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Research Article

Children's Use of Arithmetic Shortcuts: The Role of Attitudes in Strategy Choice

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Current models of strategy choice do not account for children's attitudes towards different problem solving strategies. Grade 2, 3, and 4 students solved three sets of three-term addition problems. On inversion problems (e.g., $4 + 8 - 8$), if children understand the inverse relation between the operations, no calculations are required. On associativity problems (e.g., $5 + 27 - 23$), if children understand the associative relation between the operations, problem solving can be facilitated by performing subtraction before addition. A brief intervention involving demonstrations of different problem solving strategies followed the first problem set. Shortcut use increased after the intervention, particularly for students who preferred shortcuts to the left-to-right algorithm. In the third set, children were given transfer problems (e.g., $8 + 4 - 8$, $4 - 8 + 8$, $27 + 5 - 23$). Shortcut use was similar to first set suggesting that transfer did occur. That shortcut use increased the most for students who had positive attitudes about the shortcuts suggests that attitudes have important implications for subsequent arithmetic performance.

1. Children's Use of Arithmetic: The Role of Preference in Strategy Choice

Children's understanding of the relations between addition and subtraction is considered integral for later mathematical skills [1]. Part of the research agenda needs to understand and account for the large individual differences in children's conceptual knowledge [2]. While many researchers have focussed on factors such as general mathematics skills [3] or working memory [4] to account for individual differences in understanding the relationship between addition and subtraction, one overlooked factor may be students' attitudes and beliefs towards mathematics. Attitudes are positive or negative evaluations (e.g., "I like addition") and beliefs are "thoughts" based on experience (e.g., "division is difficult") [5] and both are considered to be integral in determining success in mathematics [6]. These attitudes and beliefs could be integral in the conceptual knowledge that children use and understand when making problem solving strategy choices.

Researchers have often considered children's use of the inversion shortcut strategy on inversion problems of the form $a+b-b$ as an indicator of understanding of the inverse relation between the two operations [3]. The inversion shortcut involves stating that the answer is a without performing any calculations as the two $b$terms cancel each other out. Children's understanding of the relationship between addition and subtraction can also be assessed via children's use of the associativity shortcut on problems of the form $a + b - c$, which were originally included in studies of children's inversion as control problems [7]. When children understand that addition and subtraction are associative, they can use that information to simplify and speed up problem solving by first subtracting and then adding (e.g., $3 + 29 - 27$ would be solved by calculating $29 - 27$ and then adding 3) [4]. Both inversion and associativity three-term problems are novel and therefore children must implement spontaneous problem solving strategies [8]. Researchers have taken advantage of this novelty to gain an understanding of children's knowledge of the relations between addition and subtraction by utilizing children's use of conceptually-based shortcuts as an indicator of conceptual understanding because understanding the inverse or associative relations between addition and subtraction is required to implement the inversion and associativity shortcuts [9].

Given the importance of understanding the relations between addition and subtraction, some studies have gone beyond assessing children's spontaneous use or evaluation of the inversion shortcut by trying to promote conceptually-based shortcut use focussing exclusively on inversion problems. In these studies, 8-year-old children's conceptual understanding of the inverse relation between addition and subtraction, as assessed via shortcut use, increased after either 2 or 3 training sessions with concrete objects [10, 11] or after repeated weekly exposure to inversion problems [12].

A possible alternative to the longer interventions used in previous studies comes from a study with Grade 2, 3, and 4 students by Robinson and Dubé [13] who asked children to solve inversion and associativity problems and then had the children complete an evaluation of procedures task [9] as a second measure of conceptual understanding. In the task, for both inversion and associativity problems, the shortcut and a standard left-to-right algorithm (i.e., adding the first two numbers and then subtracting the third number) were demonstrated to participants and participants were asked if they approved of each approach. Bisanz et al. [9] argued that the evaluation of procedures task is a good measure of conceptual understanding as it permits children to demonstrate their understanding which they may have implemented during problem solving. However, in an earlier pilot study, we found that participants were very likely to approve of both the shortcut and the left-to-right algorithm for both inversion and associativity and thus a more stringent measure was needed. To gain more useful information about children's understanding of arithmetic concepts, the participants in Robinson and Dubé [13] were also asked to compare the two solution procedures for each problem type and decide which was better and why. Explicitly justifying a strategy suggests conceptual understanding because the ability to justify typically develops after the ability to assess a strategy's worth and even use the strategy during problem solving [14]. Robinson and Dubé [13] found that most of the participants (80% or more) approved of both the shortcut and the left-to-right procedure for both the inversion and associativity problems. The more stringent measure of choosing the better strategy was more sensitive as a clear majority of participants preferred the shortcut to the left-to-right algorithm for inversion problems (78%) but not for associativity problems (59%) with older students in Grades 3 and 4 more likely to prefer the associativity shortcut. Effects of the evaluation of procedures task on subsequent problem solving were not examined in that study.

The study by Robinson and Dubé [13] provided two conclusions regarding strategy instruction and attitudes about problem solving strategies. The first conclusion is that the children had distinctly positive or negative attitudes about the appropriateness of the shortcuts. For example, some participants commented on the cleverness of the shortcuts (e.g., "that's so easy, I wish I'd thought of doing it that way" while others strongly disapproved of the shortcuts "e.g., it's cheating not to do all the math"). This variability in attitudes across individuals suggests that some children may be open and flexible about learning new ways to problem solve while others may be rigid and inflexible. For example, across a series of studies, Torbeyns and colleagues [15–17] have shown that children are often resistant to using a more efficient arithmetic strategy, even when explicitly encouraged and/or instructed to use that strategy. Previous research has suggested that children's attitudes and beliefs about mathematics are critical to success in mathematics and to enrollment in advanced mathematics courses and therefore engaging children with mathematics is of critical importance [18]. During schooling, children begin to have strong attitudes and beliefs about mathematics [19]. Children who are intrinsically motivated [20], are mastery-oriented [21] or who have low levels of mathematics anxiety [22], are more likely to be successful in mathematics. Other factors relating to mathematics success include the role of peer pressure [23] and disengagement [24].

Although several studies have investigated children's attitudes and beliefs about mathematics, few studies have investigated children's attitudes about specific problem solving strategies. Ellis [25] and Verschaffel et al. [26] note that much of the research on the factors involved in children's strategy choices has focussed on the roles of speed and accuracy and has ignored other factors such as individual and sociocultural influences. The findings of Robinson and Dubé [13] suggest that children's attitudes towards the use of problem solving strategies that involve simplifying and reducing solution times are already entrenched by Grade 2 and that these attitudes may become even more firmly held by the end of the middle school years [27]. Current models of children's strategy choices emphasize the tendency for children to want to use the most efficient problem solving strategies [28, 29] and yet the findings of Robinson and Dubé [13] support Ellis' and Verschaffel et al.'s [26] proposal that other factors, in this case attitudes, may also impact strategy choices.

The second conclusion drawn from Robinson and Dubé [13] was that comparison of problem solving strategies yielded more information about children's understanding of arithmetic concepts than only asking them to evaluate strategies. Research on comparison of solution methods has been encouraging as it appears to be an effective learning tool [30–32]. In two studies, participants in Grades 7 and/or 8 were given the opportunity to compare different solution methods for solving algebraic equations [30, 31] and in Star and Rittle-Johnson [32], participants in Grades 5 and 6 compared different solution methods for computational estimation. In all of the studies, the opportunity to compare different methods led to greater increases in procedural and conceptual knowledge than simply being presented with different methods. None of the studies examined whether there were developmental changes in the effectiveness of comparison and no studies of the effects of comparison have been conducted with younger students. Extending the use of comparisons even further, Yakes and Star [33] found that teachers who participated in a workshop on how to use comparison as an instructional tool for teaching algebra changed their teaching practices. The teachers used comparison of different solution methods as a teaching tool and also began to recognize and emphasize the importance of procedural flexibility in their teaching.

Based on the findings of Robinson and Dubé [13] that there are marked individual differences in children's attitudes about the inversion and associativity shortcuts and that presenting solution methods concurrently improves comparison and is a useful learning technique for older children (e.g., [30, 31]), the first two goals of this study were to determine whether children's attitudes about the solution methods they were asked to evaluate would impact subsequent problem solving strategies and to examine whether presenting solution methods concurrently improves comparison and is an effective learning tool on inversion and associativity problems. The final two goals of the study were to examine whether attitudes and the impact of comparison would change across development and to assess the strength of the intervention task by investigating whether children would transfer their new knowledge about the shortcuts to transfer problems. When concepts such as inversion or associativity are effectively learned, then that knowledge should be applicable or transferable to new problem types [34]. Rittle-Johnson and Star [30, 31] found that comparison of solution methods promoted flexibility and Rittle-Johnson [35] proposed that explaining why a procedure works is an effective learning tool and can result in the ability to transfer knowledge to new situations. Children do not necessarily understand a concept that has been taught to them but the ability to generalize a problem-solving strategy based on conceptual knowledge is an indicator that learning of that concept has occurred [36].

2. Method

2.1. Participants. Twenty-four Grade 2 students (12 boys, 12 girls) (mean age = 7 years, 0 months), thirty-five Grade 3 students (14 boys, 21 girls) (mean age = 7 years, 11 months), and forty-three Grade 4 students (20 boys, 23 girls) (mean age = 8 years, 10 months) participated in the study. The age groups were selected to correspond with previous research [13] and to correspond to the time in which children are learning about addition and subtraction [18]. Participants were from a large Canadian city, were predominantly Caucasian, and from middle SES families. The study took place in the first half of the school year.

2.2. Design and Intervention. The same pretest-intervention-posttest design was used as in the studies by Rittle-Johnson and Star [30, 31] but the participants were tested individually throughout the study rather than being paired up for the intervention, and the posttest was divided into the first posttest with familiar problems and a second posttest with transfer problems to assess strength of learning from the intervention. Half the students per grade were randomly assigned to either the sequential condition ($n = 51$) or the concurrent condition ($n = 51$). The intervention was the evaluation of procedures task. For both inversion and associativity problems, two strategies were demonstrated. In the sequential group, for each problem type: (1) the first strategy was demonstrated and then participants were asked if they approved of the strategy; (2) the second strategy was demonstrated and then participants were asked if they approved of the second strategy; (3) participants were asked which of the two strategies they preferred. In the concurrent group, for each problem type: (1) both strategies were demonstrated consecutively; (2) participants were asked if they approved of both strategies at the same time; (3) participants were asked which of the two strategies they preferred. Problem type and strategy orders were counterbalanced across boys and girls as closely as possible within each grade. For the inversion problems, on one problem, participants were told how a fictitious child had solved the problem by using the inversion shortcut ("When X solved this problem, s/he said that the answer would be the first number because when you add and subtract by the same number, the answer is always the first number"). On the other problem, participants were told how a fictitious child had solved the problem using a left-to-right algorithmic approach ("When X solved this problem, she/he added the first two numbers together and then subtracted the third number from that answer"). For the associativity problems, on one problem, participants were told how a fictitious child had solved the problem by using the associativity shortcut ("When X solved this problem, she/he subtracted the third number from the second number and then took the answer and added it to the first number"). On the other problem, the same left-to-right algorithmic approach used on the inversion problem was used again.

2.3. Materials and Procedure. There were three sessions, which took place within a 7 day period with at least one day between each session. In the first session which was the pretest, participants solved 16 three-term addition and subtraction problems. Half the problems were inversion problems of the form $a + b - b$ and half were associativity problems of the form $a+b-c$. Half the problems of each type were small ($a, b, c < 10$) (e.g., $3+6-6$ and $3+6-4$) and half were large ($a < 10, b$ and c between 21 and 29, $b > c$) (e.g., $7 + 23 - 23$ and $7 + 23 - 21$). No more than two problems of each type or each size were presented consecutively. Problems were presented on a laptop screen using e-prime. Participants did not have the aid of paper and pencil. Solution latencies (measured from the time the problem appeared on the screen until the participant stated the answer and pressed the space bar), accuracy, and immediately retrospective verbal reports of problem solving strategy (i.e., "how did you solve that problem?") were collected for each problem. Participants who were unable to provide an answer within 30 seconds were given a "cut-off protocol" in which they were asked to report how they were trying to solve the problem before moving on to the next problem. Cut-off problems were coded as incorrect and no solution latencies were recorded.

In the second session which was the intervention and first posttest session, participants were randomly assigned to the concurrent or sequential conditions and completed the evaluation of procedures task and then solved the first posttest of a new set of 16 three-term addition and subtraction problems. The same measures and the same problem parameters were used as in the first session.

In the third session which was the second posttest session, participants solved 16 novel transfer problems. Half of the problems were inversion problems and the other half were associativity problems using the same parameters and measures as in the first and second sessions except that half of the inversion problems were of the form $b+a-b$ (e.g., $4 + 9 - 4$) and half of the form $a - b + b$ (e.g., $7 - 23 + 23$) and the associativity problems were of the form $b + a - c$ (e.g., $28 + 6 - 25$).

3. Results

Data from boys and girls were collapsed together as no significant results involving gender were found. Tukey's Honestly Significant Difference test was used to examine post-hoc effects and the alpha level was .05 or less for all significant results. Analyses of verbal report data only are reported but accuracy, solution latency, and cutoff data matched expected patterns (higher accuracy, shorter solution latencies, and fewer cutoffs when shortcuts were used, see Table 1). Although differences between the concurrent and sequential conditions were expected, no differences were found (see Table 2) so the analyses reported below are all collapsed across condition. This lack of condition effect may have been a result of such a brief intervention or of the nature of the task. Both conditions included exposure to the same information and the same problems and the children in both conditions were asked to decide which strategy they preferred (the conceptually-based shortcut or the left-to-right strategy) yielding comparable data across conditions. This data allows the investigation of how children's attitudes towards different problem solving strategies impacts subsequent conceptually-based shortcut use.

3.1. Promotion of Conceptually-Based Shortcuts across Development. As can be seen in Table 3, some children were already using the inversion and associativity shortcuts in the pretest. Shortcut use between the pretest and the first posttest (familiar problems) was compared to determine if shortcut use on familiar problems increased after the evaluation of procedures task. Two separate 3 (Grade) × 2 (Size) × 2 (Session) analyses of variance were conducted on inversion and associativity shortcut use. Shortcut use increased from the pretest to the first posttest (43.6% versus 63.2% for inversion, 20.9% versus 35.9% for associativity), $F(1, 99) = 39.75$, $MSE = 929.93$, $\eta_p^2 = .29$ and $F(1, 99) = 17.43$, $MSE = 1231.66$, $\eta_p^2 = .15$ for inversion and associativity respectively. There was less shortcut use on small than large problems (50.7% versus 56.2% for inversion, 23.86% versus 32.86% for associativity), $F(1, 99) = 11.17$, $MSE = 259.68$, $\eta_p^2 = .10$ and $F(1, 99) = 15.67$, MSE = 496.78, $\eta_p^2 = .14$ for inversion and associativity respectively. For inversion, no grade differences were found (44.0%, 62.9%, and 53.5%, for Grades 2, 3, and 4) but for associativity (17.71%, 39.46%, and 27.91% for Grades 2, 3 and 4), there was more shortcut use by Grade 3 than Grade 2 students and Grade 4 students did not differ from Grade 2 or 3 students, $F(2, 99) = 3.92$, MSE = 882.88, $\eta_p^2 = .07$, and HSD = 17.7.

If the evaluation of procedures task was successful at promoting shortcut use, then participants should also use the shortcuts on the novel transfer problems in the second posttest. Shortcut use on typical problems before and after the evaluation of procedures task and shortcut use on transfer problems after the evaluation of procedures task were analyzed using two repeated measures ANOVAs. After the evaluation of procedures task, shortcuts were used less frequently on transfer problems in the second posttest than on typical problems in the first posttest (43.8% versus 65.0%, for inversion, 16.8% versus 37.6% for associativity) but were used just as frequently as shortcuts before the evaluation of procedures task (44.0% for inversion, 21.3% for associativity), $F(2, 202) = 25.68$, MSE = 588.35, $\eta_p^2 = .20$, HSD = 14.4, $F(2, 202) = 21.24$, MSE = 576.48, $\eta_p^2 = .17$, and HSD = 14.3 for inversion and associativity problems, respectively. This suggests that participants had greater difficulty applying the shortcuts to transfer problems than to typical problems after the evaluation of procedures task. However, shortcut use on the transfer problems after the intervention was the same as shortcut use on the typical problems before the intervention suggesting that the intervention may have promoted shortcut use on the transfer problems.

Shortcut use on the transfer problems in the second posttest was analyzed in greater detail to determine the effects of grade, problem size, and transfer type. For inversion problems, two types of transfer problems were possible, and a 3 (Grade) × 2 (Size) × 2 (Transfer type) ANOVA was conducted (see Figure 1). Inversion shortcut use was higher on large problems (32.7% versus 49.9% for small and large), $F(1, 99) = 34.32$, MSE = 828.65, $\eta_p^2 = .26$. There was a two-way interaction between transfer type and grade, $F(2, 99) = 5.22$, MSE = 1119.79, and $\eta_p^2 = .10$, with no grade differences on the first type of inversion transfer problem ($a - b + b$) but higher shortcut use by the Grade 4 than the Grade 2 students on the second type of transfer problem ($b+a-b$). There were no differences between transfer type in Grades 2 or 3 but in Grade 4, shortcut use was higher on the second type. This interaction was mediated by a three-way interaction between type, grade, and size, $F(2, 99) = 3.23$, MSE = 615.43, and $\eta_p^2 = .06$. There was a main effect of problem size with more shortcut use on large problems in all grades and both types of transfer problems, except for Grade 4 students who had no size difference on the first type of problem although the means were in the expected direction.

For associativity problems, a 3 (Grade) × 2 (Size) ANOVA was conducted on the transfer problems in the second posttest (see Figure 2). Associativity shortcut use was lower on small problems (12.4% versus 18.5% for small and large), $F(1, 99) = 5.89$, MSE = 303.04, and $\eta_p^2 = .06$, and there was a two-way interaction between grade and size, $F(2, 99) = 6.35$, MSE = 303.04, and $\eta_p^2 = .11$. On small problems, Grade 2 students used the associativity shortcut less frequently than the Grade 3 students, and on large problems the Grade 2 students used the shortcut less than both the Grade 3 and 4 students. There was no main effect of problem size in Grades 2 or 3, but Grade 4 participants used the shortcut more on large problems.

TABLE 1: Accuracy (%), solution latencies, and proportion of cutoffs (%) in the pretest, first posttest, and second posttest on inversion and associativity problems.

	Session		
	1	2	3
Accuracy		Inversion	
Shortcut	98.3 (2.1)	99.4 (1.7)	98.8 (2.1)
Algorithm	40.8 (2.2)	42.0 (3.0)	56.9 (2.6)
Solution latency			
Shortcut	5174 (293)	3216 (240)	4830 (298)
Algorithm	12407 (484)	114449 (646)	10346 (480)
Cutoffs			
Shortcut	1.4 (1.9)	.4 (1.6)	.9 (2.0)
Algorithm	37.0 (2.1)	36.8 (2.8)	17.2 (2.4)
Accuracy		Associativity	
Shortcut	65.0 (3.4)	76.2 (2.3)	65.0 (3.4)
Algorithm	48.8 (1.6)	47.2 (1.8)	48.8 (1.6)
Solution Latency			
Shortcut	11927 (524)	9389 (361)	10479 (585)
Algorithm	13396 (347)	12358 (373)	13359 (327)
Cutoffs			
Shortcut	16.1 (2.8)	9.1 (2.1)	10.2 (3.2)
Algorithm	33.3 (1.5)	36.3 (1.7)	28.8 (1.5)

Note. Standard errors are in parentheses.

TABLE 2: Percentage shortcut use in the pretest, first posttest (familiar problems), and second posttest (transfer problems) for the sequential and concurrent groups in each grade.

		Session					
		Pretest	1st posttest	2nd posttest	Pretest	1st posttest	2nd posttest
Grade 2	n		Inversion			Associativity	
Sequential	13	42.3 (11.1)	51.0 (11.1)	42.3 (8.6)	15.4 (8.8)	22.1 (11.1)	6.7 (4.8)
Concurrent	11	31.8 (12.0)	50.0 (12.1)	29.5 (7.3)	11.4 (8.8)	21.6 (12.1)	5.6 (3.9)
Grade 3							
Sequential	16	57.0 (10.0)	75.8 (10.0)	53.9 (10.1)	35.2 (7.2)	48.4 (10.0)	29.7 (10.0)
Concurrent	19	51.3 (9.1)	68.4 (9.2)	40.8 (7.5)	30.9 (6.7)	44.1 (9.2)	12.5 (4.8)
Grade 4							
Sequential	23	37.0 (8.3)	70.1 (8.3)	48.9 (6.3)	18.5 (6.1)	41.3 (8.3)	21.7 (5.4)
Concurrent	20	42.5 (8.9)	64.4 (9.0)	41. (7.8)	13.8 (6.5)	37.5 (8.9)	17.5 (5.5)

Note. Standard errors are in parentheses.

3.2. The Effects of Attitudes about Conceptually-Based Shortcuts across Development. In the evaluation of procedures task, participants were asked which they preferred: the shortcut or the left-to-right algorithm for both inversion and associativity problems. For inversion problems, participants were more likely to prefer the shortcut than the algorithm (71.8% versus 28.2%), $\chi^2(1, N = 103) = 19.66$, and that preference was similar across grade (58.3%, 77.1%, and 76.7% for Grades 2, 3, and 4). For associativity problems, participants, however, were not more likely to prefer the shortcut (58.3% versus 41.7%) but older children were more likely to prefer the shortcut (29.2%, 62.9%, and 72.1% for Grades 2, 3, and 4), $\chi^2(2, N = 103) = 12.08$. Thus, the success of the evaluation of procedures task may vary depending on whether participants "bought in" to the shortcuts.

To determine whether increases in shortcut use between sessions were attributable to participants "buying in" to the training provided by the evaluation of procedures task or whether they were attributable to practice or exposure effects, changes in inversion and associativity shortcut use within the pretest and the first posttest were compared to changes in shortcut use between the pretest and the first posttest using two 4 (1st half of pretest, 2nd half of pretest, 1st half of first posttest, 2nd half of first posttest) × 2 (Preference: shortcut, algorithm) ANOVAs (see Figure 3).

For inversion, shortcut use did not change within the pretest of the first posttest (34.6% versus 42.7%, 56.6% versus 58.8% for the 1st and 2nd half of the prestest and first posttest, resp.) but did increase between the 2nd half of the pretest and the first half of the first posttest, $F(3, 300)$

Table 3: Percentage shortcut use in the pretest and first posttest on small and large problems in each grade.

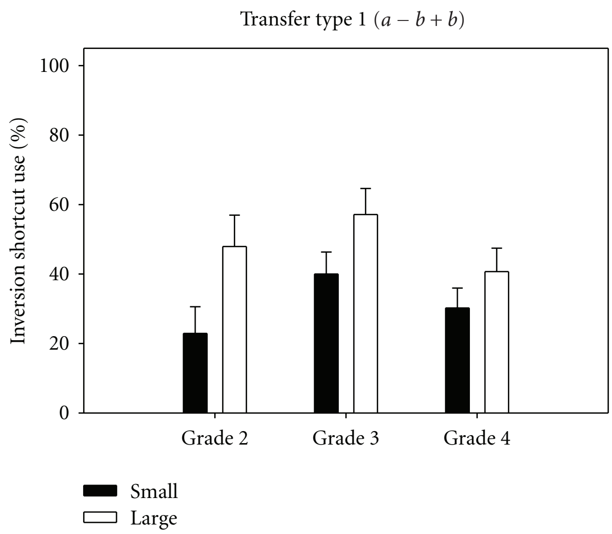
	Session	
	Pretest	First posttest
Grade 2	Inversion	
Small	37.5 (7.8)	49.0 (8.7)
Large	37.5 (8.7)	52.1 (8.2)
Grade 3		
Small	51.4 (6.4)	69.3 (7.2)
Large	56.4 (7.3)	74.3 (6.8)
Grade 4		
Small	34.3 (5.9)	62.8 (6.5)
Large	44.8 (6.5)	72.1 (6.1)
Grade 2	Associativity	
Small	11.5 (5.9)	18.8 (8.1)
Large	15.6 (7.0)	25.0 (9.0)
Grade 3		
Small	30.7 (4.9)	42.1 (6.7)
Large	35.0 (5.8)	50.0 (7.4)
Grade 4		
Small	8.7 (4.4)	31.4 (6.1)
Large	23.8 (6.7)	47.7 (6.7)

Note. Standard errors are in parentheses.

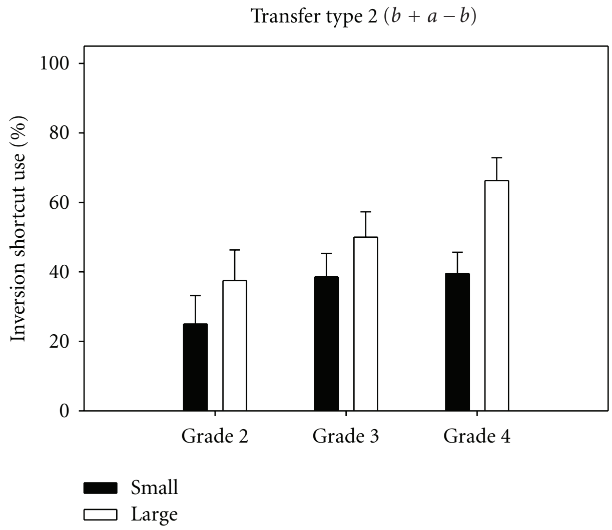
= 20.8, MSE = 516.29, η_p^2 = .17, HSD = 13.3. Participants who preferred the inversion shortcut used the shortcut more frequently than participants who preferred the algorithm (62.2% versus 34.2%, resp.), $F(1, 100) = 13.2$, MSE = 4826.17, and $\eta_{p2} = .11$.

For associativity, shortcut use did not change within the pretest of first posttest (17.3% versus 23.9%, 32.9% versus 36.8% for the 1st and 2nd half of the pretest and first posttest, resp.) nor did it change between the 2nd half of the pretest and 1st half of the first posttest, $F(3, 300) = 14.552$, MSE = 526.11, η_p^2 = .13, and HSD = 12.1. Participants who preferred the associativity shortcut used the shortcut more frequently than participants who preferred the algorithm (37.6% versus 17.9%, resp.), $F(1, 100) = 11.38$, MSE = 3387.69, and η_p^2 = .10. There was a two-way interaction between session and preference, $F(3, 300) = 9.38$, MSE = 526.11, η_p^2 = .09, and HSD = 14.2. For participants who preferred the algorithm, shortcut use did not change within the pretest or the first posttest nor did it change between the two. For participants who preferred associativity, shortcut use did not change within the pretest or the first posttest but did increase between the 2nd half of the pretest and the 1st half of the first posttest. Furthermore, the difference in shortcut use between participants who preferred the associativity shortcut and participants who preferred the algorithm was not significant in the pretest but was significant in the first posttest.

To determine if participants "buying in" to the training affected shortcut use on transfer problems, two independent sample t-tests were conducted comparing shortcut use in Session 3 between participants who preferred the shortcut and participants who preferred the algorithm. For inversion problems, participants who preferred the shortcut used shortcuts more frequently (M = 48.0%, SE = 3.8%) than participants who preferred the algorithm (M = 32.6%, SE = 5.8%), $t(100) = -2.15$. For associativity problems, participants who preferred the shortcut used the shortcut just as frequently (M = 20.2%, SE =3.6%) as participants who preferred the algorithm (M = 11.9%, SE = 3.6%), $t(100) = -1.58$, $P = .17$.



(a)



(b)

Figure 1: Inversion shortcut use on transfer problems in the second posttest for each grade and problem size Type 1 (a) problems ($a - b + b$) and Type 2 (b) problems ($b + a - b$). Error bars are SE.

4. Discussion

The brief evaluation of procedures task appears to have been successful in promoting conceptually-based shortcut use and decreasing the use of the less efficient left-to-right problem solving algorithm in many children in Grades 2,

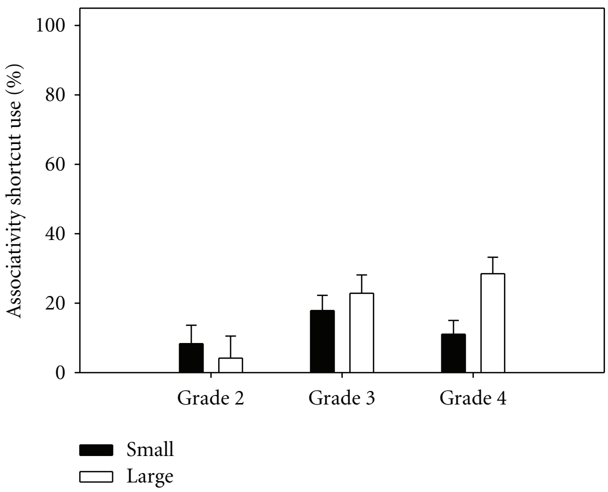


Figure 2: Associativity shortcut use on transfer problems in the second posttest for each grade and problem size. Error bars are SE.

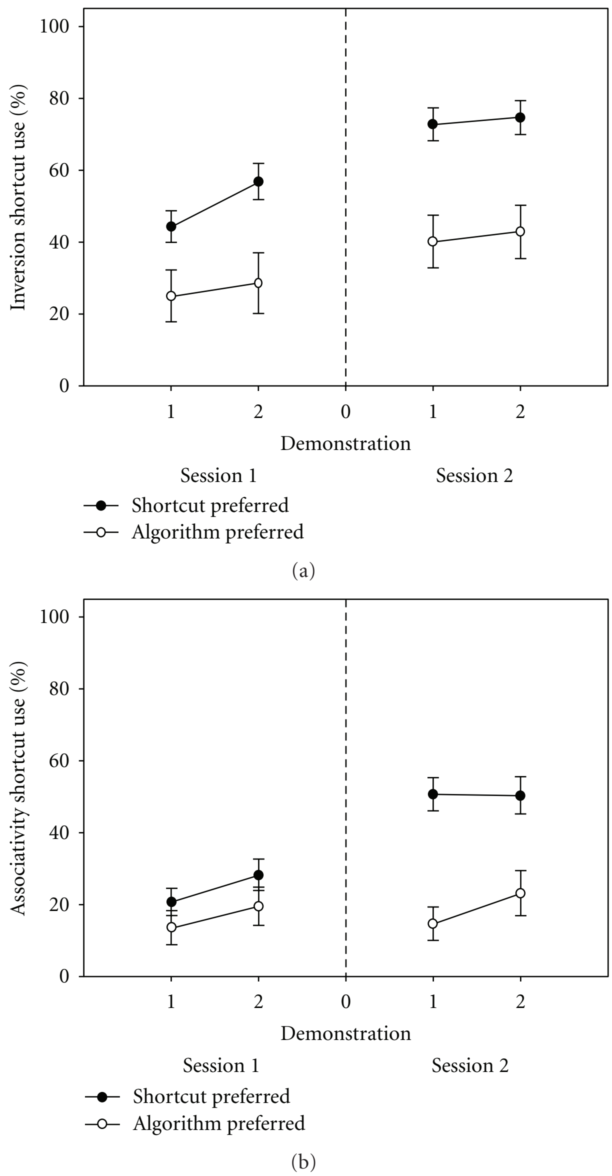


Figure 3: Shortcut use in 1st and 2nd half of the pretest and the first posttest on inversion (a) and associativity (b) problems by preference. Error bars are SE.

3, and 4. Children who use a conceptually-based shortcut are implementing an efficient problem solving strategy or procedure that is dependent on the inverse or associate relation between addition and subtraction, that is, they are using a procedure that is based on an arithmetic concept. The successful promotion of shortcuts that arise from arithmetic concepts is a finding consistent with a recent study with slightly older children showing that the task promoted more conceptually-based shortcut use in a condition similar to the present study's concurrent condition than in a control condition [37]. To learn and implement new problem solving strategies such as the inversion and associativity shortcuts is integral to increasing mathematical knowledge [8] as children with stronger conceptual knowledge of arithmetic will be able to represent problems in arithmetic and algebra more accurately [14, 38]. Children are expected to understand and discover the inversion and associativity concepts and shortcuts on their own [8] and few studies have attempted to teach children about the inverse relationship between addition and subtraction [11] and none have been conducted on teaching children about the associative relationship between addition and subtraction. The results of the current study support both the ability of many children in Grades 2, 3, and 4 to spontaneously discover and apply conceptually-based shortcuts during problem solving and the capacity of children to learn how to use these shortcuts immediately after a brief intervention where children are given demonstration of the shortcuts. Interestingly, children's attitudes towards the shortcuts were much more critical for children adopting a shortcut during subsequent problem solving. In contrast, being in the concurrent or sequential condition had little impact [30–32].

The lack of grade differences in spontaneous inversion shortcut use (see also [3]), the higher use of the inversion shortcut for all students after the evaluation of procedures task, and the more frequent use of the inversion shortcut than the associativity shortcut all suggest that the inversion shortcut is easier to discover, learn, and apply than the associativity shortcut. The associativity shortcut, on the other hand, was more likely to be used by the Grade 3 than the Grade 2 and 4 students. The shortcut requires calculation and therefore the advantage of using the shortcut may not be clear to Grade 2 students struggling with calculation, whilst the Grade 4 students may be good enough calculators that the shortcut does not seem superior. Demonstrating the conceptually-based shortcuts may be most suitable for Grade 3 students who have just enough calculation skills to be able to cope with the shortcut but not enough that the shortcut and the algorithm seem equally efficient. When typical solution methods are too difficult or inefficient, students are more likely to use alternative methods [39], which is consistent with the finding in the current study that students in all grades had higher use of shortcuts on problems with larger numbers.

When children have good conceptual knowledge of inversion and associativity, they should be able to appropriately transfer the conceptually-based shortcuts to solve novel problems [1]. Our findings suggest that the evaluation of procedures task promoted the use of conceptually-based shortcuts on similar problems and that students were also able to apply the shortcuts to transfer problems. Consistent with the findings of Siegler and Stern [12], inversion shortcut use was lower on transfer problems than in the previous session with typical inversion problems, and the pattern was similar for associativity shortcut use. However, shortcut use on the transfer problems was similar to that of spontaneous shortcut use in the first session. It is possible that, without the intervention, use of the shortcuts on the transfer problems might have been even lower and therefore future studies that investigate spontaneous use of the shortcuts on transfer problems before and after an intervention are needed. Children in all grades were able to spontaneously apply their knowledge of the shortcuts to the novel problems in the first session indicating a strong schema or strong understanding of inversion and associativity [40]. After the very brief intervention, shortcut use may not have transferred because children had learned to use the shortcut rather than understanding the concept behind the shortcut.

Contrary to expectations, the presentation format for the evaluation of procedures task, either sequential or concurrent, made little difference to the promotion of conceptually-based shortcuts. Rittle-Johnson and Star's [30–32] studies differed from the present task in that our task was briefer, the problems and solution procedures in our task were simpler, and our participants were not required to provide as much information about their mathematical reasoning. Future work could use an intervention containing more problems and include problems that are similar in structure but to which the shortcuts do not apply—this could improve the effectiveness of the instruction by showing children the limitations of the shortcuts. Our results indicate that providing multiple problem solving approaches was effective in promoting shortcut use and the presentation of the approaches and the opportunity for comparison was not critical as it was in the Rittle-Johnson and Star studies. However, Rittle-Johnson, Star, and Durkin [41] found that comparison of solution strategies was not always effective. Based on children's shortcut use before and after the evaluation of procedures task, we propose that the evaluation task used in the present study provides a quick and effective instructional tool that can be implemented in the classroom to promote children's understanding of the relations between addition and subtraction.

Children's attitudes about the demonstrated shortcuts, compared to a left-to-right algorithm on inversion and associativity problems, had a clear effect on subsequent problem solving procedures. Inversion and associativity shortcut use was higher for participants who preferred the shortcut, with associativity shortcut use significantly increasing after the demonstration for participants who preferred the shortcut. On transfer problems, preferring the inversion shortcut also led to higher shortcut use and preferring the associativity shortcut did not. This latter finding provides further evidence that the associativity shortcut is more difficult for students perhaps because it can overtax cognitive resources. The associativity transfer problems require students to pay attention to all three numbers and notice that the third number can be subtracted from the first number and then to subsequently perform that subtraction and then add the middle number while keeping track that all of the numbers have been added or subtracted. However, associativity shortcut use was similar on the transfer problems as it was in the pretest so the intervention may still have had a beneficial effect.

As most intervention studies involve providing students with information or instruction without assessing children's attitudes or beliefs about that information or instruction, the finding that children's views of problem solving procedures impact their subsequent procedure or strategy choices also needs to be further investigated and current theories of strategy choice need to include children's attitudes as a factor in strategy choice [27]. The current study used a brief intervention task and children had the opportunity to only provide brief information about their strategy preferences. As Verschaffel et al. [26] propose, more extensive interviews with children to gain deeper insight into their strategy choices and the influences on their strategy choices are needed. However, this is the first study to examine the consequences of children being more or less receptive to conceptually-based shortcuts and the consequences, particularly for the associativity shortcut, were marked. In areas outside of mathematics, research has shown that children's attitudes significantly contribute to their actions [5]. The present study indicates that children's attitudes towards mathematics, even simple preferences between two strategies, can be the basis for action during problem solving.

Researchers are increasingly focussing attention on how attitudes impact children's mathematics performance, decisions to take advanced mathematics courses, and career choices (e.g., [24]). However, there is also a need to investigate how attitudes impact smaller components of mathematics learning such as strategy choice. Conceptually-based shortcut use has important advantages. First, using shortcuts frees up cognitive resources [42] that can then be used to deal with more complex problems such as algebra problems [38]. Second, using shortcuts may make children more likely to pay attention to problem characteristics to guide strategy choices [39, 42] rather than being "mindless" problem solvers who are rigid rather than flexible [43]. Further research on why some children have more positive attitudes towards accepting strategies that are highly efficient but are novel to their current strategy repertoire of algorithmic approaches may help explain some of the large individual differences found in children's procedural and conceptual knowledge of arithmetic.

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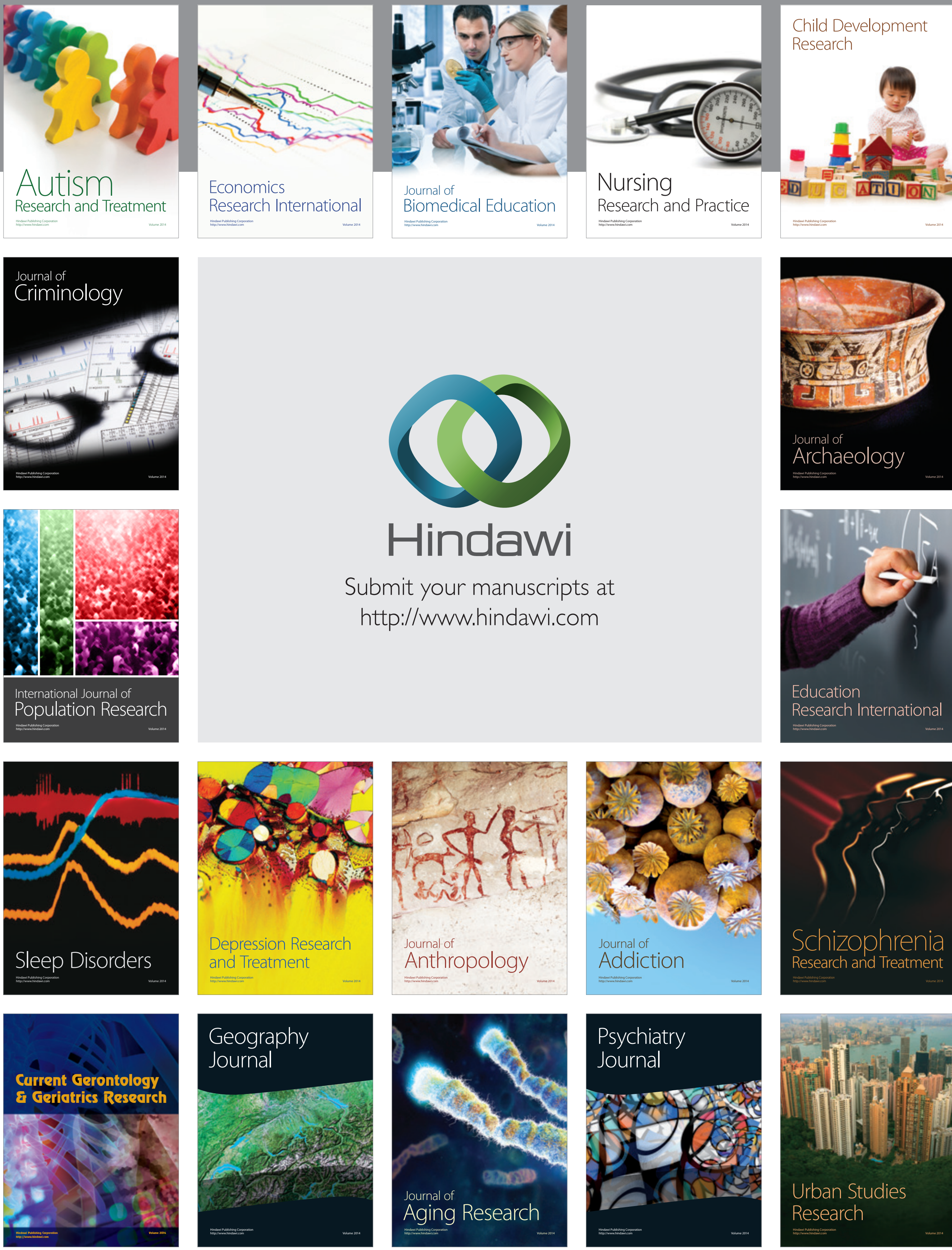
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