## Research Article

# Some Properties of Certain Subclasses of Analytic Functions with Complex Order 

Zhi-Gang Wang, ${ }^{1}$ Feng-Hua Wen, ${ }^{2}$ and Yong Sun ${ }^{3}$<br>${ }^{1}$ School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455002, China<br>${ }^{2}$ School of Econometrics and Management, Changsha University of Science and Technology, Changsha, Hunan 410114, China<br>${ }^{3}$ Department of Mathematics, Huaihua University, Huaihua, Hunan 418008, China<br>Correspondence should be addressed to Zhi-Gang Wang, zhigangwang@foxmail.com<br>Received 2 November 2011; Accepted 30 November 2011<br>Academic Editor: G. Martin

Copyright © 2012 Zhi-Gang Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main purpose of this paper is to derive some coefficient inequalities and subordination properties for certain subclasses of analytic functions involving the Salagean operator. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$
\begin{equation*}
f(z)=z+\sum_{j=2}^{\infty} a_{j} z^{j}, \tag{1.1}
\end{equation*}
$$

which are analytic in the open unit disk

$$
\begin{equation*}
\mathbb{U}:=\{z: z \in \mathbb{C},|z|<1\} . \tag{1.2}
\end{equation*}
$$

For $0 \leqq \alpha<1$, we denote by $\mathcal{S}^{*}(\alpha)$ and $\nless(\alpha)$ the usual subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike of order $\alpha$ and convex of order $\alpha$ in $\mathbb{U}$. Clearly, we know that

$$
\begin{equation*}
f \in \mathcal{K}(\alpha) \Longleftrightarrow z f^{\prime} \in \mathcal{S}^{*}(\alpha) . \tag{1.3}
\end{equation*}
$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}(\beta)$ if it satisfies the inequality

$$
\begin{equation*}
\mathfrak{R}\left(\frac{z f^{\prime}(z)}{f(z)}\right)<\beta \quad(z \in \mathbb{U}) \tag{1.4}
\end{equation*}
$$

for some $\beta(\beta>1)$. Also, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}(\beta)$ if and only if $z f^{\prime} \in \mathcal{M}(\beta)$. The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and investigated recently by Owa and Srivastava [1] (see also Nishiwaki and Owa [2], Owa and Nishiwaki [3], and Srivastava and Attiya [4]).

Sălăgean [5] introduced the operator

$$
\begin{align*}
& D^{0} f(z)=f(z), \quad D^{1} f(z)=D f(z)=z f^{\prime}(z), \\
& D^{n} f(z)=D\left(D^{n-1} f(z)\right) \quad(n \in \mathbb{N}:=\{1,2, \ldots\}) \tag{1.5}
\end{align*}
$$

We note that

$$
\begin{equation*}
D^{n} f(z)=z+\sum_{j=2}^{\infty} j^{n} a_{j} z^{j} \quad\left(n \in \mathbb{N}_{0}:=\mathbb{N} \cup\{0\}\right) \tag{1.6}
\end{equation*}
$$

Given two functions $f, g \in A$, where $f$ is given by (1.1) and $g$ is defined by

$$
\begin{equation*}
g(z)=z+\sum_{n=2}^{\infty} b_{n} z^{n} \tag{1.7}
\end{equation*}
$$

the Hadamard product (or convolution) $f * g$ is defined by

$$
\begin{equation*}
(f * g)(z):=z+\sum_{n=2}^{\infty} a_{n} b_{n} z^{n}=:(g * f)(z) \tag{1.8}
\end{equation*}
$$

For two functions $f$ and $g$, analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$, and write

$$
\begin{equation*}
f(z) \prec g(z) \quad(z \in \mathbb{U}) \tag{1.9}
\end{equation*}
$$

if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with

$$
\begin{equation*}
\omega(0)=0, \quad|\omega(z)|<1 \quad(z \in \mathbb{U}) \tag{1.10}
\end{equation*}
$$

such that

$$
\begin{equation*}
f(z)=g(\omega(z)) \quad(z \in \mathbb{U}) \tag{1.11}
\end{equation*}
$$

Indeed, it is known that

$$
\begin{equation*}
f(z) \prec g(z), \quad(z \in \mathbb{U}) \Longrightarrow f(0)=g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}) \tag{1.12}
\end{equation*}
$$

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$
\begin{equation*}
f(z) \prec g(z), \quad(z \in \mathbb{U}) \Longleftrightarrow f(0)=g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}) . \tag{1.13}
\end{equation*}
$$

In recent years, Deng [6] (see also Kamali [7], Altintaş et al. [8], Srivastava et al. [9], and Xu et al. [10]) introduced and investigated the following subclass of $\mathcal{A}$ involving the $S$ Sălăgean lagean operator and obtained the coefficient bounds for this function class.

Definition 1.1. A function $f \in \mathscr{A}$ is said to be in the class $\mathcal{S}_{n}(\lambda, \alpha, b)$ if it satisfies the inequality

$$
\begin{equation*}
\mathfrak{R}\left(1+\frac{1}{b}\left(\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right)\right)>\alpha \quad(z \in \mathbb{U}) \tag{1.14}
\end{equation*}
$$

where

$$
\begin{equation*}
n \in \mathbb{N}_{0}, \quad b \in \mathbb{C} \backslash\{0\}, \quad 0 \leqq \alpha<1, \quad 0 \leqq \lambda \leqq 1 \tag{1.15}
\end{equation*}
$$

It is easy to see that the class $S_{n}(\lambda, \alpha, b)$ includes the classes $S^{*}(\alpha)$ and $\nless K(\alpha)$ as its special cases.

Now, motivated essentially by the above-mentioned function classes, we introduce the following subclass of $\mathcal{A}$ of analytic functions.

Definition 1.2. A function $f \in \mathscr{A}$ is said to be in the class $\mathcal{M}_{n}(\lambda, \beta, b)$ if it satisfies the inequality:

$$
\begin{equation*}
\mathfrak{R}\left(1+\frac{1}{b}\left(\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right)\right)<\beta \quad(z \in \mathbb{U}) \tag{1.16}
\end{equation*}
$$

where

$$
\begin{equation*}
n \in \mathbb{N}_{0}, \quad b \in \mathbb{C} \backslash\{0\}, \quad \beta>1, \quad 0 \leqq \lambda \leqq 1 \tag{1.17}
\end{equation*}
$$

It is also easy to see that the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are special cases of the class $\mathcal{M}_{n}(\lambda, \beta, b)$.

In this paper, we aim at proving some coefficient inequalities and subordination properties for the classes $S_{n}(\lambda, \beta, b)$ and $\mathcal{M}_{n}(\lambda, \beta, b)$. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

## 2. Coefficient Inequalities

In this section, we derive some coefficient inequalities for the classes $S_{n}(\lambda, \alpha, b)$ and $\mathcal{M}_{n}(\lambda, \alpha, b)$.

Theorem 2.1. Let

$$
\begin{equation*}
n \in \mathbb{N}_{0}, \quad b \in \mathbb{C} \backslash\{0\}, \quad 0 \leqq \alpha<1, \quad 0 \leqq \lambda \leqq 1 \tag{2.1}
\end{equation*}
$$

If $f \in \mathscr{A}$ satisfies the coefficient inequality

$$
\begin{equation*}
\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right][j-1+|b|(1-\alpha)]\left|a_{j}\right| \leqq|b|(1-\alpha) \tag{2.2}
\end{equation*}
$$

then $f \in S_{n}(\lambda, \alpha, b)$.
Proof. To prove $f \in \mathcal{S}_{n}(\lambda, \alpha, b)$, it is sufficient to show that

$$
\begin{equation*}
\left|\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right|<|b|(1-\alpha) \quad(z \in \mathbb{U}) \tag{2.3}
\end{equation*}
$$

By noting that

$$
\begin{align*}
& \left|\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right| \\
& \quad=\left|\frac{\sum_{j=2}^{\infty}\left[(1-\lambda)\left(j^{n+1}-j^{n}\right)+\lambda\left(j^{n+2}-j^{n+1}\right)\right] a_{j} z^{j-1}}{1+\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right] a_{j} z^{j-1}}\right|  \tag{2.4}\\
& \quad \leqq \frac{\sum_{j=2}^{\infty}\left[(1-\lambda)\left(j^{n+1}-j^{n}\right)+\lambda\left(j^{n+2}-j^{n+1}\right)\right]\left|a_{j}\right|}{1-\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]\left|a_{j}\right|}
\end{align*}
$$

it follows from (2.2) that the above last expression is bounded by $|b|(1-\alpha)$. This completes the proof of Theorem 2.1.

Theorem 2.2. Let

$$
\begin{equation*}
n \in \mathbb{N}_{0}, \quad b \in \mathbb{C} \backslash\{0\}, \quad \beta>1, \quad 0 \leqq \lambda \leqq 1 \tag{2.5}
\end{equation*}
$$

If $f \in \mathcal{A}$ satisfies the coefficient inequality

$$
\begin{equation*}
\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right](|b-1|+j+|j-1-(2 \beta-1) b|)\left|a_{j}\right| \leqq 2|b|(\beta-1) \tag{2.6}
\end{equation*}
$$

then $f \in \mathcal{M}_{n}(\lambda, \beta)$.

Proof. To prove $f \in \mathcal{M}_{n}(\lambda, \beta, b)$, it suffices to show that

$$
\begin{align*}
\mid 1 & \left.+\frac{1}{b}\left(\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right) \right\rvert\, \\
& <\left|1+\frac{1}{b}\left(\frac{(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)}{(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)}-1\right)-2 \beta\right| \tag{2.7}
\end{align*}
$$

We consider $M \in \mathbb{R}$ defined by

$$
\begin{align*}
M:= & \left|(b-1)\left[(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)\right]+(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)\right| \\
& -\left|(1-\lambda) D^{n+1} f(z)+\lambda D^{n+2} f(z)-[(2 \beta-1) b+1]\left[(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)\right]\right| \\
= & \left|b z+\sum_{j=2}^{\infty}\left\{(b-1)\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]+\left[(1-\lambda) j^{n+1}+\lambda j^{n+2}\right]\right\} a_{j} z^{j}\right| \\
& -\left|z+\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n+1}+\lambda j^{n+2}\right] a_{j} z^{j}-[(2 \beta-1) b+1]\left(z+\sum_{j=2}^{\infty}\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right] a_{j} z^{j}\right)\right| \tag{2.8}
\end{align*}
$$

Thus, for $|z|=r<1$, we have

$$
\begin{align*}
M \leqq & |b| r+\sum_{j=2}^{\infty}\left\{|b-1|\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]+(1-\lambda) j^{n+1}+\lambda j^{n+2}\right\}\left|a_{j}\right| r^{j} \\
& -\left[(2 \beta-1)|b| r-\sum_{j=2}^{\infty}\left|\left[(1-\lambda) j^{n+1}+\lambda j^{n+2}\right]-[(2 \beta-1) b+1]\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]\right|\left|a_{j}\right| r^{j}\right] \\
< & \left(\sum _ { j = 2 } ^ { \infty } \left\{|b-1|\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]+(1-\lambda) j^{n+1}+\lambda j^{n+2}\right.\right. \\
& \left.\left.\quad+\left|\left[(1-\lambda) j^{n+1}+\lambda j^{n+2}\right]-[(2 \beta-1) b+1]\left[(1-\lambda) j^{n}+\lambda j^{n+1}\right]\right|\right\}\left|a_{j}\right|-2(\beta-1)|b|\right) r . \tag{2.9}
\end{align*}
$$

It follows from (2.6) that $M<0$, which implies that (2.7) holds, that is, $f \in \mathcal{M}_{n}(\lambda, \beta, b)$. The proof of Theorem 2.2 is evidently completed.

To prove our next result, we need the following lemma.
Lemma 2.3. Let $\beta>1$ and $b \in \mathbb{C} \backslash\{0\}$. Suppose also that the sequence $\left\{\mathbb{B}_{j}\right\}_{j=1}^{\infty}$ is defined by

$$
\begin{gather*}
\mathbb{B}_{1}=1 \quad(j=1) \\
\mathbb{B}_{j}=\frac{2|b|(\beta-1)}{j-1} \sum_{k=1}^{j-1} \mathbb{B}_{k} \quad(j \in \mathbb{N} \backslash\{1\}), \tag{2.10}
\end{gather*}
$$

then

$$
\begin{equation*}
\mathbb{B}_{j}=\frac{1}{(j-1)!} \prod_{k=0}^{j-2}[2|b|(\beta-1)+k] \quad(j \in \mathbb{N} \backslash\{1\}) \tag{2.11}
\end{equation*}
$$

Proof. We make use of the principle of mathematical induction to prove the assertion (2.11) of Lemma 2.3. Indeed, from (2.10), we know that

$$
\begin{equation*}
\mathbb{B}_{2}=2|b|(\beta-1)=\frac{1}{1!} \prod_{k=0}^{0}[2|b|(\beta-1)+k] \tag{2.12}
\end{equation*}
$$

which implies that (2.11) holds for $j=2$.
We now suppose that (2.11) holds for $j=m(m \geqq 2)$, then

$$
\begin{equation*}
\mathbb{B}_{m}=\frac{1}{(m-1)!} \prod_{k=0}^{m-2}[2|b|(\beta-1)+k] \tag{2.13}
\end{equation*}
$$

Combining (2.10) and (2.13), we find that

$$
\begin{align*}
\mathbb{B}_{m+1} & =\frac{2|b|(\beta-1)}{m} \sum_{k=1}^{m} \mathbb{B}_{k} \\
& =\frac{2|b|(\beta-1)}{m} \sum_{k=1}^{m-1} \mathbb{B}_{k}+\frac{2|b|(\beta-1)}{m} \mathbb{B}_{m} \\
& =\frac{2|b|(\beta-1)}{m} \cdot \frac{m-1}{2|b|(\beta-1)} \mathbb{B}_{m}+\frac{2|b|(\beta-1)}{m} \mathbb{B}_{m}  \tag{2.14}\\
& =\frac{2|b|(\beta-1)+m-1}{m} \mathbb{B}_{m} \\
& =\frac{1}{m!} \prod_{k=0}^{m-1}[2|b|(\beta-1)+k]
\end{align*}
$$

which shows that (2.11) holds for $j=m+1$. The proof of Lemma 2.3 is evidently completed.

Theorem 2.4. Let $f \in \mathcal{M}_{n}(\lambda, \beta, b)$, then

$$
\begin{equation*}
\left|a_{j}\right| \leqq \frac{1}{(j-1)!(1-\lambda+\lambda j) j^{n}} \prod_{k=0}^{j-2}[2|b|(\beta-1)+k] \quad(j \in \mathbb{N} \backslash\{1\}) \tag{2.15}
\end{equation*}
$$

Proof. We first suppose that

$$
\begin{equation*}
F(z):=(1-\lambda) D^{n} f(z)+\lambda D^{n+1} f(z)=z+\sum_{j=2}^{\infty} B_{j} z^{j} \quad(z \in \mathbb{U} ; f \in \mathcal{A}) \tag{2.16}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{j}=j^{n}(1-\lambda+\lambda j) a_{j} \tag{2.17}
\end{equation*}
$$

Next, by setting

$$
\begin{align*}
& h(z):=\frac{\beta-1-(1 / b)\left(z F^{\prime}(z) / F(z)-1\right)}{\beta-1}=1+h_{1} z+h_{2} z^{2}+\cdots  \tag{2.18}\\
&\left(z \in \mathbb{U} ; f \in \mathcal{M}_{n}(\lambda, \beta, b)\right)
\end{align*}
$$

we easily find that $h \in D$. It follows from (2.18) that

$$
\begin{equation*}
z F^{\prime}(z)=[1+b(\beta-1)] F(z)-b(\beta-1) h(z) F(z) \tag{2.19}
\end{equation*}
$$

We now find from (2.16), (2.18), and (2.19) that

$$
\begin{align*}
& z+2 B_{2} z^{2}+\cdots+j B_{j} z^{j}+\cdots \\
& =[1+b(\beta-1)]\left(z+B_{2} z^{2}+\cdots+B_{j} z^{j}+\cdots\right)  \tag{2.20}\\
& \quad-b(\beta-1)\left(1+h_{1} z+h_{2} z^{2}+\cdots+h_{j} z^{j}+\cdots\right)\left(z+B_{2} z^{2}+\cdots+B_{j} z^{j}+\cdots\right)
\end{align*}
$$

By evaluating the coefficients of $z^{j}$ in both the sides of (2.20), we get

$$
\begin{equation*}
j B_{j}=[1+b(\beta-1)] B_{j}-b(\beta-1)\left(h_{j-1}+h_{j-2} B_{2}+\cdots+h_{1} B_{j-1}+B_{j}\right) \tag{2.21}
\end{equation*}
$$

On the other hand, it is well known that

$$
\begin{equation*}
\left|h_{k}\right| \leqq 2 \quad(k \in \mathbb{N}) \tag{2.22}
\end{equation*}
$$

Combining (2.21) and (2.22), we easily get

$$
\begin{equation*}
\left|B_{j}\right| \leqq \frac{2|b|(\beta-1)}{j-1} \sum_{k=1}^{j-1}\left|B_{k}\right| \quad\left(B_{1}=1 ; j \in \mathbb{N} \backslash\{1\}\right) \tag{2.23}
\end{equation*}
$$

Suppose that $\beta>1$ and $b \in \mathbb{C} \backslash\{0\}$. We define the sequence $\left\{\mathbb{B}_{j}\right\}_{j=1}^{\infty}$ as follows:

$$
\begin{gather*}
\mathbb{B}_{1}=1 \quad(j=1) \\
\mathbb{B}_{j}=\frac{2|b|(\beta-1)}{j-1} \sum_{k=1}^{j-1} \mathbb{B}_{k} \quad(j \in \mathbb{N} \backslash\{1\}) . \tag{2.24}
\end{gather*}
$$

In order to prove that

$$
\begin{equation*}
\left|B_{j}\right| \leqq \mathbb{B}_{j} \quad(j \in \mathbb{N} \backslash\{1\}) \tag{2.25}
\end{equation*}
$$

we use the principle of mathematical induction. By noting that

$$
\begin{equation*}
\left|B_{2}\right| \leqq 2|b|(\beta-1) \tag{2.26}
\end{equation*}
$$

thus, assuming that

$$
\begin{equation*}
\left|B_{m}\right| \leqq \mathbb{B}_{m} \quad(m \in\{2,3, \ldots, j\}) \tag{2.27}
\end{equation*}
$$

we find from (2.23) and (2.24) that

$$
\begin{equation*}
\left|B_{j+1}\right| \leqq \frac{2|b|(\beta-1)}{j} \sum_{k=1}^{j}\left|B_{k}\right| \leqq \frac{2|b|(\beta-1)}{j} \sum_{k=1}^{j} \mathbb{B}_{k}=\mathbb{B}_{j+1} \quad(j \in \mathbb{N}) \tag{2.28}
\end{equation*}
$$

Therefore, by the principle of mathematical induction, we have

$$
\begin{equation*}
\left|B_{j}\right| \leqq \mathbb{B}_{j} \quad(j \in \mathbb{N} \backslash\{1\}) \tag{2.29}
\end{equation*}
$$

as desired.
By virtue of Lemma 2.3 and (2.24), we know that

$$
\begin{equation*}
\mathbb{B}_{j}=\frac{1}{(j-1)!} \prod_{k=0}^{j-2}[2|b|(\beta-1)+k] \quad(j \in \mathbb{N} \backslash\{1\}) \tag{2.30}
\end{equation*}
$$

Combining (2.17), (2.29), and (2.30), we readily arrive at the coefficient estimates (2.15) asserted by Theorem 2.4.

Remark 2.5. Setting $\lambda=0, b=1$, and $n=0$ or 1 in Theorem 2.4, we get the corresponding results obtained by Owa and Nishiwaki [3].

Remark 2.6. We cannot show that the result of Theorem 2.4 is sharp. Indeed, if one can prove the sharpness of Theorem 2.4, the sharpness of the corresponding result obtained by Deng [6] follows easily.

## 3. Subordination Properties

In view of Theorems 2.1 and 2.2, we now introduce the following subclasses:

$$
\begin{equation*}
\widetilde{S_{n}}(\lambda, \alpha, b) \subset S_{n}(\lambda, \alpha, b), \quad \widetilde{\mathcal{M}_{n}}(\lambda, \beta, b) \subset \mathcal{M}_{n}(\lambda, \beta, b) \tag{3.1}
\end{equation*}
$$

which consist of functions $f \in \mathscr{A}$ whose Taylor-Maclaurin coefficients satisfy the inequalities (2.2) and (2.6), respectively.

A sequence $\left\{b_{j}\right\}_{j=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f$ of the form (1.1) is analytic, univalent, and convex in $\mathbb{U}$, we have the subordination

$$
\begin{equation*}
\sum_{j=1}^{\infty} a_{j} b_{j} z^{j} \prec f(z) \quad\left(a_{1}=1 ; z \in \mathbb{U}\right) \tag{3.2}
\end{equation*}
$$

To derive the subordination properties for the classes $\widetilde{S_{n}}(\lambda, \alpha, b)$ and $\widetilde{\mathscr{M}_{n}}(\lambda, \alpha, b)$, we need the following lemma.

Lemma 3.1 (see [11]). The sequence $\left\{b_{j}\right\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if

$$
\begin{equation*}
\Re\left(1+2 \sum_{j=1}^{\infty} b_{j} z^{j}\right)>0 \quad(z \in \mathbb{U}) \tag{3.3}
\end{equation*}
$$

Theorem 3.2. If $f \in \widetilde{S_{n}}(\lambda, \alpha, b)$ and $g \in \mathcal{K}(0)$, then

$$
\begin{gather*}
\Phi(n, \lambda, \alpha, b) \cdot(f * g)(z)<g(z)  \tag{3.4}\\
\Re(f)>-\frac{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{2^{n}(1+\lambda)[1+|b|(1-\alpha)]} \tag{3.5}
\end{gather*}
$$

for

$$
\begin{equation*}
0 \leqq \lambda \leqq 1, \quad 0 \leqq \alpha<1, \quad b \in \mathbb{C} \backslash\{0\}, \quad n \in \mathbb{N}_{0} \tag{3.6}
\end{equation*}
$$

where, for convenience,

$$
\begin{equation*}
\Phi(n, \lambda, \alpha, b):=\frac{2^{n-1}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} \tag{3.7}
\end{equation*}
$$

The constant factor

$$
\begin{equation*}
\frac{2^{n-1}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} \tag{3.8}
\end{equation*}
$$

in the subordination result (3.4) cannot be replaced by a larger one.

Proof. Let $f \in \widetilde{\mathcal{S}_{n}}(\lambda, \alpha, b)$ and suppose that

$$
\begin{equation*}
g(z)=z+\sum_{j=2}^{\infty} c_{j} z^{j} \in \nless K:=\nless \not(0), \tag{3.9}
\end{equation*}
$$

then

$$
\begin{equation*}
\Phi(n, \lambda, \alpha, b) \cdot(f * g)(z)=\Phi(n, \lambda, \alpha, b) \cdot\left(z+\sum_{j=2}^{\infty} a_{j} c_{j} z^{j}\right) \tag{3.10}
\end{equation*}
$$

where $\Phi(n, \lambda, \alpha, b)$ is defined by (3.7).
If

$$
\begin{equation*}
\left\{\Phi(n, \lambda, \alpha, b) \cdot a_{j}\right\}_{j=1}^{\infty} \tag{3.11}
\end{equation*}
$$

is a subordinating factor sequence with $a_{1}=1$, then the subordination result (3.4) holds. By Lemma 3.1, we know that this is equivalent to the inequality

$$
\begin{equation*}
\mathfrak{R}\left(1+\sum_{j=1}^{\infty} \frac{2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} a_{j} z^{j}\right)>0 \quad(z \in \mathbb{U}) . \tag{3.12}
\end{equation*}
$$

Since

$$
\begin{equation*}
j^{n}(1-\lambda+\lambda j)[j-1+|b|(1-\alpha)] \quad\left(j \geqq 2 ; 0 \leqq \lambda \leqq 1 ; 0 \leqq \alpha<1 ; b \in \mathbb{C} \backslash\{0\} ; n \in \mathbb{N}_{0}\right) \tag{3.13}
\end{equation*}
$$

is an increasing function of $j$, and using Theorem 2.1, we have

$$
\begin{align*}
\Re(1 & \left.+\sum_{j=1}^{\infty} \frac{2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} a_{j} z^{j}\right) \\
= & \Re\left(1+\frac{2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} a_{1} z\right. \\
& \left.+\frac{1}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} \cdot \sum_{j=2}^{\infty} 2^{n}(1+\lambda)[1+|b|(1-\alpha)] a_{j} z^{j}\right)  \tag{3.14}\\
\geqq & 1-\frac{2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} r \\
& -\frac{1}{|b|(1-\alpha)+2^{2}(1+\lambda)[1+|b|(1-\alpha)]} \cdot \sum_{j=2}^{\infty} 2^{n}(1+\lambda)[1+|b|(1-\alpha)]\left|a_{j}\right| r^{j} \\
> & 1-\frac{2^{n}(1+\lambda)[1+|b|(1-\alpha)]}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} r-\frac{|b|(1-\alpha)}{|b|(1-\alpha)+2^{n}(1+\lambda)[1+|b|(1-\alpha)]} r \\
= & 1-r>0 \quad(|z|=r<1) .
\end{align*}
$$

This evidently proves the inequality (3.12), and hence also the subordination result (3.4), asserted by Theorem 3.2. The inequality (3.5) asserted by Theorem 3.2 follows from (3.4) by setting

$$
\begin{equation*}
g(z)=\frac{z}{1-z}=\sum_{j=1}^{\infty} z^{j} \in \nless . \tag{3.15}
\end{equation*}
$$

Finally, we consider the function $f_{0}$ defined by

$$
\begin{equation*}
f_{0}(z):=z-\frac{|b|(1-\alpha)}{2^{n}(1+\lambda)[1+|b|(1-\alpha)]} z^{2} \quad\left(n \in \mathbb{N}_{0} ; 0 \leqq \lambda \leqq 1 ; 0 \leqq \alpha<1 ; b \in \mathbb{C} \backslash\{0\}\right), \tag{3.16}
\end{equation*}
$$

which belongs to the class $\widetilde{S_{n}}(\lambda, \alpha, b)$. Thus, by (3.4), we know that

$$
\begin{equation*}
\Phi(n, \lambda, \alpha, b) \cdot f_{0}(z)<\frac{z}{1-z} \quad(z \in \mathbb{U}) \tag{3.17}
\end{equation*}
$$

Furthermore, it can be easily verified for the function $f_{0}$ given by (3.16) that

$$
\begin{equation*}
\min _{z \in \mathbb{U}}\left\{\Re\left(\Phi(n, \lambda, \alpha, b) \cdot f_{0}(z)\right)\right\}=-\frac{1}{2} \tag{3.18}
\end{equation*}
$$

We thus complete the proof of Theorem 3.2.
The proof of the following subordination result is much akin to that of Theorem 3.2. We, therefore, choose to omit the analogous details involved.

Corollary 3.3. If $f \in \widetilde{\mathcal{M}_{n}}(\lambda, \alpha, b)$ and $g \in \mathcal{K}(0)$, then

$$
\begin{gather*}
\Psi(n, \lambda, \beta, b) \cdot(f * g)(z) \prec g(z)  \tag{3.19}\\
\Re(f)>-\frac{|b|(\beta-1)+2^{n-1}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)}{2^{n-1}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)} \tag{3.20}
\end{gather*}
$$

for

$$
\begin{equation*}
0 \leqq \lambda \leqq 1, \quad \beta>1, \quad b \in \mathbb{C} \backslash\{0\}, \quad n \in \mathbb{N}_{0} \tag{3.21}
\end{equation*}
$$

where, for convenience,

$$
\begin{equation*}
\Psi(n, \lambda, \beta, b):=\frac{2^{n-2}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)}{|b|(\beta-1)+2^{n-1}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)} \tag{3.22}
\end{equation*}
$$

The constant factor

$$
\begin{equation*}
\frac{2^{n-2}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)}{|b|(\beta-1)+2^{n-1}(1+\lambda)(|b-1|+2+|1-(2 \beta-1) b|)} \tag{3.23}
\end{equation*}
$$

in the subordination result (3.19) cannot be replaced by a larger one.
Remark 3.4. Putting $\lambda=0, b=1$, and $n=0$ or 1 in Corollary 3.3, we get the corresponding results obtained by Srivastava and Attiya [4].

## Acknowledgments

The present investigation was supported by the National Natural Science Foundation under grants 11101053, 70971013, and 71171024, the Natural Science Foundation of Hunan Province under grant 09JJ1010, the Key Project of Chinese Ministry of Education under grant 211118, the Excellent Youth Foundation of Educational Committee of Hunan Province under grant 10B002, the Open Fund Project of Key Research Institute of Philosophies and Social Sciences in Hunan Universities under grant 11FEFM02, and the Key Project of Natural Science Foundation of Educational Committee of Henan Province under grant 12A110002 of China.

## References

[1] S. Owa and H. M. Srivastava, "Some generalized convolution properties associated with certain subclasses of analytic functions," Journal of Inequalities in Pure and Applied Mathematics, vol. 3, no. 3, article 42, 13 pages, 2002.
[2] J. Nishiwaki and S. Owa, "Coefficient inequalities for certain analytic functions," International Journal of Mathematics and Mathematical Sciences, vol. 29, no. 5, pp. 285-290, 2002.
[3] S. Owa and J. Nishiwaki, "Coefficient estimates for certain classes of analytic functions," Journal of Inequalities in Pure and Applied Mathematics, vol. 3, no. 5, article 72, 5 pages, 2002.
[4] H. M. Srivastava and A. A. Attiya, "Some subordination results associated with certain subclasses of analytic functions," Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 4, article 82, 6 pages, 2004.
[5] G. S. Sălăgean, "Subclasses of univalent functions," in Complex Analysis-5th Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), vol. 1013 of Lecture Notes in Mathematics, pp. 362-372, Springer, Berlin, Germany, 1983.
[6] Q. Deng, "Certain subclass of analytic functions with complex order," Applied Mathematics and Computation, vol. 208, no. 2, pp. 359-362, 2009.
[7] M. Kamali, "Neighborhoods of a new class of $p$-valently starlike functions with negative coefficients," Mathematical Inequalities \& Applications, vol. 9, no. 4, pp. 661-670, 2006.
[8] O. Altıntaş, H. Irmak, S. Owa, and H. M. Srivastava, "Coefficient bounds for some families of starlike and convex functions of complex order," Applied Mathematics Letters, vol. 20, no. 12, pp. 1218-1222, 2007.
[9] H. M. Srivastava, Q.-H. Xu, and G.-P. Wu, "Coefficient estimates for certain subclasses of spiral-like functions of complex order," Applied Mathematics Letters, vol. 23, no. 7, pp. 763-768, 2010.
[10] Q.-H. Xu, Y.-C. Gui, and H. M. Srivastava, "Coefficient estimates for certain subclasses of analytic functions of complex order," Taiwanese Journal of Mathematics, vol. 15, no. 5, pp. 2377-2386, 2011.
[11] H. S. Wilf, "Subordinating factor sequences for convex maps of the unit circle," Proceedings of the American Mathematical Society, vol. 12, pp. 689-693, 1961.


Advances in
Operations Research $=-$


The Scientific World Journal



Journal of
Applied Mathematics
-
Algebra
$\xlongequal{=}$


Journal of Probability and Statistics
$\qquad$


International Journal of Differential Equations


