Research Article

A Wavelet Method for the Cauchy Problem for the Helmholtz Equation

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We consider a Cauchy problem for the Helmholtz equation at a fixed frequency. The problem is severely ill posed in the sense that the solution (if it exists) does not depend continuously on the data. We present a wavelet method to stabilize the problem. Some error estimates between the exact solution and its approximation are given, and numerical tests verify the efficiency and accuracy of the proposed method.

1. Introduction

The Helmholtz equation is often used to approximate model wave propagation in inhomogeneous media. The demand for reliable numerical solutions to such type of problems is frequently encountered in geophysical and optoelectronic applications [1, 2]. In geophysical applications, for example, wave propagation simulations are used for the development of acoustic imaging techniques for gaining knowledge about geophysical structures deep within the Earth's subsurface [3]. In optoelectronics, the determination of a radiation field surrounding a source of radiation (e.g., a light emitting diode) is also a frequently occurring problem [4]. In many engineering problems, the boundary conditions are often incomplete, either in the form of underspecified and overspecified boundary conditions on different parts of the boundary or the solution is prescribed at some internal points in the domain. These so-called Cauchy problems are inverse problems, and it is well known that they are generally ill posed in the sense of Hadamard [5]. However, the Cauchy problem suffers from the nonexistence and instability of the solution. In this paper we consider the Cauchy problem for the Helmholtz equation in a "strip" 0 < x < 1 as follows:

$$\Delta u(x,y) + k^{2}u(x,y) = 0, \quad x \in (0,1), \ y \in \mathbb{R}^{n}, \ n \ge 1,$$
$$u(0,y) = g(y), \quad y \in \mathbb{R}^{n},$$
$$u_{x}(0,y) = 0, \quad y \in \mathbb{R}^{n},$$
(1.1)

where $\Delta = \frac{\partial^2}{\partial x^2} + \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$ is an n + 1 dimensional Laplace operator. We want to determine the solution u(x, y) for $0 < x \le 1$ from the data g(y). Due to the importance of its application, this problem has been studied by many researchers, for example, DeLillo et al. [6, 7], Jin and Zheng [8], Johansson and Martin [9], and Marin et al. [10–14].

Let S be the Schwartz space over \mathbb{R}^n , and let S' be its dual (the space of tempered distributions). Let \hat{f} denote the Fourier transform of function $f(y) \in S$ defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot y} f(y) dy, \quad \xi = (\xi_1, \dots, \xi_n), \quad y = (y_1, \dots, y_n), \quad (1.2)$$

while the Fourier transform of a tempered distribution $f \in S'$ is defined by

$$\left(\widehat{f},\phi\right) = \left(f,\widehat{\phi}\right), \quad \forall \phi \in \mathcal{S}.$$
 (1.3)

In this paper, we will consider functions depending on the variables $x \in [0, 1]$, $y \in \mathbb{R}^n$.

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all tempered distributions $f(y) \in S'$, for which $\hat{f}(\xi)(1+|\xi|^2)^{s/2}$ is a function in $L^2(\mathbb{R}^n)$. The norm on this space is given by

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 (1+|\xi|^2)^s d\xi\right)^{1/2}.$$
(1.4)

We assume there exists a unique solution u(x, y) of problem (1.1), which satisfies the problem in the classical sense and $g(\cdot)$, $u(x, \cdot) \in L^2(\mathbb{R}^n)$. Applying the Fourier transform technique to problem (1.1) with respect to the variable y yields the following problem in the frequency space:

$$\widehat{u}_{xx}(x,\xi) + (k^2 - |\xi|^2) \widehat{u}(x,\xi) = 0, \quad x \in (0,1), \ \xi \in \mathbb{R}^n, \ n \ge 1,
\widehat{u}(0,\xi) = \widehat{g}(\xi), \quad \xi \in \mathbb{R}^n,
\widehat{u}_x(0,\xi) = 0, \quad \xi \in \mathbb{R}^n.$$
(1.5)

It is easy to obtain the solution of problem (1.5) (if exists) has the form

$$\widehat{u}(x,\xi) = \cosh\left(x\sqrt{|\xi|^2 - k^2}\right)\widehat{g}(\xi),\tag{1.6}$$

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or equivalently, the solution of problem (1.1) has the representation

$$u(x,y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot y} \cosh\left(x\sqrt{|\xi|^2 - k^2}\right) \widehat{g}(\xi) d\xi.$$
(1.7)

Since $\cosh(x\sqrt{|\xi|^2 - k^2})$ increases rapidly with exponential order as $|\xi| \to \infty$, the Fourier transform of the exact data g(y) must decay rapidly. However, in practice, the data at x = 0 is often obtained on the basis of reading of physical instrument which is denoted by g_m . We assume that $g(\cdot)$ and $g_m(\cdot)$ satisfy

$$\left\|g(\cdot) - g_m(\cdot)\right\|_{H^r} \le \delta. \tag{1.8}$$

Since $g_m(\cdot)$ belong to $L^2(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ for $r \leq 0$, r should not be positive. A small perturbation in the data g(y) may cause a dramatically large error in the solution u(x, y) for $0 < x \leq 1$. Hence problem (1.1) is severely ill posed and its numerical simulation is very difficult. It is obvious that the ill-posedness of the problem is caused by the perturbation of high frequencies.

By (1.6) we know

$$\widehat{u}(1,\xi) = \cosh\left(\sqrt{|\xi|^2 - k^2}\right)\widehat{g}(\xi).$$
(1.9)

Since the convergence rates can only be given under a priori assumptions on the exact solution [15], we will formulate such an a priori assumption in terms of the exact solution at x = 1 by considering

$$\|u(1,\cdot)\|_{H^s} \le E. \tag{1.10}$$

Meyer wavelets are special because, unlike most other wavelets, they have compact support in the frequency domain but not in the time domain (however, they decay very fast). The wavelet methods have been used to solve one-dimensional heat conduction problems [16, 17] and noncharacteristic Cauchy problem for parabolic equation in one-dimensional [18] and multidimensional [19] cases, and so forth. In this paper we propose a similar wavelet method as suggested in [19] to the problem (1.1).

The paper is organized as follows. In Section 2 we describe the Meyer wavelets and discuss the properties that make them useful for solving ill-posed problems. Some error estimates between the exact solution and its approximation as well as the choice of the regularization parameter are given in Section 3. Finally, in Section 4 numerical tests verify the efficiency and accuracy of the proposed method.

2. The Meyer Wavelets

In the present paper let Φ be Meyer's orthonormal scaling function in *n* dimensions. This function is constructed from the one-dimensional scaling functions in the following way. Let

 $\phi(x)$ and $\psi(x)$ be the Meyer scaling and wavelet function in one dimension defined by their Fourier transform in [20] which satisfy

$$\operatorname{supp} \widehat{\phi} = \left[-\frac{4}{3}\pi, \frac{4}{3}\pi \right],$$

$$\operatorname{supp} \widehat{\psi} = \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi \right] \cup \left[\frac{2}{3}, \frac{8}{3}\pi \right].$$
(2.1)

It can be proved (cf. [20]) that the set of functions

$$\psi_{jk}(x) = 2^{j/2} \psi\left(2^j x - k\right), \quad j, k \in \mathbb{Z},$$

$$(2.2)$$

is an orthonormal basis of $L^2(\mathbb{R})$. Consequently, the MRA $\{V_j\}_{j\in\mathbb{Z}}$ of Meyer is generated by

$$V_{j} = \overline{\{\phi_{jk}, k \in \mathbb{Z}\}}, \quad \phi_{jk} \coloneqq 2^{j/2} \phi(2^{j}x - k), \quad j, k \in \mathbb{Z},$$
$$\operatorname{supp}(\widehat{\phi}_{jk}) = \left\{\xi; \ |\xi| \le \frac{4}{3}\pi 2^{j}\right\}.$$

$$(2.3)$$

For the construction of an *n*-dimensional MRA, we take tensor products of the spaces V_i (see [21, 22]). Then the scaling function Φ is given by

$$\Phi(x) = \prod_{k=1}^{n} \phi(x_k), \quad x \in \mathbb{R}^n,$$
(2.4)

and any basis function Ψ in W_J can be written in the form

$$\Psi(x) = 2^{nJ/2} \psi \left(2^J x_i - k_i \right) \cdot \prod_{m \neq i} \theta_m \left(2^J x_m - k_m \right), \quad x \in \mathbb{R}^n,$$
(2.5)

where $k \in \mathbb{Z}^n$, and for any $m \in \{1, ..., n\}$, θ_m stands for ϕ or ψ . Hence we obtain from (2.1) that

$$\operatorname{supp}(\widehat{\Phi}) = \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^n, \qquad (2.6)$$

$$\widehat{f}(\xi) = 0 \text{ for } \|\xi\|_{\infty} \le \frac{2}{3}\pi 2^J, \quad f \in W_J, \ J \in \mathbb{N}.$$
 (2.7)

The orthogonal projection on the space V_J is defined by

$$P_J f := \sum_{k \in \mathbb{Z}^n} (f, \Phi_{J,k}) \Phi_{J,k}, \qquad (2.8)$$

while $Q_J f$ denotes the orthogonal projection of a function f on the wavelet space W_J with $V_{J+1} = V_J \oplus W_J$. (In many contexts one will find more than one detailed space W_J , that is, $V_{J+1} = V_J \oplus W_{1,J} \oplus W_{2,J} \oplus \cdots$. Here, the space W_J is simply defined as the orthogonal complement of V_J in V_{J+1}).

Let

$$\Omega_J := 2^J \left[-\frac{2}{3} \pi, \frac{2}{3} \pi \right]^n.$$
(2.9)

Setting $\Gamma_J := \mathbb{R}^n \setminus \Omega_J$, together with (2.6), it follows for $J \in \mathbb{N}$ that

$$\widehat{P_J f}(\xi) = 0 \quad \text{for } \xi \in \Gamma_{J+1},$$

$$((I - P_J) f)^{\widehat{}}(\xi) = \widehat{Q_J f}(\xi) \quad \text{for } \xi \in \Omega_{J+1}.$$
(2.10)

We introduce the operator M_I which is defined by the equation

$$\widehat{M_J f} := (1 - \chi_J)\widehat{f}, \ J \in \mathbb{N}, \tag{2.11}$$

where χ_J denotes the characteristic function of the cube Ω_J . From (2.7) it follows that any basis function Ψ in W_j , $j \ge J$, satisfies

$$\Psi(\xi) = 0, \ \xi \in \Omega_J, \tag{2.12}$$

and we obtain

$$(f,\Psi) = \left(\widehat{f},\widehat{\Psi}\right) = \left((1-\chi_J)\widehat{f},\widehat{\Psi}\right) = (M_J f,\Psi).$$
(2.13)

And it follows for $J \in \mathbb{N}$ that

$$Q_J = Q_J M_J,$$

$$I - P_J = (I - P_J) M_J.$$
(2.14)

3. Wavelet Regularization and Error Estimates

We list the following two lemmas given in [19, 23] which are useful to our proof.

Lemma 3.1 (see [19, 23]). Let $\{V_J\}_{J \in \mathbb{Z}}$ be an *m*-regular MRA, and let $r, s \in \mathbb{R}$ be such that -m < r < s < m. Then for each function $f \in H^s(\mathbb{R}^n)$ and $J \in \mathbb{N}$, the following inequality holds:

$$\|f - P_J f\|_{H^r} \le C_1 2^{-J(s-r)} \|f\|_{H^r}.$$
(3.1)



Figure 1: (a) Exact solution $u(1, \cdot)$; (b) unregularized solution reconstructed from g_m for x = 1.

Lemma 3.2 (see [19]). Let $\{V_J\}_{J \in \mathbb{Z}}$ be Meyer's (tensor-) MRA, and suppose $J \in \mathbb{N}$, $r \in \mathbb{R}$. Then for all $f \in V_J$, one has

$$\left\|\frac{\partial^l}{\partial x_i^l}f\right\|_{H^r} \le C_2 2^{(J-1)l} \left\|f\right\|_{H^r}, \quad i=1,\ldots,n, \ l\in\mathbb{N}.$$
(3.2)

Define an operator $T_x : g(y) \mapsto u(x, y)$ by (1.6), that is,

$$T_x g = u(x, y), \quad 0 < x \le 1,$$
 (3.3)

or equivalently,

$$\widehat{T_x g}(\xi) = \cosh\left(x\sqrt{|\xi|^2 - k^2}\right)\widehat{g}(\xi), \quad 0 < x \le 1.$$
(3.4)

Then we have

Theorem 3.3. Let $\{V_J\}_{J \in \mathbb{Z}}$ be Meyer's MRA and suppose $r \in \mathbb{R}$ and $J \in \mathbb{N}$ which satisfies $2^J > k$, $0 \le x \le 1$. Then for all $f \in V_J$, one has

$$\|T_x f\|_{H^r} \le \left(C_5 e^{x\sqrt{2^{2(j-1)}-k^2}} + 1\right) \|f\|_{H^r}.$$
(3.5)

Proof. For $f \in V_J$, by definition (1.4) and formula (3.4), from Lemma 3.2, we have

$$\begin{split} \|T_{x}f\|_{H'} &= \left(\int_{\mathbb{R}^{n}} \left| \cosh(x\sqrt{|\xi|^{2} - k^{2}}) \hat{f} \right|^{2} (1 + |\xi|^{2})^{r} d\xi \right)^{1/2} \\ &\leq \left(\int_{|\xi| > k} \left| \frac{\cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right)}{\cosh(x|\xi|)} \cosh(x|\xi|) \hat{f} \right|^{2} (1 + |\xi|^{2})^{r} d\xi \right)^{1/2} \\ &+ \left(\int_{|\xi| < k} \left| \cos\left(x\sqrt{k^{2} - |\xi|^{2}}\right) \hat{f} \right|^{2} (1 + |\xi|^{2})^{r} d\xi \right)^{1/2} \\ &\leq \sup_{\xi \in \Omega_{l+1}} \left| \frac{\cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right)}{\cosh(x|\xi|)} \right| \left(\int_{|\xi| > k} \left| \cosh(x|\xi|) \hat{f} \right|^{2} (1 + |\xi|^{2})^{r} d\xi \right)^{1/2} + \|f\|_{H'} \\ &\leq 2 \sup_{\xi \in \Omega_{l+1}} \frac{e^{x(\sqrt{|\xi|^{2} - k^{2}} - |\xi|)}}{\cosh(x|\xi|)} \left(\int_{|\xi| > k} \left| \frac{\sum_{l=0}^{x^{2l}} \frac{x^{2l}}{(2l)!} |\xi|^{2l} \hat{f} \right|^{2} (1 + |\xi|^{2})^{r} d\xi \right)^{1/2} + \|f\|_{H'} \\ &\leq 2 C_{3} e^{x(\sqrt{2^{2(l-1)} - k^{2} - 2^{l-1})}} \sum_{l=0}^{\infty} \frac{x^{2l}}{(2l)!} \left\| (\Delta_{y})^{l} f \right\|_{H'} + \|f\|_{H'} \\ &\leq \left(C_{4} e^{x(\sqrt{2^{2(l-1)} - k^{2} - 2^{l-1})} \sum_{l=0}^{\infty} \frac{x^{2l}}{(2l)!} \cdot n2^{2(l-1)l} + 1 \right) \|f\|_{H'} \\ &\leq \left(C_{5} e^{x(\sqrt{2^{2(l-1)} - k^{2} - 2^{l-1})}} \cosh(x2^{l-1}) + 1 \right) \|f\|_{H'} \\ &\leq \left(C_{5} e^{x(\sqrt{2^{2(l-1)} - k^{2} - 2^{l-1})}} \cosh(x2^{l-1}) + 1 \right) \|f\|_{H'} . \end{split}$$
(3.6)

Since the Cauchy data are given inexactly by g_m , we need a stable algorithm to approximate the solution of (1.1). Our method is as follows. Consider the operator

$$T_{x,J} := P_J T_x P_J, \tag{3.7}$$

and show that it approximates T_x in a stable way for an appropriate choice for $J \in \mathbb{N}$ depending on δ and E. By the triangle inequality we know

$$\|T_{x}g - T_{x,J}g_{m}\|_{H^{r}} \leq \|(T_{x} - T_{x,J})g\|_{H^{r}} + \|T_{x,J}(g - g_{m})\|_{H^{r}}.$$
(3.8)

From (1.8) and Theorem 3.3, the second term on the right-hand side of (3.8) satisfies

$$\|T_{x,J}(g-g_m)\|_{H^r} = \|P_J T_x P_J(g-g_m)\|_{H^r} \le \|T_x P_J(g-g_m)\|_{H^r}$$

$$\le \left(C_5 e^{x\sqrt{2^{2(J-1)}-k^2}} + 1\right)\delta.$$
(3.9)

For the first one we have

$$\|(T_x - T_{x,J})g\|_{H^r} \le \|(I - P_J)T_xg\|_{H^r} + \|P_JT_x(I - P_J)g\|_{H^r}.$$
(3.10)

By Lemma 3.1, (1.9), (1.10), (2.14), and (3.4), we get

$$\begin{split} \| (I - P_J) T_x g \|_{H^r} &= \| (I - P_J) M_J T_x g \|_{H^r} \le C_1 2^{-J(s-r)} \| M_J T_x g \|_{H^s} \\ &= C_1 2^{-J(s-r)} \left(\int_{\Gamma_J} \left| \frac{\cosh \left(x \sqrt{|\xi|^2 - k^2} \right)}{\cosh \left(\sqrt{|\xi|^2 - k^2} \right)} u(1, \cdot) \right|^2 \left(1 + |\xi|^2 \right)^s \right)^{1/2} \\ &\le C_1 2^{-J(s-r)} \sup_{\xi \in \Gamma_J} \left| \frac{\cosh \left(x \sqrt{|\xi|^2 - k^2} \right)}{\cosh \left(\sqrt{|\xi|^2 - k^2} \right)} \right| \cdot \| u(1, \cdot) \|_{H^s} \\ &\le 2C_1 \left(\left(\frac{3}{2} \pi 2^J \right)^2 - k^2 \right)^{-(s-r)/2} e^{-(1-x)\sqrt{((2/3)\pi 2^J)^2 - k^2}} E \\ &\le 2C_1 \left(2^{2J} - k^2 \right)^{-(s-r)/2} e^{-(1-x)\sqrt{2^{2J-k^2}}} E. \end{split}$$
(3.11)

On the other hand, due to (2.10), we know

$$\begin{split} \|P_{J}T_{x}(I-P_{J})g\|_{H^{r}} &\leq \|T_{x}(I-P_{J})g\|_{H^{r}} \\ &\leq \left(\int_{\Omega_{J+1}} \left|\cosh\left(x\sqrt{|\xi|^{2}-k^{2}}\right)(Q_{J}g)^{\widehat{}}(\xi)\right|^{2} \left(1+|\xi|^{2}\right)^{r} d\xi\right)^{1/2} \\ &+ \left(\int_{\Gamma_{J+1}} \left|\cosh\left(x\sqrt{|\xi|^{2}-k^{2}}\right)\widehat{g}(\xi)\right|^{2} \left(1+|\xi|^{2}\right)^{r} d\xi\right)^{1/2} \\ &=: I_{1}+I_{2}. \end{split}$$
(3.12)

We estimate the two parts at the right-hand side of (3.12) separately. For I_2 we have

$$\begin{split} I_{2} &= \left(\int_{\Gamma_{J+1}} \left| \cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right) \widehat{g}(\xi) \right|^{2} \left(1 + |\xi|^{2}\right)^{r} d\xi \right)^{1/2} \\ &= \left(\int_{\Gamma_{J+1}} \left| \frac{\cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right)}{\cosh\left(\sqrt{|\xi|^{2} - k^{2}}\right)} u(1, \cdot) \right|^{2} \left(1 + |\xi|^{2}\right)^{r} d\xi \right)^{1/2} \\ &\leq \sup_{\xi \in \Gamma_{J+1}} \left| \frac{\cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right)}{\cosh\left(\sqrt{|\xi|^{2} - k^{2}}\right)} \frac{1}{\left(1 + |\xi|^{2}\right)^{(s-r)/2}} \right| \times \left(\int_{\Gamma_{J+1}} \left| u(1, \cdot) \left(1 + |\xi|^{2}\right)^{s/2} \right|^{2} d\xi \right)^{1/2} \\ &\leq 2 \cdot \left(\frac{4}{3}\pi 2^{J}\right)^{-(s-r)} e^{-(1-x)\sqrt{n((4/3)\pi 2^{J})^{2} - k^{2}}} \cdot \|u(1, \cdot)\|_{H^{s}} \\ &\leq 2 \cdot (2^{2J} - k^{2})^{-(s-r)/2} e^{-(1-x)\sqrt{2^{2J} - k^{2}}} E. \end{split}$$

$$(3.13)$$

Now we turn to I_1 . There holds

$$I_{1} = \left(\int_{\Omega_{J+1}} \left| \cosh\left(x\sqrt{|\xi|^{2} - k^{2}}\right) (Q_{J}g)^{r}(\xi) \right|^{2} \left(1 + |\xi|^{2}\right)^{r} d\xi \right)^{1/2}$$

$$\leq \|T_{x}Q_{J}g\|_{H^{r}} \leq \left(C_{5}e^{x\sqrt{2^{2J}-k^{2}}} + 1\right) \|Q_{J}g\|_{H^{r}},$$
(3.14)

since $Q_J g \in V_{J+1}$. Furthermore, from (2.14), it follows that

$$\begin{aligned} \|Q_{J}g\|_{H^{r}} &= \|Q_{J}M_{J}g\|_{H^{r}} \leq \|M_{J}g\|_{H^{r}} \\ &= \left(\int_{\Gamma_{J}} |\widehat{g}(\xi)|^{2} \left(1+|\xi|^{2}\right)^{r} d\xi\right)^{1/2} \\ &= \left(\int_{\Gamma_{J}} \left|\frac{\widehat{u}(1,\xi)}{\cosh\left(\sqrt{|\xi|^{2}-k^{2}}\right)}\right|^{2} \left(1+|\xi|^{2}\right)^{r} d\xi\right)^{1/2} \\ &\leq 2 \cdot 2^{-J(s-r)} e^{-\sqrt{n((2/3)\pi 2^{J})^{2}-k^{2}}} \|u(1,\cdot)\|_{H^{s}} \\ &\leq 2 \cdot (2^{2J}-k^{2})^{-(s-r)/2} e^{-\sqrt{2^{2J}-k^{2}}} E. \end{aligned}$$

$$(3.15)$$

Therefore,

$$\|P_{J}T_{x}(I-P_{J})g\|_{H^{r}} \leq 2(1+C_{5})\left(2^{2J}-k^{2}\right)^{-(s-r)/2}e^{-(1-x)\sqrt{2^{2J}-k^{2}}}E$$

$$+2\left(2^{2J}-k^{2}\right)^{-(s-r)/2}e^{-\sqrt{2^{2J}-k^{2}}}E.$$
(3.16)

Combining (3.11) and (3.16) with (3.10), we have

$$\|(T_{x} - T_{x,J})g\|_{H^{r}} \le 2(C_{1} + 1 + C_{5}) \left(2^{2J} - k^{2}\right)^{-(s-r)/2} e^{-(1-x)\sqrt{2^{2J}-k^{2}}} E + 2\left(2^{2J} - k^{2}\right)^{-(s-r)/2} e^{-\sqrt{2^{2J}-k^{2}}} E.$$
(3.17)

Then from (3.9) and (3.17) we finally arrive at

$$\|T_{x}g - T_{x,J}g_{m}\|_{H^{r}} \leq \left(C_{5}e^{x\sqrt{2^{2(J-1)}-k^{2}}} + 1\right)\delta + 2\left(2^{2J}-k^{2}\right)^{-(s-r)/2}e^{-\sqrt{2^{2J}-k^{2}}}E + 2(C_{1}+1+C_{5})\left(2^{2J}-k^{2}\right)^{-(s-r)/2}e^{-(1-x)\sqrt{2^{2J}-k^{2}}}E.$$
(3.18)

In order to show some stability estimates of the Hölder type for our method using (3.18), we use the following lemma which appeared in [24] for choosing a proper regularization parameter *J*.

Lemma 3.4. Let the function $f(\lambda) : [0, a] \to \mathbb{R}$ be given by

$$f(\lambda) = \lambda^{b} \left(d \ln \frac{1}{\lambda} \right)^{-c}$$
(3.19)

with a constant $c \in \mathbb{R}$ and positive constants a < 1, b, and d. Then for the inverse function $f^{-1}(\lambda)$, one has

$$f^{-1}(\lambda) = \lambda^{1/b} \left(\frac{d}{b} \ln \frac{1}{\lambda}\right)^{c/b} (1 + o(1)) \quad \text{for } \lambda \longrightarrow 0.$$
(3.20)

Based on this lemma, we can choose the regularization parameter J by minimizing the right-hand side of (3.18).

Denote

$$e^{-\sqrt{2^{2/}-k^2}} = \lambda \in (0,1), \tag{3.21}$$

and let $C = C_5/2(1 + C_1 + C_5)$ and

$$C\lambda^{-x}\delta = \lambda^{1-x} \left(\ln \frac{1}{\lambda} \right)^{-(s-r)} E, \qquad (3.22)$$

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that is,

$$\frac{C\delta}{E} = \lambda \left(\ln \frac{1}{\lambda} \right)^{-(s-r)}.$$
(3.23)

Then by Lemma 3.4 we obtain that

$$\lambda = \frac{C\delta}{E} \left(\ln \frac{1}{C\delta/E} \right)^{s-r} (1+o(1)) \quad \text{for } \frac{C\delta}{E} \longrightarrow 0$$

$$= \frac{C\delta}{E} \left(\ln \frac{E}{C\delta} \right)^{s-r} (1+o(1)) \quad \text{for } \delta \longrightarrow 0.$$
(3.24)

Taking the principal part of λ , we get

$$J^* = \frac{1}{2} \log_2 \left(\ln^2 \left(\frac{E}{C\delta} \left(\ln \frac{E}{C\delta} \right)^{-(s-r)} \right) + k^2 \right), \tag{3.25}$$

due to (3.21). Now, summarizing above inference process, we obtain the main result of the present paper.

Theorem 3.5. For $s \ge r$, suppose that conditions (1.8) and (1.10) hold. If one takes

$$J^{**} = [J^*], (3.26)$$

where J^* was defined in (3.25), [a] with square bracket denotes the largest integer less than or equal to $a \in R$. Then there holds the following stability estimate:

$$\|T_{x}g - T_{x,J}g_{m}\| \leq (2(C_{1} + 1 + C_{5})E)^{x}(C_{5}\delta)^{1-x} \left(\ln\frac{E}{C\delta}\right)^{-(s-r)x} \\ \times \left(1 + \left(\frac{\ln E/C\delta}{\ln E/C\delta + \ln(\ln E/C\delta)^{-(s-r)}}\right)^{s-r}\right) \\ + \left(1 + 2C\left(\frac{\ln E/C\delta}{\ln E/C\delta + \ln(\ln E/C\delta)^{-(s-r)}}\right)^{s-r}\right)\delta \\ = (2(C_{1} + 1 + C_{5})E)^{x}(C_{5}\delta)^{1-x} \left(\ln\frac{E}{C\delta}\right)^{-(s-r)x}(1 + o(1)),$$
(3.27)

for $\delta \to 0$.



Figure 2: (a) The regularized solution at x = 1; (b) difference between the regularization and the exact solution; (1), (2), (3) correspond to $J^* = 3, 4, 5$, respectively.

Remark 3.6. In general, the a priori bound *E* and the coefficients C_1 - C_5 and *C* are not exactly known in practice. In this case, with

$$J^{\star} = \left[\frac{1}{2}\log_2\left(\ln^2\left(\frac{1}{\delta}\left(\ln\frac{1}{\delta}\right)^{-(s-r)}\right) + k^2\right)\right],\tag{3.28}$$

it holds that

$$\begin{split} \|T_{x}g - T_{x,J}g_{m}\| &\leq \delta^{1-x} \left(\ln\frac{1}{\delta}\right)^{-(s-r)x} \left(C_{5} + 2(C_{1} + 1 + C_{5})E\left(\frac{\ln 1/\delta}{\ln 1/\delta + \ln (\ln 1/\delta)^{-(s-r)}}\right)^{s-r}\right) \\ &+ \left(C_{5} + 2E\left(\frac{\ln 1/\delta}{\ln 1/\delta + \ln (\ln 1/\delta)^{-(s-r)}}\right)^{s-r}\right)\delta \\ &= \delta^{1-x} \left(\ln\frac{1}{\delta}\right)^{-(s-r)x} (1 + o(1)), \end{split}$$
(3.29)

for $\delta \to 0$.

Remark 3.7. The proposed wavelet method can also be used to solve the following Cauchy problem for the modified Helmholtz equation (i.e., the Yukawa equation [25])

$$\Delta v(x, y) + k^{2}v(x, y) = 0, \quad x \in (0, 1), \ y \in \mathbb{R}^{n}, \ n \ge 1,$$
$$v(0, y) = g(y), \quad y \in \mathbb{R}^{n},$$
$$v_{x}(0, y) = 0, \quad y \in \mathbb{R}^{n},$$
(3.30)

where $\Delta = \partial^2 / \partial x^2 + \sum_{i=1}^n \partial^2 / \Delta y_i^2$ is the same as in (1.1). It is easy to know that the exact solution of problem (3.30) is

$$\upsilon(x,y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot y} \cosh\left(x\sqrt{|\xi|^2 + k^2}\right) \widehat{g}(\xi) d\xi.$$
(3.31)

Define an operator $\widetilde{T}_x : g(y) \mapsto v(x, y)$ such that

$$\widehat{\widetilde{T}_x g}(\xi) = \cosh\left(x\sqrt{|\xi|^2 + k^2}\right)\widehat{g}(\xi), \quad 0 < x \le 1,$$
(3.32)

and the approximate solution is

$$v_J^{\delta} = \widetilde{T}_{x,J} g_m, \tag{3.33}$$

where *g* and g_m satisfy (1.8), and $\tilde{T}_{x,J} = P_J \tilde{T}_x P_J$. If we select the regularization parameter

$$J^{\dagger} = \left[\frac{1}{2}\log_2\left(\ln^2\left(\frac{1}{\delta}\left(\ln\frac{1}{\delta}\right)^{-(s-r)}\right) - k^2\right)\right],\tag{3.34}$$

then there holds

$$\begin{split} \left\| v(x, \cdot) - v_{J}^{\delta}(x, \cdot) \right\| &\leq \delta^{1-x} \left(\ln \frac{1}{\delta} \right)^{-(s-r)x} \\ &\times \left(C_{5} + 2(C_{1} + 1 + C_{5})E\left(\frac{\ln 1/\delta}{\ln 1/\delta + \ln (\ln 1/\delta)^{-(s-r)}}\right)^{s-r} \right) \\ &+ \left(C_{5} + 2E\left(\frac{\ln 1/\delta}{\ln 1/\delta + \ln (\ln 1/\delta)^{-(s-r)}}\right)^{s-r} \right) \delta \\ &= \delta^{1-x} \left(\ln \frac{1}{\delta} \right)^{-(s-r)x} (1 + (o(1))), \end{split}$$
(3.35)

for $\delta \to 0$.

4. Numerical Aspect

4.1. Numerical Implementation

We want to discuss some numerical aspects of the proposed method in this section.

We consider the case when n = 2. Supposing that the sequence $\{g(y_{1,i}, y_{2,j})\}_{i,j=1}^N$ represents samples from the function $g(y_1, y_2)$ on an equidistant grid in the square $[a, b]^2$, and N is even, then we add a random uniformly distributed perturbation to each data and obtain the perturbation data

$$g_m = g + \mu \operatorname{randn}(\operatorname{size}(g)). \tag{4.1}$$

Then the total noise δ can be measured in the sense of *root mean square error* according to

$$\delta := \left\| g_m - g \right\|_{l^2} = \sqrt{\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left(g_m(y_{1,i}, y_{2,j}) - g(y_{1,i}, y_{2,j}) \right)^2}, \tag{4.2}$$

where "randn(·)" is a normally distributed random variable with zero mean and unit standard deviation and e dictates the level of noise. "randn(size(g))" returns an array of random entries that is the same size as g.

For the function $g_m(y_1, y_2)$, we have

$$u_{I}^{b}(x,y) = T_{x,J}g_{m} = P_{J}T_{x}P_{J}g_{m}.$$
(4.3)

Hence, by using it with J^* being given in (3.28), we can obtain the approximate solution.

We will use DMT as a short form of the "discrete Meyer (wavelet) transform." Algorithms for discretely implementing the Meyer wavelet transform are described in [21]. These algorithms are based on the fast Fourier transform (FFT), and computing the DMT of a vector in \mathbb{R} requires $\mathcal{O}(N\log_2^2 N)$ operations.



Figure 3: (a) and (b) correspond to k = 5 and k = 100, respectively; (1), (2), (3) correspond to the exact solution, the regularized solution and the difference between the regularization and the exact solution, respectively.

4.2. Numerical Tests

In this section some numerical tests are presented to demonstrate the usefulness of the approach. The tests were performed using Matlab and the wavelet package WaveLab 850, which was downloaded from http://www-stat.stanford.edu/~wavelab/. Throughout this section, we set $\mu = 10^{-3}$, a = -5, b = 5, and $N = 2^6$.



Figure 4: (a) Exact solution $v(1, \cdot)$; (b) unregularized solution reconstructed from g_m for x = 1; (c) regularized solution reconstructed from g_m for x = 1 and $J^{\dagger} = 4$; (d) the difference between the regularization and the exact solution.

Example 4.1. Take n = 2 and $g(y) = e^{-y^2} \in \mathcal{S}(\mathbb{R}^2)$, where $y = (y_1, y_2)$ and $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz function space.

Since $\hat{g}(\xi) \in \mathcal{S}(\mathbb{R}^2)$, $\xi = (\xi_1, \xi_2)$ decays rapidly, and the formula (1.7) can be used to calculate u(x, y) with exact data directly, that is,

$$u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi_1 y_1 + \xi_2 y_2)} \cosh\left(x\sqrt{\xi_1^2 + \xi_2^2 - k^2}\right) \widehat{g}(\xi_1,\xi_2) d\xi_1 d\xi_2.$$
(4.4)

In Figure 1 we give the exact solution at x = 1, that is, $u(1, y_1, y_2)$, and the reconstructed solution $u^{\delta}(1, y_1, y_2)$ from the noisy data $g_m(y_1, y_2)$ without regularization. We see that u^{δ} does not approximate the solution and some regularization procedure is necessary.

Letting k = 1, the regularized solutions and the corresponding errors $u - u_{J^*}^{\delta}$ defined by the regularization parameter $J^* = 3, 4, 5$ are illustrated in Figure 2. We can see that in V_3 the approximation is very poor since the frequencies are cut off excessively by the projection P_3 . If J^* is taken to be too large, the noise in the function g_m is not damped enough by P_{J^*} , and thus the high frequencies of \hat{g}_m are so extremely magnified that they destroy the approximated solution. The approximation parameter $J^* = 4$ seems to be the optimal choice for this example.

In Figure 3 we display the exact solution, its approximation, and corresponding errors for k = 5 and 100, respectively. We see that the proposed method is useful for different wave number k.

Figure 4 shows that the proposed method for the Cauchy problem for the modified Helmholtz equation is also effective.

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