

Research Article

Differential Transcendancy in the Theory of Linear Differential Systems with Constant Coefficients

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We consider a reduction of a nonhomogeneous linear system of first-order operator equations to a totally reduced system. Obtained results are applied to Cauchy problem for linear differential systems with constant coefficients and to the question of differential transcendancy.

1. Introduction

Linear systems with constant coefficients are considered in various fields (see [1–5]). In our paper [5] we use the rational canonical form and a certain sum of principal minors to reduce a linear system of first-order operator equations with constant coefficients to an equivalent, so called partially reduced, system. In this paper we obtain more general results regarding sums of principal minors and a new type of reduction. The obtained formulae of reduction allow some new considerations in connection with Cauchy problem for linear differential systems with constant coefficients and in connection with the differential transcendancy of the solution coordinates.

2. Notation

Let us recall some notation. Let K be a field, and let $B \in K^{n \times n}$ be an n -square matrix. We denote by

$$(\delta_k) \quad \delta_k = \delta_k(B) \text{ the sum of its principal minors of order } k (1 \leq k \leq n),$$

(δ_k^i) $\delta_k^i = \delta_k^i(B)$ the sum of its principal minors of order k containing i th column ($1 \leq i, k \leq n$).

Let v_1, \dots, v_n be elements of K . We write $B^i(v_1, \dots, v_n) \in K^{n \times n}$ for the matrix obtained by substituting $\vec{v} = [v_1 \dots v_n]^T$ in place of i th column of B . Furthermore, it is convenient to use

$$\delta_k^i(B; \vec{v}) = \delta_k^i(B; v_1, \dots, v_n) = \delta_k^i\left(B^i(v_1, \dots, v_n)\right), \quad (2.1)$$

and the corresponding vector

$$\vec{\delta}_k(B; \vec{v}) = \left[\delta_k^1(B; \vec{v}) \dots \delta_k^n(B; \vec{v}) \right]^T. \quad (2.2)$$

The characteristic polynomial $\Delta_B(\lambda)$ of the matrix $B \in K^{n \times n}$ has the following form:

$$\Delta_B(\lambda) = \det(\lambda I - B) = \lambda^n + d_1 \lambda^{n-1} + \dots + d_{n-1} \lambda + d_n, \quad (2.3)$$

where $d_k = (-1)^k \delta_k(B)$, $1 \leq k \leq n$; see [6, page 78].

Denote by $\bar{B}(\lambda)$ the adjoint matrix of $\lambda I - B$, and let $B_0, B_1, \dots, B_{n-2}, B_{n-1}$ be n -square matrices over K determined by

$$\bar{B}(\lambda) = \text{adj}(\lambda I - B) = \lambda^{n-1} B_0 + \lambda^{n-2} B_1 + \dots + \lambda B_{n-2} + B_{n-1}. \quad (2.4)$$

Recall that $(\lambda I - B)\bar{B}(\lambda) = \Delta_B(\lambda)I = \bar{B}(\lambda)(\lambda I - B)$.

The recurrence $B_0 = I$; $B_k = B_{k-1}B + d_k I$ for $1 \leq k < n$ follows from equation $\bar{B}(\lambda)(\lambda I - B) = \Delta_B(\lambda)I$; see [6, page 91].

3. Some Results about Sums of Principal Minors

In this section we give two results about sums of principal minors.

Theorem 3.1. For $B \in K^{n \times n}$ and $\vec{v} = [v_1 \dots v_n]^T \in K^{n \times 1}$, one has:

$$\delta_k^i \left(B; \sum_{j=1}^n b_{1j} v_j, \dots, \sum_{j=1}^n b_{nj} v_j \right) + \delta_{k+1}^i(B; v_1, \dots, v_n) = \delta_k(B) v_i. \quad (3.1)$$

Remark 3.2. The previous result can be described by

$$\delta_k^i(B; B\vec{v}) + \delta_{k+1}^i(B; \vec{v}) = \delta_k(B) v_i, \quad 1 \leq i \leq n, \quad (3.2)$$

or simply by the following vector equation:

$$\vec{\delta}_k(B; B\vec{v}) + \vec{\delta}_{k+1}(B; \vec{v}) = \delta_k(B) \vec{v}. \quad (3.3)$$

Proof of Theorem 3.1. Let $\vec{e}_s \in K^{n \times 1}$ denote the column whose only nonzero entry is 1 in s th position. We also write $B_{\downarrow s}$ for s th column of the matrix B and $[B]_{\bar{s}}$ for a square matrix of order $n - 1$ obtained from B by deleting its s th column and row. According to the notation used in (2.1), let $[B^i(B_{\downarrow s})]_{\bar{s}}$ stand for the matrix of order $n - 1$ obtained from B by substituting s th column $B_{\downarrow s}$ in place of i th column, and then by deleting s th column and s th row of the new matrix. By applying linearity of $\vec{\delta}_k(B; \vec{v})$ with respect to \vec{v} , we have

$$\begin{aligned}
\delta_k^i(B; B\vec{v}) + \delta_{k+1}^i(B; \vec{v}) &= \delta_k^i\left(B; \sum_{s=1}^n v_s B_{\downarrow s}\right) + \delta_{k+1}^i\left(B; \sum_{s=1}^n v_s \vec{e}_s\right) \\
&= \sum_{s=1}^n v_s \delta_k^i(B; B_{\downarrow s}) + \sum_{s=1}^n v_s \delta_{k+1}^i(B; \vec{e}_s) \\
&= \sum_{s=1}^n v_s \left(\delta_k^i(B; B_{\downarrow s}) + \delta_{k+1}^i(B; \vec{e}_s)\right) \\
&= v_i \left(\delta_k^i(B; B_{\downarrow i}) + \delta_{k+1}^i(B; \vec{e}_i)\right) + \sum_{\substack{s=1 \\ s \neq i}}^n v_s \left(\delta_k^i(B; B_{\downarrow s}) + \delta_{k+1}^i(B; \vec{e}_s)\right).
\end{aligned} \tag{3.4}$$

First, we compute $v_i(\delta_k^i(B; B_{\downarrow i}) + \delta_{k+1}^i(B; \vec{e}_i)) = v_i(\delta_k^i(B) + \delta_k([B]_{\bar{i}})) = v_i \delta_k(B)$. Then, it remains to show that $\sum_{s=1, s \neq i}^n v_s(\delta_k^i(B; B_{\downarrow s}) + \delta_{k+1}^i(B; \vec{e}_s)) = 0$.

It suffices to prove that each term in the sum is zero, that is,

$$\delta_k^i(B; B_{\downarrow s}) + \delta_{k+1}^i(B; \vec{e}_s) = 0, \quad \text{for } s \neq i. \tag{3.5}$$

Suppose that $s \neq i$. We now consider minors in the sum $\delta_k^i(B; B_{\downarrow s})$. All of them containing s th column are equal to zero, so we deduce

$$\delta_k^i(B; B_{\downarrow s}) = \delta_k^i(B^i(B_{\downarrow s})) = \delta_k^i\left([B^i(B_{\downarrow s})]_{\bar{s}}\right). \tag{3.6}$$

If $s \neq i$, then each minor in the sum $\delta_{k+1}^i(B; \vec{e}_s)$ necessarily contains s th row and i th column. By interchanging i th and s th column, we multiply each minor by -1 . We now proceed by expanding these minors along s th column to get -1 times the corresponding k th order principal minors of matrix $B^i(B_{\downarrow s})$ which do not include s th column. Hence, $\delta_{k+1}^i(B; \vec{e}_s) = -\delta_k^i([B^i(B_{\downarrow s})]_{\bar{s}})$, and the proof is complete. \square

In the following theorem, we give some interesting correspondence between the coefficients B_k of the matrix polynomial $\bar{B}(\lambda) = \text{adj}(\lambda I - B)$ and sums of principal minors $\vec{\delta}_{k+1}(B; \vec{v})$, $0 \leq k < n$.

Theorem 3.3. Given an arbitrary column $[v_1 \ \dots \ v_n]^T \in K^{n \times 1}$, it holds

$$B_k \vec{v} = B_k \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (-1)^k \begin{bmatrix} \delta_{k+1}^1(B; v_1, \dots, v_n) \\ \delta_{k+1}^2(B; v_1, \dots, v_n) \\ \vdots \\ \delta_{k+1}^n(B; v_1, \dots, v_n) \end{bmatrix} = (-1)^k \vec{\delta}_{k+1}(B; \vec{v}). \quad (3.7)$$

Proof. The proof proceeds by induction on k . It is being obvious for $k = 0$. Assume, as induction hypothesis (IH), that the statement is true for $k - 1$. Multiplying the right side of the equation $B_k = B_{k-1}B + d_k I$, by the vector \vec{v} , we obtain that

$$\begin{aligned} B_k \vec{v} &= B_{k-1}(B\vec{v}) + d_k \vec{v} \stackrel{\text{(IH)}}{=} (-1)^{k-1} \vec{\delta}_k(B; B\vec{v}) + d_k \vec{v} \\ &= (-1)^{k-1} (\vec{\delta}_k(B; B\vec{v}) - \delta_k \vec{v}) \stackrel{(5)}{=} (-1)^k \vec{\delta}_{k+1}(B; \vec{v}). \end{aligned} \quad (3.8)$$

□

Remark 3.4. Theorem 3.3 seems to have an independent application. Taking $\vec{v} = \vec{e}_j, 1 \leq j \leq n$, we prove formulae (8)–(10) given in [7].

4. Formulae of Total Reduction

We can now obtain a new type of reduction for the linear systems with constant coefficients from [5] applying results of previous section. For the sake of completeness, we introduce some definition.

Let K be a field, V a vector space over field K , and let $A : V \rightarrow V$ be a linear operator. We consider a linear system of first-order A -operator equations with constant coefficients in unknowns $x_i, 1 \leq i \leq n$, is

$$\begin{aligned} A(x_1) &= b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n + \varphi_1 \\ A(x_2) &= b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n + \varphi_2 \\ &\vdots \\ A(x_n) &= b_{n1}x_1 + b_{n2}x_2 + \dots + b_{nn}x_n + \varphi_n \end{aligned} \quad (4.1)$$

for $b_{ij} \in K$ and $\varphi_i \in V$. We say that $B = [b_{ij}]_{i,j=1}^n \in K^{n \times n}$ is the system matrix, and $\vec{\varphi} = [\varphi_1 \ \dots \ \varphi_n]^T \in V^{n \times 1}$ is the free column.

Let $\vec{x} = [x_1 \ \dots \ x_n]^T$ be a column of unknowns and $\vec{A} : V^{n \times 1} \rightarrow V^{n \times 1}$ be a vector operator defined componentwise $\vec{A}(\vec{x}) = [A(x_1) \ \dots \ A(x_n)]^T$. Then system (4.1) can be written in the following vector form:

$$\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi}. \quad (4.2)$$

Any column $\vec{v} \in V^{n \times 1}$ which satisfies the previous system is its solution.

Powers of operator A are defined as usual $A^i = A^{i-1} \circ A$ assuming that $A^0 = I : V \rightarrow V$ is the identity operator. By n th order linear A -operator equation with constant coefficients, in unknown x , we mean

$$A^n(x) + b_1 A^{n-1}(x) + \cdots + b_n I(x) = \varphi, \quad (4.3)$$

where $b_1, \dots, b_n \in K$ are coefficients and $\varphi \in V$. Any vector $v \in V$ which satisfies (4.3) is its solution.

The following theorem separates variables of the initial system.

Theorem 4.1. *Assume that the linear system of first-order A -operator equations is given in the form (4.2), $\vec{A}(\vec{x}) = B\vec{x} + \vec{\varphi}$, and that matrices B_0, \dots, B_{n-1} are coefficients of the matrix polynomial $\vec{B}(\lambda) = \text{adj}(\lambda I - B)$. Then it holds the following:*

$$\left(\Delta_B(\vec{A}) \right)(\vec{x}) = \sum_{k=1}^n B_{k-1} \vec{A}^{n-k}(\vec{\varphi}). \quad (4.4)$$

Proof. Let $L_B : V^{n \times 1} \rightarrow V^{n \times 1}$ be a linear operator defined by $L_B(\vec{x}) = B\vec{x}$. Replacing λI by \vec{A} in the equation $\Delta_B(\lambda)I = \vec{B}(\lambda)(\lambda I - B)$, we obtain that

$$\begin{aligned} \Delta_B(\vec{A}) &= \vec{B}(\vec{A}) \circ (\vec{A} - L_B), \text{ hence} \\ \Delta_B(\vec{A})(\vec{x}) &= \vec{B}(\vec{A})\left((\vec{A} - L_B)(\vec{x})\right) \\ &= \vec{B}(\vec{A})\left(\vec{A}(\vec{x}) - B\vec{x}\right) \stackrel{(8)}{=} \vec{B}(\vec{A})(\vec{\varphi}) = \sum_{k=1}^n B_{k-1} \vec{A}^{n-k}(\vec{\varphi}). \end{aligned} \quad (4.5)$$

□

The next theorem is an operator generalization of Cramer's rule.

Theorem 4.2 (the theorem of total reduction-vector form). *Linear system of first-order A -operator equations (4.2) can be reduced to the system of n th order A -operator equations*

$$\left(\Delta_B(\vec{A}) \right)(\vec{x}) = \sum_{k=1}^n (-1)^{k-1} \vec{\delta}_k(B; \vec{A}^{n-1}(\vec{\varphi})). \quad (4.6)$$

Proof. It is an immediate consequence of Theorems 4.1 and 3.3 as follows:

$$\Delta_B(\vec{A})(\vec{x}) \stackrel{(10)}{=} \sum_{k=1}^n B_{k-1} \vec{A}^{n-k}(\vec{\varphi}) \stackrel{(6)}{=} \sum_{k=1}^n (-1)^{k-1} \vec{\delta}_k(B; \vec{A}^{n-k}(\vec{\varphi})). \quad (4.7)$$

□

We can now rephrase the previous theorem as follows.

Theorem 4.3 (the theorem of total reduction). *Linear system of first-order A -operator equations (4.1) implies the system, which consists of n th order A -operator equations as follows:*

$$\begin{aligned}
 (\Delta_B(A))(x_1) &= \sum_{k=1}^n (-1)^{k-1} \delta_k^1 (B; \vec{A}^{n-k}(\vec{\varphi})) \\
 &\quad \vdots \\
 (\Delta_B(A))(x_i) &= \sum_{k=1}^n (-1)^{k-1} \delta_k^i (B; \vec{A}^{n-k}(\vec{\varphi})) \\
 &\quad \vdots \\
 (\Delta_B(A))(x_n) &= \sum_{k=1}^n (-1)^{k-1} \delta_k^n (B; \vec{A}^{n-k}(\vec{\varphi})).
 \end{aligned} \tag{4.8}$$

Remark 4.4. System (4.8) has separated variables, and it is called totally reduced. The obtained system is suitable for applications since it does not require a change of base. This system consists of n th order linear A -operator equations which differ only in the variables and in the nonhomogeneous terms.

Transformations of the linear systems of operator equations into independent equations are important in applied mathematics [1]. In the following two sections: we apply our theorem of total reduction to the specific linear operators A .

5. Cauchy Problem

Let us assume that A is a differential operator on the vector space of real functions and that system (4.1) has initial conditions $x_i(t_0)=c_i$, for $1 \leq i \leq n$. Then the Cauchy problem for system (4.1) has a unique solution. Using form (4.2), we obtain additional $n - 1$ initial conditions of i th equation in system (4.8). Consider

$$\left(A^j(x_i) \right) (t_0) = \left[B^j \vec{x}(t_0) \right]_i + \sum_{k=0}^{j-1} \left[B^{j-1-k} \vec{A}^k(\vec{\varphi})(t_0) \right]_i, \quad 1 \leq j \leq n-1, \tag{5.1}$$

where $[\]_i$ denotes i th coordinate. Then each equation in system (4.8) has a unique solution under given conditions and additional conditions (5.1), and these solutions form a unique solution to system (4.1). Therefore, formulae (4.8) can be used for solving systems of differential equations.

It is worth pointing out that the above method can be also extended to systems of difference equations.

6. Differential Transcendancy

Now suppose that V is the vector space of meromorphic functions over the complex field C and that A is a differential operator, $A(x) = (d/dz)(x)$. Let us consider system (4.1) under these assumptions.

Recall that a function $x \in V$ is differentially algebraic if it satisfies a differential algebraic equation with coefficients from C ; otherwise, it is differentially transcendental (see [2–4, 8–10]).

Let us consider nonhomogenous linear differential equation of n th order in the form (4.3), where $b_1, \dots, b_n \in C$ are constants and $\varphi \in V$. If x is differentially transcendental then $\Delta_B(A)(x)$ is also a differentially transcendental function. On the other hand, if φ is differentially transcendental, then, based on Theorem 2.8. from [10], the solution x of (4.3) is a differentially transcendental function. Therefore, we obtain the equivalence.

Theorem 6.1. *Let x be a solution of (4.3), and then x is a differentially transcendental function if and only if φ is a differentially transcendental function.*

We also have the following statement about differential transcendancy.

Theorem 6.2. *Let φ_j be the only differentially transcendental component of the free column $\vec{\varphi}$. Then for any solution \vec{x} of system (4.2), the corresponding entry x_j is also a differentially transcendental function.*

Proof. The sum $\sum_{k=1}^n (-1)^{k-1} \delta_k^j(B; \vec{A}^{n-k}(\vec{\varphi}))$ must be a differentially transcendental function. The previous theorem applied to the following equation:

$$\Delta_B(A)(x_j) = \sum_{k=1}^n (-1)^{k-1} \delta_k^j(B; \vec{A}^{n-k}(\vec{\varphi})) \quad (6.1)$$

implies that x_j is a differentially transcendental function too. \square

Let us consider system (4.1), and let φ_1 be the only differentially transcendental component of the free column $\vec{\varphi}$. Then, the coordinate x_1 is a differentially transcendental function too. Whether the other coordinates x_k are differentially algebraic depends on the system matrix B . From the formulae of total reduction and Theorem 6.1. we obtain the following statement.

Theorem 6.3. *Let φ_1 be the only differentially transcendental component of the free column $\vec{\varphi}$ of system (4.1). Then the coordinate x_k , $k \neq 1$, of the solution \vec{x} , is differentially algebraic if and only if in the sum $\sum_{j=1}^n (-1)^{j-1} \delta_j^k(B; \vec{A}^{n-j}(\vec{\varphi}))$ appears no function $A^{n-j}(\varphi_1)$ ($j = 1, \dots, n$).*

Example 6.4. Let us consider system (4.1) in the form (4.8) in dimensions $n = 2$ and $n = 3$ with φ_1 as the only differentially transcendental component. The function x_1 is differentially transcendental. For $n = 2$ the function x_2 is differentially algebraic if and only if $b_{21} = 0$. For $n = 3$ the function x_2 is differentially algebraic if and only if $b_{21} = 0 \wedge b_{31} \cdot b_{23} = 0$ and the function x_3 is differentially algebraic iff $b_{31} = 0 \wedge b_{31} \cdot b_{22} = 0$.

Let us emphasize that if we consider two or more differentially transcendental components of the free column $\vec{\varphi}$, then the differential transcendency of the solution coordinates also depends on some kind of their differential independence (see e.g., [8]).

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