemma will play a crucial role in the proof of the main result.

Lemma 3.1. *Let $k \in \mathbb{Z}^N$ such that $k \neq (0, \dots, 0)$. Then,*

$$\int_V \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} \leq \frac{C_0}{|k|}, \quad (3.10)$$

where C_0 is a positive constant.

Proof. We write

$$\int_V \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} = \int_{|v \cdot (k/|k|)| < \alpha} \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} + \int_{|v \cdot (k/|k|)| \geq \alpha} \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2}, \quad (3.11)$$

where $\alpha < 1$ is to be chosen later. It is clear that

$$\int_{|v \cdot (k/|k|)| < \alpha} \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} \leq \frac{1}{(\lambda + \lambda^*)^2} \mu \left\{ v \in V, \left| v \cdot \frac{k}{|k|} \right| < \alpha \right\} \leq \frac{C\alpha}{(\lambda + \lambda^*)^2}. \quad (3.12)$$

Now, let β be the image of μ under the orthogonal projection on the direction $k/|k|$. Hence,

$$\begin{aligned} \int_{|v \cdot (k/|k|)| \geq \alpha} \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} &= \int_{\alpha}^1 \frac{d\beta(t)}{(\lambda + \lambda^*)^2 + |k|^2 t^2} \\ &= \frac{1}{|k|^2} \int_{\alpha}^1 \frac{|k|^2 t^2}{(\lambda + \lambda^*)^2 + |k|^2 t^2} \frac{d\beta(t)}{t^2} \\ &\leq \frac{1}{|k|^2} \int_{\alpha}^1 \frac{d\beta(t)}{t^2}. \end{aligned} \quad (3.13)$$

Let $\zeta(t) = \int_{\alpha}^t d\beta(s)$ for $t > \alpha$. An integration by parts yields

$$\int_{\alpha}^1 \frac{d\beta(t)}{t^2} = \left[\frac{\zeta(t)}{t^2} \right]_{\alpha}^1 + 2 \int_{\alpha}^1 \frac{\zeta(t)}{t^3} dt. \quad (3.14)$$

Since $\zeta(t) = \beta([\alpha, t]) \leq \beta([-t, t]) \leq Ct$ by (3.2), it follows that

$$\int_{\alpha}^1 \frac{d\beta(t)}{t^2} \leq C + \frac{2C}{\alpha} - 2C \leq \frac{2C}{\alpha}. \quad (3.15)$$

By choosing $\alpha = \varepsilon/|k|$ with $\varepsilon < 1$, we obtain

$$\int_V \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} \leq \frac{C_0}{|k|}, \quad (3.16)$$

where C_0 is a positive constant. This achieves the proof of the lemma. $\square$

Theorem 3.2. Under assumptions (A7)-(A8), the operator $\mathcal{N}_\lambda$ ($\lambda > -\lambda^*$) maps continuously $L^2(dx \otimes d\mu(v))$ into $H_x^1(dx \otimes d\mu(v))$, where

$$H_x^1(dx \otimes d\mu(v)) = \left\{ \psi \in L^2(dx \otimes d\mu(v)), \frac{\partial \psi}{\partial x_i} \in L^2(dx \otimes d\mu(v)), \text{ for } 1 \leq i \leq N \right\}. \quad (3.17)$$

Moreover, $\|\mathcal{N}_\lambda\|_{\mathcal{L}(L^2(dx \otimes d\mu(v)); H_x^1(dx \otimes d\mu(v)))}$ is locally bounded with respect to $\lambda > -\lambda^*$.

Proof. Let $\psi \in L^2(dx \otimes d\mu(v))$. We recall that

$$\mathcal{N}_\lambda \psi(x, v) = \sum_{k \in \mathbb{Z}^N} \left[\int_V f_k(v') d\mu(v') \int_V \frac{G(v'', v, v')}{\bar{\sigma}(v'') + i(v'' \cdot k)} d\mu(v'') \right] e^{i(x \cdot k)}. \quad (3.18)$$

Since

$$\int_V \frac{G(v'', v, v')}{\bar{\sigma}(v'') + i(v'' \cdot k)} d\mu(v'') = \int_V \frac{\bar{\sigma}(v'') G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v'') - i \int_V \frac{(v'' \cdot k) G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v''), \quad (3.19)$$

and thanks to the evenness of $\sigma(\cdot)$, $G(\cdot, v, v')$ together with the fact that $d\mu$ is invariant by symmetry with respect to the origin, the last integral vanishes. Therefore,

$$\mathcal{N}_\lambda \psi(x, v) = \sum_{k \in \mathbb{Z}^N} \left[\int_V f_k(v') d\mu(v') \int_V \frac{\bar{\sigma}(v'') G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v'') \right] e^{i(x \cdot k)}, \quad (3.20)$$

and consequently

$$\frac{\partial}{\partial x_j} \mathcal{N}_\lambda \psi(x, v) = \sum_{k \in \mathbb{Z}^N} i k_j \left[\int_V f_k(v') d\mu(v') \int_V \frac{\bar{\sigma}(v'') G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v'') \right] e^{i(x \cdot k)}. \quad (3.21)$$

Now, (A8) and Lemma 3.1 lead to

$$\begin{aligned} \left| \int_V \frac{\bar{\sigma}(v'') G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v'') \right| &\leq (\lambda + \|\sigma\|_\infty) G^\infty(v, v') \int_V \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} \\ &\leq \frac{C_0(\lambda + \|\sigma\|_\infty) G^\infty(v, v')}{|k|}. \end{aligned} \quad (3.22)$$

By the Hölder inequality, we obtain

$$\begin{aligned} \left| k_j \int_V f_k(v') d\mu(v') \int_V \frac{\bar{\sigma}(v'') G(v'', v, v')}{\bar{\sigma}(v'')^2 + (v'' \cdot k)^2} d\mu(v'') \right| \\ \leq C_0(\lambda + \|\sigma\|_\infty) \left(\int_V |f_k(v')|^2 d\mu(v') \right)^{1/2} \left(\int_V G^\infty(v, v')^2 d\mu(v') \right)^{1/2} \end{aligned} \quad (3.23)$$

and by the Parseval formula

$$\left\| \frac{\partial}{\partial x_i} \mathcal{M}_\lambda \psi \right\|_{L^2(dx \otimes d\mu(v))} \leq C_0(\lambda + \|\sigma\|_\infty) \|G^\infty\|_{L^2(V \times V)} \|\psi\|_{L^2(dx \otimes d\mu(v))}. \quad (3.24)$$

This ends the proof of the theorem. $\square$

Remark 3.3. The measure $d\mu(v) = d\alpha(\rho) \otimes dS(w)$, with $\int_0^1 (d\alpha(\rho)/\rho) < +\infty$, satisfies (3.2). Thus, it covers the continuous models as well as the multigroup models in periodic Transport.

4. H^1 Regularity of the Operator $\mathcal{M}_\lambda = K(\lambda - T_p)^{-1}$

Our notations and assumptions on $d\mu$ are the same as in the preceding section. Let $k(\cdot, \cdot)$ be a measurable function on $V \times V$ such that

$$K : \varphi \longrightarrow \int_V k(\cdot, v') \varphi(v') d\mu(v') \in \mathcal{L}(L^2(V; d\mu(v))). \quad (4.1)$$

It is announced without proof, in [10], that $\mathcal{M}_\lambda$ maps continuously $L^2(dx \otimes d\mu(v))$ into $H^{1/2}$, where $d\mu$ is the Lebesgue measure and $k(\cdot, \cdot) = 1$. We propose a proof for this result (see Theorem 4.1) for a class of measures $d\mu$ satisfying (3.2). The proof is inspired from [9] but is not a consequence of the results of [9] because the traces of $\varphi \in D(T_p)$ do not lie in $L^2(\partial D \times V; |v \cdot n(x)| d\sigma(x) d\mu(v))$, but in a certain greatest weighted L^2 space (see [1, 2, 4] for details).

We define

$$\|\varphi\|_{1/2} = (2\pi)^{N/2} \left[\sum_{k \in \mathbb{Z}^N} |f_k|^2 (1 + |k|^2)^{1/2} \right]^{1/2}, \quad (4.2)$$

where

$$\varphi(x) = \sum_{k \in \mathbb{Z}^N} f_k e^{i(x \cdot k)} \in L^2(D). \quad (4.3)$$

Theorem 4.1. *Assume that $k(\cdot, \cdot) = 1$. Then, for any $\lambda > -\lambda^*$, the operator $\mathcal{M}_\lambda$ maps continuously $L^2(dx \otimes d\mu(v))$ into $H^{1/2}(D)$.*

Proof. Let $\varphi \in L^2(dx \otimes d\mu(v))$. We expand φ into the Fourier series:

$$\varphi(x, v) = \sum_{k \in \mathbb{Z}^N} f_k(v) e^{i(x \cdot k)}. \quad (4.4)$$

We have

$$K(\lambda - T_p)^{-1} \varphi(x, v) = \sum_{k \in \mathbb{Z}^N} \left(\int_V \frac{f_k(v)}{\bar{\sigma}(v) + i(v \cdot k)} d\mu(v) \right) e^{i(x \cdot k)}. \quad (4.5)$$

By the Hölder inequality and Lemma 3.1

$$\begin{aligned}
& \left(1 + |k|^2\right)^{1/2} \left| \int_V \frac{f_k(v)}{\overline{\sigma}(v) + i(v \cdot k)} d\mu(v) \right|^2 \\
& \leq \left(1 + |k|^2\right)^{1/2} \left(\int_V |f_k(v)|^2 d\mu(v) \right) \int_V \frac{d\mu(v)}{(\lambda + \lambda^*)^2 + (v \cdot k)^2} \\
& \leq C_0 \frac{\left(1 + |k|^2\right)^{1/2}}{|k|} \int_V |f_k(v)|^2 d\mu(v).
\end{aligned} \tag{4.6}$$

The proof is achieved by the Parseval formula and the boundedness of $(1 + |k|^2)^{1/2}/|k|$. $\square$

Remark 4.2. Assumption (3.1) is unnecessary in the proof of Theorem 4.1 but it plays a key role in the following theorem.

Now, we are going to prove the H^1 regularity of $\mathcal{M}_\lambda$ when restricted to a subspace of $L^2(dx \otimes d\mu)$ consisting of even source term.

Theorem 4.3. *Let (A7) be satisfied. In addition, suppose that*

(A9)

$$\begin{aligned}
& k(v, v') = k(v, -v'), \quad d\mu(v) \text{ a.e.}, \\
& \text{ess-sup}_{v' \in V} |k(\cdot, v')| = k^\infty(\cdot) \in L^2(V; d\mu(v)).
\end{aligned} \tag{4.7}$$

Then, for any $\lambda > -\lambda^*$, the operator $\mathcal{M}_\lambda$ maps continuously $\mathcal{O}$ into $H_x^1(dx \otimes d\mu(v))$, where

$$\mathcal{O} = \left\{ \psi \in L^2(dx \otimes d\mu(v)), \quad \psi(x, v) = \psi(x, -v), \quad \left(\sup_{v \in V} |f_k(v)| \right)_{k \in \mathbb{Z}^N} \in l^2 \right\} \tag{4.8}$$

and $f_k(\cdot) = (1/(2\pi)^N) \int_D \psi(x, \cdot) e^{-i(x \cdot k)} dx$.

Moreover, $\|\mathcal{M}_\lambda\|_{\mathcal{L}(L^2(dx \otimes d\mu(v)); H_x^1(dx \otimes d\mu(v)))}$ is locally bounded with respect to $\lambda > -\lambda^*$.

Proof. We proceed as in the proof of Theorem 3.2. We expand $\psi \in \mathcal{O}$ into the Fourier series:

$$\psi(x, v) = \sum_{k \in \mathbb{Z}^N} f_k(v) e^{i(x \cdot k)}. \tag{4.9}$$

We have

$$K(\lambda - T_p)^{-1} \psi(x, v) = \sum_{k \in \mathbb{Z}^N} \left(\int_V \frac{k(v, v') f_k(v')}{\overline{\sigma}(v') + i(v' \cdot k)} d\mu(v') \right) e^{i(x \cdot k)}. \tag{4.10}$$

According to (A7) and (A9)

$$\frac{\partial}{\partial x_j} \mathcal{M}_\lambda \psi(x, v) = \sum_{k \in \mathbb{Z}^N} i k_j \left(\int_V \frac{\bar{\sigma}(v') k(v, v') f_k(v')}{\bar{\sigma}(v')^2 + (v' \cdot k)^2} d\mu(v') \right) e^{i(x \cdot k)}. \quad (4.11)$$

The use of Lemma 3.1 gives

$$\begin{aligned} \left| k_j \int_V \frac{\bar{\sigma}(v') k(v, v') f_k(v')}{\bar{\sigma}(v')^2 + (v' \cdot k)^2} d\mu(v') \right| &\leq |k_j| (\lambda + \|\sigma\|_\infty) k^\infty(v) \left(\sup_{v \in V} |f_k(v)| \right) \\ &\quad \times \left(\int_V \frac{d\mu(v')}{(\lambda + \lambda^*)^2 + (v' \cdot k)^2} \right) \\ &\leq C_0 (\lambda + \|\sigma\|_\infty) k^\infty(v) \sup_{v \in V} |f_k(v)|. \end{aligned} \quad (4.12)$$

Since $(\sup_{v \in V} |f_k(v)|)_{k \in \mathbb{Z}^N} \in l^2$, we deduce from the Parseval formula that $(\partial/\partial x_j) \mathcal{M}_\lambda \psi \in L^2(dx \otimes d\mu(v))$, which concludes the proof. $\square$

Remark 4.4. Arguing as in the proof of Lemma 3.1, one sees that

$$\int_V \frac{d\mu(v')}{[(\lambda + \lambda^*)^2 + (v' \cdot k)^2]^2} \leq \frac{\widehat{C}_0}{|k|}, \quad (4.13)$$

where $\widehat{C}_0$ is a positive constant. So, by the Hölder inequality Theorem 4.3 is still true if we replace $\mathcal{O}$ by

$$\left\{ \psi(x, v) = \sum_{j \in J, J \text{ is finite}} \varphi_j(x) \varphi_j(v), \varphi_j \in L^2(dx), \varphi_j \in L^\infty(d\mu) \text{ and } \varphi_j(v) = \varphi_j(-v) d\mu \text{ a.e.} \right\}. \quad (4.14)$$

which is of interest for neutron transport approximations.

The subspace $\mathcal{O}$ can be also replaced by

$$\left\{ \psi \in L^2(dx \otimes d\mu(v)), \varphi(x, v) = \varphi(x, -v) \text{ and } \sum_{k \in \mathbb{Z}^N} (1 + |k|^2)^{1/2} \int_{v \in V} |f_k(v)|^2 d\mu < \infty \right\}, \quad (4.15)$$

where $f_k(\cdot) = (1/2\pi^N) \int_D \psi(x, \cdot) e^{-i(x \cdot k)} dx$.

5. H^2 Regularity of the Operator $\mathcal{M}_\lambda = K(\lambda - T_p)^{-1}$

In this section we focus our attention on the smoothing effect of the velocity averages (H^2) under some particular assumptions on the abstract measure $d\mu$. For technical reasons, we treat separately the cases $N = 1$ and $N > 1$.

5.1. H_x^2 of Regularity $\mathcal{M}_\lambda$ in One Dimension

Let $d\alpha$ be a bounded measure (not necessarily positive) on $(-1, 1)$ satisfying the following assumptions:

(A10)

$$d\alpha \text{ is invariant by symmetry with respect to zero,} \quad (5.1)$$

(A11)

$$\int_0^1 \frac{d|\alpha|(\mu)}{\mu^2} < \infty, \quad (5.2)$$

where $d|\alpha|$ is the absolute value of the measure $d\alpha$,

(A12)

$$\begin{aligned} \sigma &\in L^\infty((-1, 1); d|\alpha|), \\ \sigma &\text{ is even } d|\alpha| \text{ a.e.} \end{aligned} \quad (5.3)$$

Let $k(\cdot, \cdot)$ be a $d|\alpha|$ measurable function such that

(A13)

$$\begin{aligned} k(\cdot, \mu') &= k(\cdot, -\mu') d|\alpha| \text{ a.e.,} \\ d|\alpha| \text{ess-} \sup_{\mu' \in (-1, 1)} |k(\cdot, \mu')| &= k^\infty(\cdot) \in L^2((-1, 1); d|\alpha|). \end{aligned} \quad (5.4)$$

Let us introduce the following subspace of $L^2((0, 2\pi) \times (-1, 1); dx \otimes (d|\alpha|/\mu^2))$:

$$\hat{\mathcal{O}} = \left\{ \psi \in L^2\left((0, 2\pi) \times (-1, 1); dx \otimes \frac{d|\alpha|}{\mu^2}\right), \psi(\cdot, \mu) = \psi(\cdot, -\mu) d|\alpha| \text{ a.e.} \right\}. \quad (5.5)$$

Then, we have the following.

Theorem 5.1. *Let assumptions (A10)–(A13) be satisfied. Then, for any operator $\lambda > -\lambda^* = -d|\alpha|_{\text{ess-inf}} \sigma(\cdot)$ the operator $\mathcal{M}_\lambda$ maps continuously $\hat{\mathcal{O}}$ into*

$$H_x^2(dx \otimes d|\alpha|) = \left\{ \psi \in L^2(dx \otimes d|\alpha|), \frac{\partial^2 \psi}{\partial x^2} \in L^2(dx \otimes d|\alpha|) \right\}. \quad (5.6)$$

Moreover, $\|\mathcal{M}_\lambda\|_{\mathcal{L}(L^2(dx \otimes d|\alpha|(\mu)/\mu^2)); H_x^2(dx \otimes d|\alpha|)}$ is locally bounded with respect to $\lambda > -\lambda^*$.

Proof. For $\psi \in \hat{\mathcal{O}}$ we have

$$K(\lambda - T_p)^{-1} \psi(x, v) = \sum_{k \in \mathbb{Z}} \left(\int_{-1}^1 \frac{k(\mu, \mu') f_k(\mu')}{\lambda + \sigma(\mu') + i\mu'k} d\alpha(\mu') \right) e^{ixk}. \quad (5.7)$$

Thus,

$$\begin{aligned} \frac{\partial^2 \mathcal{M}_\lambda \psi}{\partial x^2}(x, v) &= - \sum_{k \in \mathbb{Z}} k^2 \left(\int_{-1}^1 \frac{(\lambda + \sigma(\mu')) k(\mu, \mu') f_k(\mu')}{\lambda + \sigma(\mu') + i\mu'k} d\alpha(\mu') \right) e^{ixk} \\ &= - \sum_{k \in \mathbb{Z}} k^2 \left(\int_{-1}^1 \frac{(\lambda + \sigma(\mu')) k(\mu, \mu') f_k(\mu')}{(\lambda + \sigma(\mu'))^2 + \mu'^2 k^2} d\alpha(\mu') \right. \\ &\quad \left. + ik \int_{-1}^1 \frac{\mu' k(\mu, \mu') f_k(\mu')}{(\lambda + \sigma(\mu'))^2 + \mu'^2 k^2} d\alpha(\mu') \right) e^{ixk}. \end{aligned} \quad (5.8)$$

Note that the last integral of (5.8) vanishes because its integrand is odd, in view of assumptions (A10), (A12)–(A13) and the evenness of $f_k(\cdot)$. Now, using the Hölder inequality together with assumption (A10), we obtain

$$\begin{aligned} &k^2 \left| \int_{-1}^1 \frac{k(\mu, \mu') f_k(\mu')}{\lambda + \sigma(\mu') + i\mu'k} d\alpha(\mu') \right| \\ &\leq 2(\lambda + \|\sigma\|_{L^\infty(d|\alpha|)}) k^\infty(\mu) \int_0^1 \frac{k^2 \mu^2}{(\lambda + \sigma(\mu))^2 + \mu^2 k^2} |f_k(\mu)| \frac{d|\alpha|(\mu)}{\mu^2} \\ &\leq 2(\lambda + \|\sigma\|_{L^\infty(d|\alpha|)}) k^\infty(\mu) \left(\int_0^1 \frac{d|\alpha|(\mu)}{\mu^2} \right)^{1/2} \left(\int_0^1 |f_k(\mu)|^2 \frac{d|\alpha|(\mu)}{\mu^2} \right)^{1/2}. \end{aligned} \quad (5.9)$$

Next, since $k^\infty(\cdot) \in L^2((-1, 1); d|\alpha|)$ and $\sum_{k \in \mathbb{Z}} \int_0^1 |f_k(\mu)|^2 (d|\alpha|(\mu)/\mu^2) < \infty$, it follows from the Parseval formula that

$$\left\| \frac{\partial^2 \mathcal{M}_\lambda \psi}{\partial x^2} \right\|_{L^2(dx \otimes d|\alpha|)} \leq C \|\psi\|_{L^2(dx \otimes (d|\alpha|/\mu^2))}, \quad (5.10)$$

where

$$C = 2 \left(\lambda + \|\sigma\|_{L^\infty(d|\alpha|)} \right) \|k^\infty\|_{L^2(d|\alpha|)} \left(\int_0^1 \frac{d|\alpha|(\mu)}{\mu^2} \right)^{1/2} \quad (5.11)$$

is locally bounded in $\lambda > -\lambda^*$, and the proof is complete. $\square$

5.2. H^2 Regularity of $\mathcal{M}_\lambda = M(\lambda - T_p)^{-1}$ ($N > 1$)

Let $d\mu(v) = d\alpha(\rho) \otimes dS(w)$, where $dS(w)$ is the Lebesgue measure on $\mathbb{S}^{N-1}$ and $d\alpha$ is a signed measure on $[0, 1]$ satisfying the following conditions:

(A14)

$$\int_0^1 \frac{d\alpha(\rho)}{\rho} = 0, \quad (5.12)$$

(A15)

$$\int_0^1 \frac{d|\alpha|(\rho)}{\rho^2} < \infty. \quad (5.13)$$

The more technical assumption (A14) plays a key role for our analysis. In the sequel, we show the optimality of this assumption. We need also the following assumption:

(A16)

$$\begin{aligned} \sigma(v) &= \sigma(|v|), \\ \sigma(\cdot) &\in W^{1,\infty}([0, 1]). \end{aligned} \quad (5.14)$$

Theorem 5.2. *Let the hypotheses (A14)–(A16) be satisfied. Then, for any $\lambda > -\lambda^* = -d|\alpha|_{\text{ess-inf}} \sigma(\cdot)$, the operator $\mathcal{M}_\lambda = M(\lambda - T_p)^{-1}$ maps continuously $L^2(D)$ into $H^2(D)$, where*

$$M : \varphi \in L^2(dx \otimes d\mu) \longrightarrow \int_V \varphi(x, v) d\mu(v) \in L^2(D), \quad (5.15)$$

is the averaging operator.

Proof. Let $\varphi \in L^2(D)$. We use the Fourier series of $\varphi(x) = \sum_{k \in \mathbb{Z}^N} f_k e^{i(x \cdot k)}$. We recall that

$$\mathcal{M}_\lambda \varphi(x, v) = \sum_{k \in \mathbb{Z}^N} f_k \int_0^1 d\alpha(\rho) \int_{\mathbb{S}^{N-1}} \frac{dS(w)}{\bar{\sigma}(\rho) + i\rho(w \cdot k)} e^{i(x \cdot k)}, \quad (5.16)$$

where $\bar{\sigma}(\rho) = \lambda + \sigma(\rho)$.

Let us recall [22] that for all fixed $w_0 \in \mathcal{S}^{N-1}$

$$\int_{\mathcal{S}^{N-1}} f(w \cdot w_0) dS(w) = |S^{N-2}| \int_{-1}^1 f(t) (1-t^2)^{(N-3)/2} dt. \quad (5.17)$$

Accordingly,

$$\begin{aligned} \mathcal{M}_\lambda \psi(x, v) &= f_0 |S^{N-1}| \int_0^1 \frac{d\alpha(\rho)}{\bar{\sigma}(\rho)} \\ &+ 2 |S^{N-2}| \sum_{k \neq 0} f_k \left(\int_0^1 d\alpha(\rho) \int_0^1 (1-t^2)^{(N-3)/2} \frac{\bar{\sigma}(\rho)}{\bar{\sigma}(\rho)^2 + \rho^2 |k|^2 t^2} dt \right) e^{i(x \cdot k)}. \end{aligned} \quad (5.18)$$

Set

$$F_0 = f_0 |S^{N-1}| \int_0^1 \frac{d\alpha(\rho)}{\bar{\sigma}(\rho)}. \quad (5.19)$$

We give the proof for $N > 3$ (the other cases are similar).

An integration by parts with respect to t yields

$$\begin{aligned} \mathcal{M}_\lambda \psi(x, v) &= F_0 + 2(N-3) |S^{N-2}| \\ &\times \sum_{k \neq 0} \frac{f_k}{|k|} \left(\int_0^1 t (1-t^2)^{(N-5)/2} dt \int_0^1 \arctan \left(\frac{\rho |k| t}{\bar{\sigma}(\rho)} \right) \frac{d\alpha(\rho)}{\rho} \right) e^{i(x \cdot k)}. \end{aligned} \quad (5.20)$$

Thanks to (A14), (5.20) becomes

$$\begin{aligned} \mathcal{M}_\lambda \psi(x, v) &= F_0 - 2(N-3) |S^{N-2}| \\ &\times \sum_{k \neq 0} \frac{f_k}{|k|} \left(\int_0^1 t (1-t^2)^{(N-5)/2} dt \int_0^1 \arctan \left(\frac{\bar{\sigma}(\rho)}{\rho |k| t} \right) \frac{d\alpha(\rho)}{\rho} \right) e^{i(x \cdot k)} \\ &+ 2\pi(N-3) |S^{N-2}| \int_0^1 \frac{d\alpha(\rho)}{\rho} \int_0^1 t (1-t^2)^{(N-5)/2} dt \sum_{k \neq 0} \frac{f_k}{|k|} e^{i(x \cdot k)} \\ &= F_0 - 2(N-3) |S^{N-2}| \\ &\times \sum_{k \neq 0} \frac{f_k}{|k|} \left(\int_0^1 t (1-t^2)^{(N-5)/2} dt \int_0^1 \arctan \left(\frac{\bar{\sigma}(\rho)}{\rho |k| t} \right) \frac{d\alpha(\rho)}{\rho} \right) e^{i(x \cdot k)}. \end{aligned} \quad (5.21)$$

Set

$$P(\rho) = \int_0^\rho \frac{d\alpha(s)}{s}. \quad (5.22)$$

An integration by parts for the Stieltjes measures yields

$$\begin{aligned} \int_0^1 \arctan\left(\frac{\bar{\sigma}(\rho)}{\rho|k|t}\right) dP(\rho) &= -\frac{1}{|k|t} \int_0^1 P(\rho) \left(\frac{\bar{\sigma}(\rho)}{\rho}\right)' \frac{\rho^2|k|^2 t^2}{\bar{\sigma}(\rho)^2 + \rho^2|k|^2 t^2} d\rho \\ &\quad + \left[P(\rho) \arctan\left(\frac{\bar{\sigma}(\rho)}{\rho|k|t}\right) \right]_{\rho=0}^{\rho=1}. \end{aligned} \quad (5.23)$$

Now, thanks to (A14), the last term of (5.23) vanishes. Set

$$Z(\rho) = P(\rho) \left(\frac{\bar{\sigma}(\rho)}{\rho}\right)'. \quad (5.24)$$

Accordingly, (5.20) becomes

$$\begin{aligned} \mathcal{M}_\lambda \psi(x, v) &= F_0 + 2(N-3) \left| S^{N-2} \right| \sum_{k \neq 0} \frac{f_k}{|k|^2} e^{i(x \cdot k)} \\ &\quad \times \left(\int_0^1 (1-t^2)^{(N-5)/2} dt \int_0^1 Z(\rho) \frac{\rho^2|k|^2 t^2}{\bar{\sigma}(\rho)^2 + \rho^2|k|^2 t^2} d\rho \right). \end{aligned} \quad (5.25)$$

Consequently,

$$\begin{aligned} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\mathcal{M}_\lambda \psi(x, v)) &= -2(N-3) \left| S^{N-2} \right| \sum_{k \neq 0} \frac{k_i k_j}{|k|^2} f_k e^{i(x \cdot k)} \\ &\quad \times \left(\int_0^1 (1-t^2)^{(N-5)/2} dt \int_0^1 Z(\rho) \frac{\rho^2|k|^2 t^2}{\bar{\sigma}(\rho)^2 + \rho^2|k|^2 t^2} d\rho \right). \end{aligned} \quad (5.26)$$

Since

$$\begin{aligned} &2(N-3) \left| S^{N-2} \right| \left| \frac{k_i k_j}{|k|^2} \int_0^1 (1-t^2)^{(N-5)/2} dt \int_0^1 Z(\rho) \frac{\rho^2|k|^2 t^2}{\bar{\sigma}(\rho)^2 + \rho^2|k|^2 t^2} d\rho \right| \\ &\leq (N-2) \left| S^{N-2} \right| (\lambda + \|\sigma\|_\infty) \left(\int_0^1 \frac{d|\alpha|(\rho)}{\rho^2} \right), \end{aligned} \quad (5.27)$$

we deduce from the Parseval formula that $(\partial/\partial x_i)(\partial/\partial x_j)(\mathcal{M}_\lambda \psi) \in L^2(D)$. This achieves the proof. $\square$

Remark 5.3. (1) Expression (5.21) shows the optimality of assumption (A14) because $(|k|f_k)_{k \in \mathbb{Z}^N}$ is not necessary in l^2 .

(2) Note that in [19, 20] the use of velocity averages in the context of inverse problems has been studied. The problems consist in the explicit determination of the spatial parts of

internal sources from two suitable moments (velocity averages) of the solution of integro-differential transport equations for classical vacuum boundary conditions (see [19]) or periodic boundary conditions (see [20]) by means of appropriate signed measures.

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