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Research Article

On g-Semisymmetric Rings

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We introduce right (left) g-semisymmetric ring as a new concept to generalize the well-known concept: symmetric ring. Examples are given to show that these classes of rings are distinct. They coincide under some conditions. It is shown that R is bounded right g-semisymmetric with boundary 1 from right if and only if R is symmetric, whenever R is regular. It is shown that a ring R is strongly regular if and only if R is regular and bounded right g-semisymmetric with boundary 1 from right. For a right $p.p.$-ring R it is shown that R is reduced if and only if R is symmetric, if and only if R is bounded right g-semisymmetric ring with boundary 1 from left, if and only if R is IFP, if and only if R is abelian. We prove that there is a special subring of the ring of 3×3 matrices over a ring without zero divisors which is bounded right g-semisymmetric with boundary 2 from left and boundary 2 from right. Also we show that flat left modules over bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right are bounded left g-semisymmetric with boundaries 1 from left and 1 from right.

1. Introduction

Throughout this paper, all rings are associated with identity and all modules are unitary. For a subset X of R, the left (right) annihilator of X in R is denoted by $l(X)$ $(r(X))$. If $X = \{a\}$, we usually abbreviate $l(a)$ $(r(a))$. According to Lambic [1], a ring R is called *symmetric* if $abc = 0$ then $acb = 0$ for $a, b \in R$. A ring R is called reduced if it has no nonzero nilpotent elements. Reduced rings are symmetric according to [2, Theorem 1.3]. According to Lee and Zhou [3], a left R-module M is reduced if $a^2m = 0$ implies $aRm = 0$, for all $a \in R$, $m \in M$. Abelian rings are rings in which each idempotent is central. According to Buhphang and Rege [4], a left R-module M is semicommutative, if $am = 0$ implies $aRm = 0$, for all $a \in R$, $m \in M$. Reduced rings are symmetric [2, Theorem 1.3]. Commutative rings are symmetric. Semicommutative rings are abelian [5, Lemma 2.7]. Several examples in the indicated references were given to show that the converse of these implications is not necessary to be true, for example, [2, Example II.5] is an example of noncommutative nonreduced symmetric ring. g-semisymmetric rings are defined and studied herein. A ring R is called right g-semisymmetric if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers $n = n(c)$, $l = l(b)$ such that $ac^n b^l = 0$. A ring R is called bounded right g-semisymmetric with boundary n from left if for $a, b, c \in R$ with $abc = 0$, there exists two positive integers n, $l = l(b)$ such that $ac^s b^l = 0$, for all $s \geq n$. Clearly, symmetric rings are right g-semisymmetric. Examples 2.2 and 2.21 are given to show that there exist right g-semisymmetric rings which are not symmetric. Bounded right g-semisymmetric ring with boundary 1 from left is abelian. This is false for rings without identity, by Example 2.2. Also its converse is not necessary true as shown from Example 2.17. The converse holds if R is right $p.p.$-ring, by Theorem 2.19.

2. G-Semisymmetric Rings

Definition 2.1. (1) A right R-module M is called g-semisymmetric if for $m \in M$ and $a, b \in R$ with $mab = 0$, there exist two positive integers $n = n(b)$, $l = l(a)$ such that $mb^n a^l = 0$. A ring R is called right g-semisymmetric if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers $n = n(c)$, $l = l(b)$ such that $ac^n b^l = 0$.

(2) A left R-module M is called g-semisymmetric if for $m \in M$ and $a, b \in R$ with $abm = 0$, there exist two positive integers $n = n(b)$, $l = l(a)$ such that $b^n a^l m = 0$. A ring R is called left g-semisymmetric if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers $n = n(b)$, $l = l(a)$ such that $b^n a^l c = 0$.

(3) A right R-module M is called bounded g-semisymmetric with boundary n from left if for $m \in M$ and $a, b \in R$ with $mab = 0$, there exist two positive integers n, $l = l(a)$ such that $mb^s a^l = 0$, for all $s \geq n$. A ring R is called bounded right g-semisymmetric with boundary n from left if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers n, $l = l(b)$ such that $ac^s b^l = 0$, for all $s \geq n$.

(4) A right R-module M is called bounded g-semisymmetric with boundary l from right if for $m \in M$ and $a, b \in R$ with $mab = 0$, there exist two positive integers $n = n(b)$, l such that $mb^n a^s = 0$, for all $s \geq l$. A ring R is called bounded right g-semisymmetric with boundary l from right if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers $n = n(c)$, l such that $ac^n b^s = 0$, for all $s \geq l$.

(5) A left R-module M is called bounded g-semisymmetric with boundary n from left if for $m \in M$ and $a, b \in R$ with $abm = 0$, there exist two positive integers n, l such that $abm = 0$, then $b^s a^l m = 0$, for all $s \geq n$. A ring R is called bounded left g-semisymmetric with boundary n from left if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers n, $l = l(a)$ such that $b^s a^l c = 0$, for all $s \geq n$.

(6) A left R-module M is called bounded g-semisymmetric with boundary l from right if for $m \in M$ and $a, b \in R$ with $abm = 0$, there exist two positive integers $n = n(b)$, l such that $abm = 0$, then $b^n a^s m = 0$, for all $s \geq l$. A ring R is called bounded left g-semisymmetric with boundary l from right if for $a, b, c \in R$ with $abc = 0$, there exist two positive integers $n = n(b)$, l such that $b^n a^s c = 0$, for all $s \geq l$.

Every symmetric ring is right g-semisymmetric ring, the converse is not true as illustrated by the following example, due originally to Bell [6, Example 9] with changes in its multiplications.

Example 2.2. Let $S = \{a, b\}$ be the semigroup with multiplication $a^2 = ba = a$, $b^2 = ab = b$. Put $T = F_2S$, which is a four-element semigroup ring without identity. This ring T is bounded right g-semisymmetric ring with boundary 2 from right and with boundary 1 from left, but T is neither symmetric ring nor reversible.

Remark 2.3. (1) A ring R is left g-semisymmetric if and only if the module ${}_RR$ is g-semisymmetric. (2) A ring R is right g-semisymmetric if and only if the module R_R is g-semisymmetric.

Proposition 2.4. *The following conditions are equivalent for a right R-module M.*

(1) *M is g-semisymmetric.*

(2) *All cyclic submodules of M are g-semisymmetric.*

Proof. (1) $\Rightarrow$ (2) Let $N = mR$ be a cyclic submodules of M, and let $m' \in N$. Since M is g-semisymmetric, then for $a, b \in R$ with $m'ab = 0$, it implies that $m'b^m a^n = 0$, and some positive integers $m = m(b)$,$n = n(a)$. Hence N is g-semisymmetric.

(2) $\Rightarrow$ (1) Let $a, b \in R$, $m \in M$ such that $mab = 0$. Since the cyclic R-module mR is semisymmetric, then there exist positive integers $m = m(b)$, $n = n(a)$ such that $mb^m a^n = 0$. Therefore M is g-semisymmetric. □

Proposition 2.5. *The following conditions are equivalent for a ring R.*

(i) *R is strongly regular.*

(ii) *Every right R-module is flat and g-semisymmetric with boundary 1 from right.*

(iii) *Every cyclic right R-module is flat and g-semisymmetric with boundary 1 from right.*

(iv) *R is regular and bounded right g-semisymmetric with boundary 1 from right.*

Proof. $(i) \Rightarrow (ii)$ Let R be a strongly regular ring, and let M be a right R-module. Then M is flat module. Let $m \in M$ and $r, s \in R$ with $mrs = 0$, and let $I = \{x \in R \mid mx = 0\}$. Since R is strongly regular, then the right ideal I of R is a two-sided ideal and R has no nilpotent elements. Hence $\overline{R} = R/I$ has no nilpotent elements. Since $rs \in I$, then $((\overline{s})^m(\overline{r})^n)^2 = \overline{0}$ and hence $(\overline{s})^m\overline{r} = \overline{0}$. This shows that $ms^m r = 0$. Therefore M is bounded g-semisymmetric with boundary 1 from right.

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (iv)$ Suppose that every cyclic right R-module is flat and g-semisymmetric with boundary 1 from right. Since every cyclic right R-module is flat, then R is a regular ring [7, Theorem 4.21]. Since every cyclic right R-module is g-semisymmetric with boundary 1 from right, then R_R is g-semisymmetric with boundary 1 from right proving that the ring R is bounded right g-semisymmetric with boundary 1 from right.

$(iv) \Rightarrow (i)$ Let R be regular and bounded right g-semisymmetric with boundary 1 from right. Suppose that $x \in R$ with $x^2 = 0$. Since R is regular, then there exists $y \in R$ such that $x = xyx$. Since R is bounded right g-semisymmetric ring with boundary 1 from right and $y(x)(xy) = 0$, then $y(xy)^n x^l = 0$ for all $l \geq 1$. Since $x = xyx = xyxyx = xyxyxyx = \cdots = (xy)^n x$, then $yx = y(xy)^n x = 0$. Therefore $x = xyx = 0$. Hence R has no nonzero nilpotent element and R is strongly regular ring. □

Corollary 2.6. *If a ring R is regular and bounded right g-semisymmetric with boundary 1 from right, then R is reduced.*

A one-sided ideal I of a ring R is said to have the insertion-of-factors principle (or simply IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. Hence the ring R is called IFP ring if the zero ideal of R has the IFP. Such rings are also known as semicommutative rings or rings satisfying SI condition or ZI rings, see [6, 8–10]. The equivalences of (1), (2), (4), (5), and (6) in the following proposition are in [11, Proposition 2.7 (7)]. By Corollary 2.6 and the fact that every symmetric ring is bounded right g-semisymmetric with boundary 1 from right, we state without proof the following proposition.

Proposition 2.7. *Let R be a von Neumann regular ring. Then the following conditions are equivalent:*

(1) *R is right (left) duo,*

(2) *R is reduced,*

(3) *R is bounded right g-semisymmetric with boundary 1 from right,*

(4) *R is symmetric,*

(5) *R is IFP,*

(6) *R is abelian.*

Proposition 2.8. *(1) The class of right g-semisymmetric rings is closed under subrings.*

(2) The class of bounded right g-semisymmetric rings with boundaries 1 from left and 1 from right is closed under direct products.

(3) A ring is semiperfect and bounded right g-semisymmetric with boundary 1 from left if and only if R is a finite direct sum of local bounded right g-semisymmetric rings from left.

(4) A ring R is strongly regular if and only if R is regular and bounded right g-semisymmetric ring with boundary 1 from left if and only if R is regular and bounded right g-semisymmetric ring with boundary 1 from right.

Proof. (1) Trivial.

(2) Assume that R is a direct product of bounded right g-semisymmetric rings R_i, $i \in I$ with boundaries 1 from left and 1 from right. Let $x_i, y_i, z_i \in R_i$, $i \in I$ with $(x_1, x_2, \ldots)(y_1, y_2, \ldots)(z_1, z_2, \ldots) = (0, 0, \ldots)$. Then $x_i y_i z_i = 0$, $i = 1, 2, \ldots$. Since R_i, $i \in I$ are bounded right g-semisymmetric rings with boundaries 1 from left and 1 from right, then $x_i z_i^{s_i} y_i^{n_i} = 0$, for all $s_i \geq 1$, and $n_i \geq 1$, $i = 1, 2 \ldots$. Therefore $(x_1, x_2, \ldots)(z_1, z_2, \ldots)^s (y_1, y_2, \ldots)^n = 0$, for all $s \geq 1$ and $n \geq 1$. Hence R is bounded right g-semisymmetric ring with boundaries 1 from left and 1 from right.

(3) Assume that R is semiperfect bounded right g-semisymmetric ring with boundary 1 from left. Since R is semiperfect, R has a finite orthogonal set of local idempotents whose sum is 1 [1, Proposition 3.7.2]. Hence we consider $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since R is bounded right g-semisymmetric rings with boundary 1 from left, then R is abelian by Lemma 2.16, whence every e_i is central and $e_i R$ is an ideal of R, $i = 1, 2, \ldots, n$. Thus $e_i R = e_i R e_i$, for all $i = 1, 2, \ldots, n$. It follows that each $e_i R$ is bounded right g-semisymmetric ring with boundary 1 from left, by (1).

Conversely, suppose that R is a finite direct sum of local bounded right g-semisymmetric rings with boundary 1 from left. Then, by (2), and the fact that local rings are semiperfect, R is bounded right g-semisymmetric ring with boundary 1 from left.

(4) By Lemma 2.16, every bounded right g-semisymmetric ring with boundary 1 from left with identity is abelian. Moreover, as R is regular, then this is equivalent to R be strongly regular by [12, Theorem 3.7] which is equivalent to the condition R is regular and bounded right g-semisymmetric ring with boundary 1 from right, by Proposition 2.5. $\square$

Proposition 2.9. *Let $\theta : R \to A$ be a ring homomorphism and M a left A-module; then M is a left R-module via $r \cdot m = \theta(r) \cdot m$. Moreover,*

(1) *If $_AM$ is g-semisymmetric, then so is $_RM$,*

(2) *If θ is onto and $_RM$ is g-semisymmetric, then so is $_AM$.*

Proof. (1) Suppose $_AM$ is g-semisymmetric, and let $a, b \in R$, $m \in M$ such that $abm = 0$. Then $0 = abm = \theta(ab)m = \theta(a)\theta(b)m$. Since $_AM$ is g-semisymmetric, then there exist positive integers $s = s(b)$, $t = t(a)$ such that $\theta(b)^s\theta(a)^t m = 0$. Hence $b^s a^t m = \theta(b^s a^t)m = \theta(b^s)\theta(a^t)m = \theta(b)^s(\theta(a))^t m = 0$. Therefore $_RM$ is g-semisymmetric.

(2) Let $a, b \in A$, $m \in M$ such that $abm = 0$. Since θ is onto, there exists $r, s \in R$ such that $\theta(r) = a$, $\theta(s) = b$. Now $0 = abm = \theta(r)\theta(s)m = rsm$. Since $_RM$ is g-semisymmetric, then there exist positive integers $t = t(s)$, $n = n(r)$ such that $s^t r^n m = 0$ and $b^t a^n m = \theta(s^t)(\theta(r))^n m = \theta(s^t)\theta(r^n)m = s^t r^n m = 0$. Hence $_AM$ is g-semisymmetric. □

Lemma 2.10 (see [10, Proposition 2.6]). *Suppose that M is a flat left R-module. Then for every exact sequence $0 \to K \to F \to M \to 0$ where F is R-free, one has $(IF) \cap K = IK$ for each right ideal I of R; in particular, one has $xF \cap K = xK$ for each element x of R.*

Lemma 2.11. *Let R be a bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right, then every free left R-module M is bounded g-semisymmetric with boundaries 1 from left and 1 from right.*

Proof. Since M is free module, then M is isomorphic to a (possibly infinite) direct sum of copies of R, see [7]. Since R is bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right, then $_RM$ is bounded g-semisymmetric with boundaries 1 from left and 1 from right, by Proposition 2.8. □

Now we are ready to prove the following proposition.

Proposition 2.12. *Flat left modules over bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right are bounded left g-semisymmetric with boundaries 1 from left and 1 from right.*

Proof. Let $_RM$ be a flat module over bounded left g-semisymmetric ring R with boundaries 1 from left and 1 from right. Let $m \in M$ and $a \in R$ be such that $abm = 0$. Suppose that for the epimorphism $\beta : F \to M$ the sequence $0 \to K \to F \to M \to 0$ is exact. Now there exists $y \in F$ such that $\beta(y) = m$. This implies that $\beta(aby) = abm = 0$. Hence $aby \in \ker(\beta) = \operatorname{Im} K = K$. Therefore $aby \in (abF) \cap K = abK$, by Lemma 2.10. Hence for some $k \in K$, $aby = abk$, yielding $ab(y - k) = 0$. Since F is free R-module over bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right, then $_RF$ is bounded g-semisymmetric with boundaries 1 from left and 1 from right, by Lemma 2.11. Therefore $b^n a^s(y - k) = 0$, for all $n \geq 1$ and $s \geq 1$. Hence $b^n a^s y = b^n a^s k$ and so $\beta(b^n a^s y) = \beta(b^n a^s k)$ gives $b^n a^s \beta(y) = b^n a^s \beta(k)$, for all $n \geq 1$ and $s \geq 1$. Since $k \in \ker \beta$, then $b^n a^s \beta(k) = 0$ implies $b^n a^s \beta(y) = 0$, for all $n \geq 1$ and $s \geq 1$. Hence $b^n a^s m = 0$, for all $n \geq 1$ and $s \geq 1$. Thus $_RM$ is bounded g-semisymmetric with boundaries 1 from left and 1 from right. □

In the following propositions $E(M)$ denotes the R-endomorphism ring of M. The associativity is deduced from the generalized associativity situation in the standard Morita context $(R, M, M^*, E(M))$ without explicit mention, where M^* is the left $E(M)$-,right R-bimodule $\operatorname{Hom}_R(M, R)$.

A torsionless R-module M is an R-module M such that M is a direct product of copies of R, or, equivalently, if $0 \neq m \in M$, then there exists $q \in M^*$ such that $mq \neq 0$. If M is faithful R-module, then R is a submodule of a direct product of copies of M. The following proposition is an application of Remark 2.3 and Proposition 2.8.

Proposition 2.13. *The following conditions are equivalent.*

(1) *R is a bounded left g-semisymmetric ring with boundaries 1 from left and 1 from right.*

(2) *Every torsionless left R-module is bounded g-semisymmetric with boundaries 1 from left and 1 from right.*

(3) *Every submodule of a free left R-module is bounded g-semisymmetric with boundaries 1 from left and 1 from right.*

(4) *There exists a faithful, bounded g-semisymmetric left R-module with boundaries 1 from left and 1 from right.*

An application of Propositions 2.13 and 2.9 yields the following proposition.

Proposition 2.14. *For an R-module M, let $\overline{R}$ denote the ring $R/ann(M)$. Then one has the following.*

(1) *The left R-module M is g-semisymmetric if and only if the left $\overline{R}$-module M is g-symmetric.*

(2) *If the left $\overline{R}$-module M is bounded g-semisymmetric with boundaries 1 from left and 1 from right, then $\overline{R}$ is bounded left g-semisymmetric with boundaries 1 from left and 1 from right.*

(3) *If the right $E(M)$-module M is bounded g-semisymmetric from left, then the ring $E(M)$ is bounded right g-semisymmetric with boundaries 1 from left and 1 from right.*

An application of Proposition 2.9 yields (1); since the left R-, right $E(M)$-bimodule M is faithful as a left $\overline{R}$-module and is also faithful as a right $E(M)$-module, applying (4) $\Rightarrow$ (1) of Proposition 2.13 we get (2) and (3).

Let M be a right R-module. Then as in [10] M is called

(1) reduced if $ma^2 = 0$, then $mRa = 0$, $a \in R, m \in M$;

(2) ZI (zero-insertive ring) if $ma = 0$, then $mRa = 0$, $a \in R, m \in M$.

Proposition 2.15. *Let R be a right R-module M. Then,*

(1) *if M is reduced, then M is symmetric [10, Proposition 2.2],*

(2) *if M is symmetric, then M is ZI [10, Proposition 2.2].*

Lemma 2.16. *If R is bounded right g-semisymmetric ring with boundary 1 from left, then R is abelian.*

Proof. Assume that R is bounded right g-semisymmetric ring with boundary 1 from left and e is an idempotent. Then $e - e^2 = 0$ gives $e(1-e) = 0$. Hence for all $x \in R$ there exists a positive integer n such that $ex^s(1-e)^n = 0$ for all $s \geq 1$. Therefore $ex = exe$. And since $(1-e)e = 0$, then $xe = exe$. Therefore e is central. □

The previous lemma is false for rings without identity. Indeed, the ring T in Example 2.2 is a ring without identity and it is a bounded right g-semisymmetric ring with boundary 2 from right and 1 from left which is nonabelian ring. Also its converse is not necessary true as shown from the following example.

Example 2.17. *We use [1, Example 2.10], as a counter example. Let* $R = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2\times 2}(\mathbb{Z}) : a \equiv d(\mathrm{mod}2), b = c \equiv 0(\mathrm{mod}2)\}$, *where* $M_{2\times2}(\mathbb{Z})$ *is the full matrix ring over the ring of integers. Since the zero and the identity matrices are only the idempotent elements in* R, *then* R *is abelian ring. Since* $\left(\begin{smallmatrix} 0 & 0 \\ -2 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 4 \\ 0 & 2 \end{smallmatrix}\right) = 0$ *and* $\left(\begin{smallmatrix} 0 & 0 \\ -2 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 4 \\ 0 & 2 \end{smallmatrix}\right)^m\left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right)^n \neq 0$ *for any positive integers* m *and* n, *then* R *is not right g-semisymmetric ring.*

A ring R be a right $p.p.$-ring if for any $a \in R$, $r(a) = eR$ for some idempotent e of R.

Proposition 2.18. *Let* R *be a right* $p.p.$*-ring. If* R *is abelian, then* R *is reduced.*

Proof. Let R be abelian right $p.p.$-ring. Let $a^2 = 0$. Since R is right $p.p.$-ring, then $r(a) = eR$, for some idempotent e of R. Since R is abelian and $a \in r(a)$, then $a = ea = ae = 0$ and hence R is reduced. □

Since every reduced ring is symmetric, bounded right g-semisymmetric ring with boundary 1 from left and IFP, since every bounded right g-semisymmetric ring with boundary 1 from left is abelian, by Lemma 2.16 and since every reduced ring, symmetric ring, and IFP ring are abelian, then we deduce the following theorem from the above proposition.

Theorem 2.19. *Let* R *be right* $p.p.$*-ring. Then the following are equivalent.*

(1) *R is reduced.*

(2) *R is symmetric.*

(3) *R is bounded right g-semisymmetric ring with boundary 1 from left.*

(4) *R is IFP.*

(5) *R is abelian.*

Theorem 2.20. *Let* S *be a ring without zero divisors and* $R = \left\{\left(\begin{smallmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{smallmatrix}\right) \mid a, b, c, d \in S\right\}$. *Then* R *is bounded right g-semisymmetric with boundary 2 from left and boundary 2 from right.*

Proof. Suppose that $0 \neq A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}$, $0 \neq B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}$, $0 \neq C = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} \in R$ such that $ABC = 0$. Then,

$$\begin{aligned}
&a_1a_2a_3 = 0,\\
&a_1a_2b_3 + a_1a_3c_3 + a_2a_3b_1 = 0,\\
&a_1a_2d_3 + a_3b_1d_2 + a_2a_3d_1 = 0,\\
&a_1a_2c_3 + a_1b_2d_3 + a_1a_3c_2 + a_2b_1d_3 + a_3b_1d_2 + a_2a_3c_1 = 0.
\end{aligned}$$

Therefore we have the following cases:

(1) if $a_1 = 0$, $a_2 \neq 0$, $a_3 \neq 0$, then $A = 0$, impossible,

(2) if $a_1 \neq 0$, $a_2 = 0$, $a_3 \neq 0$, then $B = \left(\begin{smallmatrix} 0 & b_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, and $C = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$; in this case, $ACB^2 = 0$,

(3) if $a_1 \neq 0$, $a_2 \neq 0$, $a_3 = 0$, then $C = 0$, impossible,

(4) if $a_1 = 0$, $a_2 = 0$, $a_3 \neq 0$, then $A = \left(\begin{smallmatrix} 0 & 0 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, and $C = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix}$; hence $ACB^2 = 0$,

(5) if $a_1 = 0$, $a_2 \neq 0$, $a_3 = 0$, then $A = \left(\begin{smallmatrix} 0 & 0 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}$, and $C = \left(\begin{smallmatrix} 0 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ which implies that $AC^2B = 0$,

(6) if $a_1 \neq 0$, $a_2 = 0$, $a_3 = 0$, then $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & c_2 \\ 0 & 0 & d_2 \\ 0 & 0 & a \end{pmatrix}$, and $C = \begin{pmatrix} 0 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which implies that $ACB^2 = 0$.

These cases prove that R is bounded right g-semisymmetric ring with boundary 2 from left and right. □

The following example gives a bounded right g-semisymmetric ring with boundary 2 from left and right which is not symmetric.

Example 2.21. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$. *Then R is a bounded right g-semisymmetric ring with boundary 2 from left and boundary 2 from right which is not symmetric.*

Since $\mathbb{Z}$ is a ring without zero divisors, then R is bounded right g-semisymmetric ring with boundary 2 from left and boundary 2 from right, by the above theorem. This ring is not symmetric, indeed; suppose $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then $ABC = 0$ and $ACB \neq 0$, and hence R is not symmetric ring. Also we notice that $AB^2 = 0$ and $ACB \neq 0$ and therefore R_R is not reduced.

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