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Research Article

Unsteady Reversed Stagnation-Point Flow over a Flat Plate

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This paper investigates the nature of the development of two-dimensional laminar flow of an incompressible fluid at the reversed stagnation-point. Proudman and Johnson (1962) first studied the flow and obtained an asymptotic solution by neglecting the viscous terms. Robins and Howarth (1972) stated that this is not true in neglecting the viscous terms within the total flow field. Viscous terms in this analysis are now included, and a similarity solution of two-dimensional reversed stagnation-point flow is investigated by solving the full Navier-Stokes equations.

1. Introduction

The full Navier-Stokes equations are difficult or impossible to obtain an exact solution in almost every real situation because of the analytic difficulties associated with the nonlinearity due to convective acceleration. The existence of exact solutions is fundamental in their own right not only as solutions of particular flows, but also as agreeable in accuracy checks for numerical solutions.

In some simplified cases, such as a fluid traveling through a rigid body (e.g., missile, sports ball, automobile, spaceflight vehicle), or in oil recovery industry, crude oil that can be extracted from an oil field is achieved by gas injection, as shown in Figure 1 or, equivalently, an external flow impinges on a stationary point called stagnation-point that is on the surface of a submerged body in a flow, of which the velocity at the surface of the submerged object is zero. A stagnation-point flow develops and the flow in the vicinity of this stagnation point is governed by Navier-Stokes equations. The classic problems of two-dimensional stagnation-point flows can be analyzed exactly by Hiemenz [1]. The result is an exact solution for flow

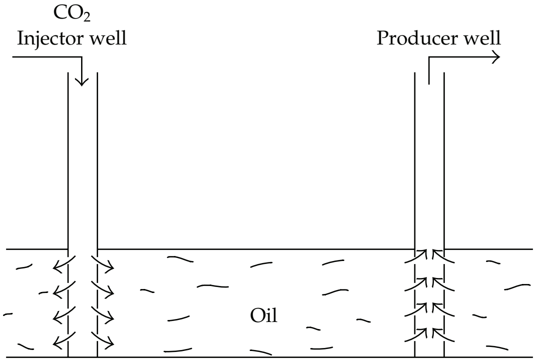

Figure 1: Oil recovery industry.

directed perpendicular to an infinite flat plate. Howarth [2] and Davey [3] extended the two-dimensional and axisymmetric flows to three dimensions, which are exact solutions of the Navier-Stokes equations, and Wang [4] studied the case for obliquely-impacting jets.

On the contrary, a solution against an infinite flat plane does not exist if the potential flow at infinity is reversed. The flow in the vicinity of this reversed stagnation point is governed by boundary-layer separation and vorticity generation and the reversed stagnation-point flow develops. Reversed stagnation-point flow is a flow in which the component of velocity normal to a wall is outward the wall everywhere in the region concerned, so that the vorticity created at the wall will be convected toward the wall, in opposition to viscous diffusion away from it.

Proudman and Johnson [5] suggested that the convection terms dominate in considering the inviscid equation in the body of the fluid. By introducing a very simple function of a particular similarity variable and neglecting the viscous forces in their analytic result for region sufficient far from the wall, they obtained an asymptotic solution in reversed stagnation-point flow, describing the development of the region of separated flow for large time t. Robins and Howarth [6] have recently extended the asymptotic solution, finding the higher order terms by singular perturbation methods. They indicated that the viscous forces cannot be ignored in the governing equation because of a consistent asymptotic expansion in both this outer inviscid region and also in the inner region near the plane. Smith [7] generalized the solution of Proudman and Johnson with both viscous and convection terms in balance by considering the monotonic potential flow when the time is relatively large. Shapiro [8] obtained a solution for unsteady backward stagnation-point flow with injection or suction.

These unsteady flows fit within a class of similarity transformations originally identified by Birkhoff using a group-theoretic approach.

Numerical simulation of reversed stagnation-point flow with full Navier-Stokes equations has been studied in [9]. In the present study, the unsteady reversed stagnation-point flow is investigated. The flow is started impulsively in motion with a constant velocity away from near the stagnation point. A similarity solution of full Navier-Stokes equations is solved by applying numerical method.

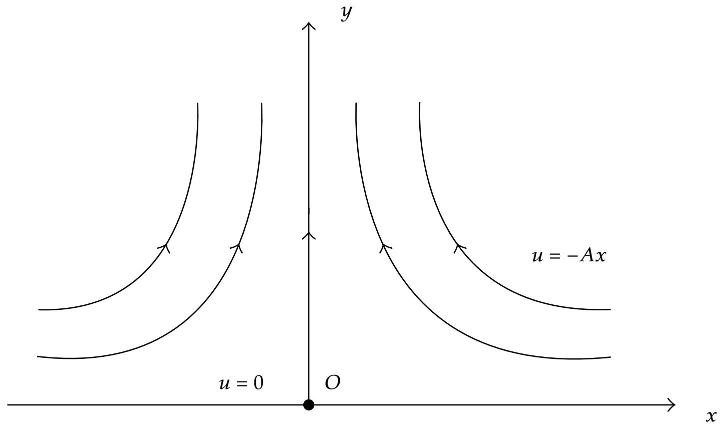

Figure 2: Coordinate system of reversed stagnation-point flow.

2. Flow Analysis Model

The viscous fluid flows in a rectangular Cartesian coordinates (x, y, z), Figure 2, illustrating the motion of external flow directly moving perpendicular out of an infinite flat plane wall. The origin is the so-called stagnation point and z is the normal to the plane.

By conservation of mass principle with constant physical properties, the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.1}$$

We consider the two-dimensional reversed stagnation-point flow in unsteady state and the flow is bounded by an infinite plane $y = 0$, the fluid remains at rest when time $t < 0$. At $t = 0$, it starts impulsively in motion which is determined by the stream function

$$\psi = -\alpha x y. \tag{2.2}$$

At large distances far above the planar boundary, the existence of the potential flow implies an inviscid boundary condition. It is given by

$$\begin{aligned} u &= -\alpha x. \\ v &= V_0, \end{aligned} \tag{2.3}$$

where u and v are the components of flow velocity, A is a constant proportional to V_0/L, V_0 is the external flow velocity removing from the plane, and L is the characteristic length. We have $u = 0$ at $x = 0$ and $v = 0$ at $y = 0$, but the no-slip boundary at wall $(y = 0)$ cannot be satisfied.

Since for a viscous fluid the flow motion is determined by only two factors, the kinematic viscosities ν and α, we consider the following modified stream function:

$$\psi = -\sqrt{A\nu}\,x f(\eta,\tau),$$
$$\eta = \sqrt{\frac{A}{\nu}}\,y, \tag{2.4}$$
$$\tau = At,$$

where η is the nondimensional distance from wall and τ is the nondimensional time. Note that the stream function automatically satisfies equation of continuity (2.1). The Navier-Stokes equations [10] governing the unsteady flow with constant physical properties are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.5}$$

where u and v are the velocity components along x and y axes, and ρ is the density.

Substituting u and v into the governing equations results in a simplified partial differential equation. From the definition of the stream function, we have

$$u = \frac{\partial \psi}{\partial y} = -Axf_{\eta},$$
$$v = -\frac{\partial \psi}{\partial x} = \sqrt{A\nu}\,f. \tag{2.6}$$

The governing equations can be simplified by a similarity transformation when several independent variables appear in specific combinations, in flow geometries involving infinite or semi-infinite surfaces. This leads to rescaling, or the introduction of dimensionless variables, converting the original system of partial differential equations into the following pair of partial differential equations:

$$-A^2 x f_{\eta\tau} + A^2 x (f_{\eta})^2 - A^2 x f f_{\eta\eta} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - A^2 x f_{\eta\eta\eta}, \tag{2.7a}$$

$$A\sqrt{A\nu}\,f_{\tau} + A\sqrt{A\nu}\,f f_{\eta} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A\sqrt{A\nu}\,f_{\eta\eta}. \tag{2.7b}$$

The pressure gradient can be again reduced by a further differentiation equation (2.7b) with respect to x. That is

$$\frac{\partial^2 p}{\partial x \partial y} = 0 \tag{2.8}$$

and (2.7a) reduces to

$$\left[f_{\eta\tau} - (f_\eta)^2 + f f_{\eta\eta} - f_{\eta\eta\eta}\right]_\eta = 0. \tag{2.9}$$

The initial and boundary conditions are

$$\begin{gathered} f(\eta, 0) \equiv \eta \quad (\eta \neq 0), \\ f(0, \tau) = f_\eta(0, \tau) = 0 \quad (t \neq 0), \\ f(\infty, \tau) \sim \eta. \end{gathered} \tag{2.10}$$

The last condition reduces the above differential equation (2.9) to the form

$$f_{\eta\tau} - (f_\eta)^2 + f f_{\eta\eta} - f_{\eta\eta\eta} = -1, \tag{2.11}$$

with the boundary conditions

$$\begin{gathered} f(0, \tau) = f_\eta(0, \tau) = 0, \\ f_\eta(\infty, \tau) = 1. \end{gathered} \tag{2.12}$$

Equation (2.11) is the similarity equation of the full Navier-Stokes equations at a two-dimensional reversed stagnation point. The coordinates x and y are replaced by a dimensionless variable η. Under the boundary conditions $f_\eta(\infty, \tau) = 1$, when the flow is in steady state such that $f_{\eta\tau} \equiv 0$, the differential equation has no solution.

3. Similarity Analysis

3.1. Asymptotic Solution

When τ is relatively small, Proudman and Johnson [5] first considered the early stages of the diffusion of the initial vortex sheet at $y = 0$. They suggested that when the flow is near the wall region, the viscous forces are dominant, and the viscous term in the governing Navier-Stokes equations is important only near the boundary.

On the contrary, the viscous forces were neglected away from the wall. The convection terms dominate the motion of external flow in considering the inviscid equation in the fluid. They considered the similarity of the inviscid equation

$$f_{\eta\tau} - (f_\eta)^2 + f f_{\eta\eta} + 1 = 0. \tag{3.1}$$

Proudman and Johnson obtained that a similarity solution of (3.1) is in the form

$$f(\eta, \tau) = e^\tau F(\gamma), \tag{3.2}$$

$$F(\gamma) = \gamma - \frac{2}{c}\left(1 - e^{-c\gamma}\right), \tag{3.3}$$

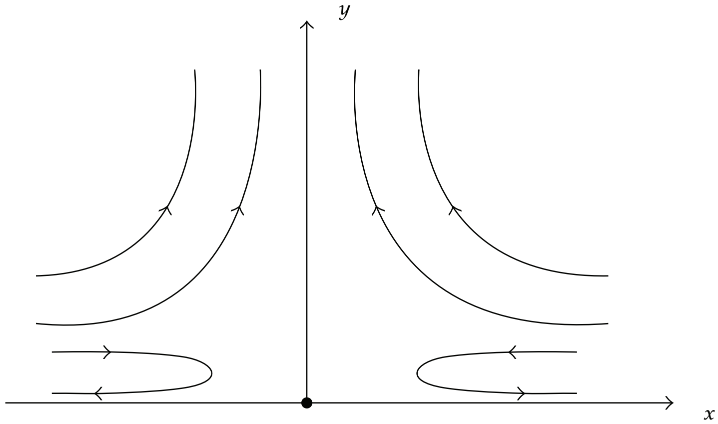

Figure 3: Streamlines of reversed stagnation-point flow.

where $\gamma = \eta e^{-\tau}$ and c is a constant of integration. Robins and Howarth [6] estimated the value of c to be 3.51. This solution describes the flow in the outer region, moving away from the plane with a constant velocity. It can be checked that the viscous term $f_{\eta\eta\eta}$ is still small compared to the convective terms, so that their assumption of neglecting the viscous term is still valid.

In the inner region, the viscous term cannot be neglected and the no-slip condition must be satisfied on the wall. When $\tau \to \infty$ and η/e^{τ} is relatively small, the solution (3.3) yields

$$F \sim -\gamma = -\eta e^{-\tau}, \qquad f = -\eta, \qquad f' = -1. \tag{3.4}$$

Substituting in (2.11) yields

$$\begin{gathered} f''' - ff'' + (f')^2 - 1 = 0, \\ f(0) = f'(0) = 0, \\ f'(\infty) = -1. \end{gathered} \tag{3.5}$$

This is exactly the classic stagnation-point problem (Hiemenz [1]) by changing the sign in f. It is a third-order nonlinear ordinary differential equation and does not have an analytic solution, and thus it is necessary to solve it numerically. The general features of the predicted streamline are sketched in Figure 3.

Although an asymptotic solution was obtained, it can easily been observed that this is not valid when the viscous term $f_{\eta\eta\eta}$ is neglected within the total flow field. Robins and Howarth [6] indicated that there is a consistent asymptotic expansion in both outer inviscid region and also in the inner region that must exist close to the wall where the viscous forces need to be included. No exact solutions in both outer and inner regions were discovered.

3.2. Particular Solution

In our two-dimensional model, the fluid remains at rest when time $t < 0$ and is set in motion at $t > 0$ such that at large distances far above the planar boundary the potential flow is a constant V_0 for all value of t. Both Proudman and Johnson [5], and Robins and Howarth [6] have set $V_0 = 1$ and the corresponding boundary condition $f_\eta(\infty, \tau) = 1$.

When the flow is in steady state such that $f_{\eta\tau} \equiv 0$, it was proven that the similarity velocity $f_\eta(\eta)$ cannot ultimately approach to 1. The differential equation has no solution. If the potential flow V_0 is restricted not to be a constant, the boundary condition $f_\eta(\infty, \tau)$ results in a time dependent function and then we obtain another approach of similarity solution in reversed stagnation-point flow. As with the governing equation of reversed stagnation-point flow, we can write the stream function as

$$\psi = -\sqrt{A\nu}\, x f(\eta, \tau),$$
$$\eta = \sqrt{\frac{A}{\nu}}\, y, \tag{3.6}$$
$$\tau = At,$$

where A is a constant proportional to $V_0(\tau)/L$, $V_0(\tau)$ is the external flow velocity removing from the plane, and L is the characteristic length. These result in the governing equation (2.9)

$$\left[f_{\eta\tau} - (f_\eta)^2 + f f_{\eta\eta} - f_{\eta\eta\eta}\right]_\eta = 0. \tag{3.7}$$

Integrating with respect to η, we have

$$f_{\eta\tau} - (f_\eta)^2 + f f_{\eta\eta} - f_{\eta\eta\eta} = -C(\tau). \tag{3.8}$$

Under the boundary conditions $f_\eta(\infty, \tau) = 1$, the value of $C(\tau)$ should be a constant and equal to 1. If the boundary condition $f_\eta(\infty, \tau)$ is restricted not to be a constant, a particular time-dependence function $C(\tau)$ may be expressed in the form

$$C(\tau) = \frac{c}{\tau^2}, \tag{3.9}$$

where c is an arbitrary constant. The partial differential equation can be simplified by a similarity transformation when a new similarity variable is introduced. This converts the original partial differential equation into an ordinary differential equation. For a time dependent function, we introduce the diffusion variable transformation [7]

$$\varsigma = \eta\sqrt{\frac{1}{\tau}},$$
$$f(\eta, \tau) = \frac{1}{\sqrt{\tau}} F(\varsigma). \tag{3.10}$$

Here ς is the time combined nondimensional variable and $F(\varsigma)$ are the nondimensional velocity functions. Substitution of the similarity transformation yields an ordinary differential equation

$$-\frac{1}{2}\varsigma F'' - F' - F'^2 + FF'' - F''' = -c, \tag{3.11}$$

where the prime denotes the derivative with respect to the variable ς.

Equation (3.11) is a third-order nonlinear ordinary differential equation and a key step in obtaining an analytical solution is to reversedrange the equation in an autonomous differential equation. In mathematics, an autonomous differential equation is a system of ordinary differential equations which does not explicitly depend on the independent variable.

In order to omitting the variable ς in the differential equation, it is recognized a change of variable

$$Q = F - \frac{1}{2}\varsigma \tag{3.12}$$

and the equation becomes an autonomous differential equation

$$QQ'' - 2Q' - Q'^2 - Q''' = -c + \frac{3}{4}. \tag{3.13}$$

In our analysis, $P = Q'$ is the dependent variable and Q is the independent variable. Equation (3.13) is reversed ranged as

$$QP' - 2P - P^2 - P'' = -c + \frac{3}{4} \tag{3.14}$$

and the chain rule reduces equation (3.14) to a second-order ordinary differential equation

$$QP\frac{dP}{dQ} - 2P - P^2 - P\frac{d}{dQ}\left(P\frac{dP}{dQ}\right) = -c + \frac{3}{4}. \tag{3.15}$$

Equation (3.15) is analytically solvable that the solution might be expressed as a low order polynomial. It is suggested that

$$P = a + bQ + dQ^2, \tag{3.16}$$

and substituting into equation (3.14) and comparing the coefficients in the powers of Q results in a system of linear algebraic equations

$$\begin{gathered}
2a^2d + ab^2 + 2a - a^2 = -c + \frac{3}{4}, \\
8abd + b^3 + 2b + ab = 0, \\
8ad^2 + \left(7b^2 + 2\right)d = 0, \\
12bd^2 - bd = 0, \\
6d^3 - d^2 = 0.
\end{gathered} \tag{3.17}$$

Solving the related algebraic equation, we have

$$a = -\frac{3}{2}, \qquad b = 0, \qquad c = \frac{3}{4}, \qquad d = \frac{1}{6}. \tag{3.18}$$

Substituting the constant into (3.16) yields the first-order differential equation

$$Q' = -\frac{3}{2} + \frac{1}{6}Q^2. \tag{3.19}$$

Equation (3.19) is Riccati equation, which is any ordinary differential equation that is quadratic in the unknown function. The standard form of Riccati equation is

$$Q' = RQ^2 + SQ + T. \tag{3.20}$$

The solution of Riccati equation can be obtained by a change of dependent variable, where the dependent variable y is changed to q by [11]

$$Q = -\frac{q'}{q}\frac{1}{R}. \tag{3.21}$$

By identifying $R = 1/6$, $S = 0$, and $T = -3/2$, the change of variables in (3.19) becomes

$$Q = -\frac{q'}{(1/6)q} = -\frac{6q'}{q} \tag{3.22}$$

so (3.19) becomes an second-order linear differential equation

$$q'' - \frac{1}{4} = 0. \tag{3.23}$$

The general solution to this equation is

$$q = A\cosh\frac{\varsigma}{2} + B\sinh\frac{\varsigma}{2}, \tag{3.24}$$

where A and B are arbitrary constants. Applying this solution in (3.12) leads to the general solution of (3.11)

$$F(\varsigma) = \frac{\varsigma}{2} - \frac{3A\sinh\varsigma/2 + 3B\cosh\varsigma/2}{A\cosh\varsigma/2 + B\sinh\varsigma/2}. \tag{3.25}$$

Application of the impermeability condition $F(0) = 0$ leads to the determination of the constant $B = 0$, so the exact solution becomes

$$F(\varsigma) = \frac{\varsigma}{2} - 3\tanh\frac{\varsigma}{2}. \tag{3.26}$$

Collecting results, the velocity functions become

$$f(\eta,\tau) = \frac{1}{\sqrt{\tau}}\left(\frac{\varsigma}{2} - 3\tanh\frac{\varsigma}{2}\right), \tag{3.27}$$

where $\varsigma = (\sqrt{A/\nu\tau})y$ is the nondimensional distance from the plate. In view of (3.27), the flow far from the boundary $(x,\varsigma \to \infty)$ becomes

$$f(\eta,\tau) \longrightarrow \frac{1}{\sqrt{\tau}}\left(\frac{\varsigma}{2} - 3\right). \tag{3.28}$$

We obtain a particular solution of the unsteady reversed stagnation-point flow. The above solution is obtained in the similarity framework for unsteady viscous flows. The appearance of this positive factor in the first terms of (3.28) shows that this remote flow is directed toward the axis of symmetry and away from the plate. The second term in (3.28) describes a uniform velocity directed toward the plate. An adverse pressure gradient near the wall region leads to boundary-layer separation and associated flow reversal.

The particular solution is noteworthy in that it is completely analytical, but it is limited to the region far away from the plate in the presence of nonzero term $F'(0) = -1$. Another no-slip boundary condition $F'(0) = 0$ is not satisfied completely near the wall region.

3.3. Numerical Solution

Since the analytical solution does not satisfy the no-slip condition $F'(0) = 0$, it is convenient to solve the similarity equation numerically. The similarity equation and the relevant boundary conditions are

$$-\frac{1}{2}\varsigma F'' - F' - F'^2 + FF'' - F''' = -c,$$
$$F(0) = F'(0) = 0, \tag{3.29}$$
$$F'(\infty) = \frac{1}{2},$$

where $c = 3/4$ in order to satisfy the unsteady viscous flows in the outer region.

This equation is a third-order nonlinear ordinary differential equation. It is convenient when solving an ODE system numerically to describe the problem in terms of a system of first-order equations in MATLAB[12].

For example when solving an nth-order problem numerically is a common practice to reduce the equation to a system of n first-order equations. Then, by defining $y_1 = F$, $y_2 = F'$, $y_3 = F''$, the ODE reduces to the form

$$\frac{dy}{dx} = \begin{bmatrix} y_2 \\ y_3 \\ c - \frac{1}{2}\varsigma y_3 - y_2 - y_2^2 + y_1 * y_3 \end{bmatrix}. \tag{3.30}$$

The first task is to reduce the equation above to a system of first-order equations and define in MATLAB a function to return these. Later, we need to change the boundary value into initial value, because $ode45$, anodesolver in MATLAB, can only solve the initial value problem. From (3.29), we gauss the value of $F''(0) = -1.54306$ such that $F'(\infty) = 1/2$.

The numerical solution for two-dimensional stagnation-point flow is shown in Figure 4.

This solution is a similarity solution of the reversed stagnation-point flow over a flat plate. It describes an unsteady viscous flow in both outer and inner regions. A single dividing streamline plane separates streamlines approaching the plate from external flow streamlines. Though the viscous term F''' is still small compared to the convective terms in the outer region, this is not true in neglecting the viscous terms within the total flow field. The similarity velocity field is shown in Figure 5.

4. Conclusion

The foregoing study constitutes a similarity solution of the unsteady Navier-Stokes equations for reversed stagnation-point flow in the idealized case of an infinite plane boundary. In order to analyze the flow for nonzero values of x, it is required to convert the full Navier-Stokes equations. This problem is now being studied by applying numerical method. The solution is obtained in the classical similarity framework for unsteady viscous flows.

In the case of numerical methods, a brief analysis of the new solution is discussed. When the flow was near the plane wall region, the viscous forces were dominant,

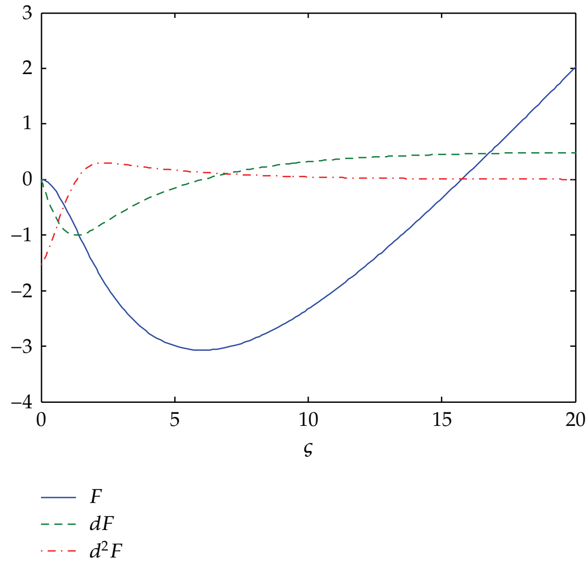

Figure 4: Numerical solutions of reversed stagnation-point flow.

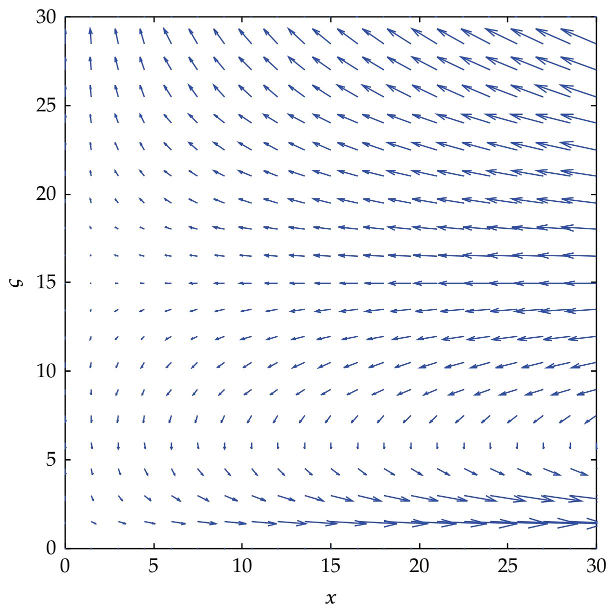

Figure 5: Similarity velocity field as a function of ς.

and the viscous term in the governing Navier-Stokes equations was important only near the boundary. On the contrary, the viscous forces were negligible when they were away from the wall.

In this paper we have examined the case of two-dimensional reversed stagnation-point flow. However, with the establishment of this frame work, the more important practical properties in engineering and technology application, like the velocity of wall is function of time, the temperature of wall is function of time and distance from wall, can be investigated and they would be the next phase of this study.

Acknowledgments

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