Research Article

The Spectral Method for Solving Sine-Gordon Equation Using a New Orthogonal Polynomial

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We have presented a numerical scheme for solving one-dimensional nonlinear sine-Gordon equation. We apply the spectral method with a basis of a new orthogonal polynomial which is orthogonal over the interval $[0, 1]$ with weighting function one. The results show the accuracy and efficiency of the proposed method.

1. Introduction

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation (PDE). It was originally considered in the nineteenth century in the course of surface of constant negative curvature. This equation attracted a lot of attention in the 1970s due to the presence of soliton solutions [1, 2]. The sine-Gordon equation appears in a number of physical applications [3–5], including applications in the chain of coupled pendulums and modelling the propagation of transverse electromagnetic (TEM) wave on a superconductor transmission system. Consider the one-dimensional nonlinear sine-Gordon equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2} - \sin(u(x, t)), \quad (x, t) \in [0, 1] \times [t_0, T], \quad (1.1)$$

with the initial conditions

$$u(x, t_0) = f_0(x), \quad \frac{\partial u}{\partial t}(x, t_0) = f_1(x), \quad x \in [0, 1], \quad (1.2)$$

and the boundary conditions

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t), \quad t \geq t_0. \quad (1.3)$$

In the last decade, several numerical schemes have been developed for solving (1.1), for instance, high-order solution of one-dimensional sine-Gordon equation using compact finite difference and DIRKN methods [3]. The authors of [6] proposed a numerical scheme for solving (1.1) using collocation and radial basis functions. Also, the boundary integral equation approach is used in [7]. Bratsos has proposed a numerical method for solving one-dimensional sine-Gordon equation and a third-order numerical scheme for the two-dimensional sine-Gordon equation in [8, 9], respectively. Also, in [10], a numerical scheme using radial basis function for the solution of two-dimensional sine-Gordon equation is used. In addition, several authors proposed spectral methods and Fourier pseudospectral method for solving nonlinear wave equation using a discrete Fourier series and Chebyshev orthogonal polynomials [11–13].

In this paper, we have proposed a numerical scheme for solving (1.1) using spectral method with a basis of a new orthogonal polynomial. This polynomial is introduced by Chelyshkov in [14].

The outline of this paper is as follows. In Section 2, we introduce a new orthogonal polynomial. In Section 3, we apply spectral method for solving (1.1). In Section 4, we show the numerical results. Finally, a brief conclusion is given in Section 5.

2. The New Orthogonal Polynomial

In this section, we briefly introduce the new orthogonal polynomial, that is introduced in [14].

Let n be a fixed whole number, and P_n is a sequence of polynomial, that is,

$$P_n = \{P_{nk}\}_{k=n}^0, \quad P_{nk} \equiv P_{nk}(x) = \sum_{l=k}^n \tau_{nkl} x^l, \quad (2.1)$$

that satisfy the orthogonality relationship

$$\int_0^1 P_{nk} P_{nl} dx = \begin{cases} 0, & k \neq l, \\ \frac{1}{(k+l+1)}, & k = l, \end{cases} \quad (2.2)$$

and standardization

$$\text{sign}(\tau_{nkn}) = (-1)^{n-k}. \quad (2.3)$$

The coefficient τ_{nkl} of the polynomial P_{nk} is defined by requirements (2.2), (2.3), and the Gram-Schmidt orthogonalization procedure (without normalization).

The explicit definition of the polynomial P_{nk} is as follows:

$$P_{nk}(x) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \binom{n+k+1+j}{n-k} x^{k+j}, \quad k = 0, 1, \dots, n. \quad (2.4)$$

This yields *Rodrigue's* type representation as follows:

$$P_{nk}(x) = \frac{1}{(n-k)!} \frac{1}{x^{k+1}} \frac{d^{n-k}}{dx^{n-k}} \left(x^{n+k+1} (1-x)^{n-k} \right), \quad k = 0, 1, \dots, n, \quad (2.5)$$

and the orthogonality relationship (2.2) is confirmed by applying (2.5). It also follows from (2.5) that

$$\int_0^1 P_{nk}(x) dx = \int_0^1 x^n dx = \frac{1}{n+1}. \quad (2.6)$$

This polynomial can be connected to a fixed set of the Jacobi polynomials $P_n^{(\alpha, \beta)}(\xi)$ [15], that is,

$$P_{nk} = x^n P_{k-n}^{(2n, 0)}(1-2x). \quad (2.7)$$

3. The Proposed Method

In this section, the introduced orthogonal polynomial is applied as the basis for spectral method to solve sine-Gordon equation.

The approximate solution $u_N(x, t)$ to the exact solution $u(x, t)$ can be written in the form

$$u(x, t) \cong U^N(x, t) = \sum_{n=0}^N a_n(t) P_{n0}(x), \quad (3.1)$$

where a_n is the time-dependent quantities which must be determined. In this paper, we apply orthogonal polynomial P_{nk} for $k = 0$. From (3.1), we get

$$\begin{aligned} U_{tt}^N &= \sum_{n=0}^N \ddot{a}_n(t) P_{n0}(x), \\ U_{xx}^N &= \sum_{n=0}^N a_n(t) \ddot{P}_{n0}(x). \end{aligned} \quad (3.2)$$

Using (3.2) and substituting in (1.1), we obtain

$$\sum_{n=0}^N \ddot{a}_n(t) P_{n0}(x_i) = \sum_{n=0}^N a_n(t) \ddot{P}_{n0}(x_i) - \sin(U^N(x_i, t)), \quad i = 1, 2, \dots, N-1, \quad (3.3)$$

where $\ddot{a}_n$ and $\ddot{P}_{n0}$ denote the second-order derivatives of a_n and P_{n0} with respect to t and x , respectively, wherein the collocation points are as $x_i = i/N$, $i = 0, 1, \dots, N$.

Equation (13) is rewritten as follows:

$$M\ddot{A}(t) = KA(t) - S_1. \quad (3.4)$$

For displaying this equation in matrix form, for $i = 1, 2, \dots, N-1$, and $n = 0, 1, \dots, N$, we define

$$\begin{aligned} A(t) &= [a_0(t), a_1(t), \dots, a_N(t)]^T, \\ M &= M_{\text{in}} = P_{n0}(x_i), \\ K &= K_{\text{in}} = \ddot{P}_{n0}(x_i), \\ S_1 &= [\sin(U_1), \sin(U_2), \dots, \sin(U_{N-1})]^T, \\ U_i &= U^N(x_i, t). \end{aligned} \quad (3.5)$$

The matrices M and K are positive definite matrices of order $(N-1) \times (N-1)$. The system given in (3.4) consists of $(N-1)$ equations in $(N+1)$ unknowns. To obtain a unique solution for this system, we need two additional equations. For this, we add two boundary conditions, which are given by (1.3), in the following forms:

$$\begin{aligned} U^N(0, t) &= \sum_{n=0}^N a_n(t) P_{n0}(0) = g_0(t), \quad t \geq t_0, \\ U^N(1, t) &= \sum_{n=0}^N a_n(t) P_{n0}(1) = g_1(t), \quad t \geq t_0. \end{aligned} \quad (3.6)$$

From (3.6), we get

$$\begin{aligned} \sum_{n=0}^N \ddot{a}_n(t) P_{n0}(0) &= \frac{d^2 g_0(t)}{dt^2}, \quad t \geq t_0, \\ \sum_{n=0}^N \ddot{a}_n(t) P_{n0}(1) &= \frac{d^2 g_1(t)}{dt^2}, \quad t \geq t_0. \end{aligned} \quad (3.7)$$

Therefore, by using (3.6), and (3.7), a new system is obtained as follows:

$$M\ddot{A}(t) = F(t, A(t)), \quad F(t, A(t)) = KA(t) - S + G(t), \quad (3.8)$$

where $A(t)$, $F(t, A(t))$, S , and $G(t)$ are vectors of order $(N + 1)$ with $(N + 1)$ components. The matrices M and K are of order $(N + 1) \times (N + 1)$, which are defined as follows:

$$M = \begin{pmatrix} p_{00}(0) & p_{10}(0) & \cdots & p_{N0}(0) \\ p_{00}(x_1) & p_{10}(x_1) & \cdots & p_{N0}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ p_{00}(x_{N-1}) & p_{10}(x_{N-1}) & \cdots & p_{N0}(x_{N-1}) \\ p_{00}(1) & p_{10}(1) & \cdots & p_{N0}(1) \end{pmatrix}, \quad (3.9)$$

$$K = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \ddot{p}_{00}(x_1) & \ddot{p}_{10}(x_1) & \cdots & \ddot{p}_{N0}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{p}_{00}(x_{N-1}) & \ddot{p}_{10}(x_{N-1}) & \cdots & \ddot{p}_{N0}(x_{N-1}) \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (3.10)$$

Also, the vectors S and $G(t)$ are defined as follows:

$$S = [0, \sin(U_1), \sin(U_2), \dots, \sin(U_{N-1}), 0]^T, \\ G(t) = \left[\frac{d^2 g_0(t)}{dt^2}, 0, \dots, 0, \frac{d^2 g_1(t)}{dt^2} \right]^T. \quad (3.11)$$

The nonlinear second-order system of ODEs (3.8) can be solved numerically using the fourth-order A-stable DIRKN method [3, 16]. From the initial conditions (1.2), we determine the initial vectors A_0 and $\dot{A}_0$. Using (3.1), we get

$$U^N(x_i, 0) = \sum_{n=0}^N a_n(t_0) P_{n0}(x_i), \quad i = 0, 1, \dots, N, \\ \frac{\partial U^N}{\partial t}(x_i, 0) = \sum_{n=0}^N \dot{a}_n(t_0) P_{n0}(x_i), \quad i = 0, 1, \dots, N. \quad (3.12)$$

Equation (3.12) is system of $(N + 1)$ equations in $(N + 1)$ unknowns. These equations can be written in the matrix form $MA_0 = b_1$ and $M\dot{A}_0 = b_2$, where

$$b_1 = [f_0(x_0), \dots, f_0(x_N)]^T, \\ b_2 = [f_1(x_0), \dots, f_1(x_N)]^T. \quad (3.13)$$

Table 1: Four-stage A-stable DIRKN method.

0.6582	0.24			
0.9300	0.186484967952	0.24		
0.0700	-3.148664086195	2.945628022183	0.24	
0.3418	1.491936101409	-1.816304852835	0.103007001689	0.24
	0.108572237906	0.012764608972	0.169586947770	0.209076205352
	0.317648443258	0.182351556742	0.182351556742	0.317648443258

3.1. The DIRKN Method

For solving (3.8), we use DIRKN method [3, 16]:

$$\begin{aligned}
 MQ_i &= MA_n + c_i \Delta t M \dot{A}_n + \Delta t^2 \sum_{j=1}^s a_{ij} F(t_n + c_j \Delta t, Q_j), \quad i = 1, \dots, s, \\
 MA_{n+1} &= MA_n + c_i \Delta t M \dot{A}_n + \Delta t^2 \sum_{j=1}^s \bar{b}_i F(t_n + c_j \Delta t, Q_j), \\
 M \dot{A}_{n+1} &= M \dot{A}_n + \Delta t \sum_{j=1}^s b_i F(t_n + c_j \Delta t, Q_j),
 \end{aligned} \tag{3.14}$$

where Q_i is the internal stages, A_{n+1} and $\dot{A}_{n+1}$ represent approximations to $A(t_{n+1})$ and $\dot{A}(t_{n+1})$, respectively. Also we define

$$\Delta t = t_0 + j \Delta t, \quad j = 0, 1, \dots, M, \tag{3.15}$$

where $\Delta t = (T - t_0)/M$. The RKN method can be denoted in *Butcher's* notation by the table of coefficients:

$$\begin{array}{c|ccc}
 c_1 & a_{11} & \cdots & a_{1s} \\
 \vdots & \vdots & \ddots & \vdots \\
 c_s & a_{s1} & \cdots & a_{ss} \\
 \hline
 & \bar{b}_1 & \cdots & \bar{b}_s \\
 \hline
 & b_1 & \cdots & b_s
 \end{array} \tag{3.16}$$

In the DIRKN method, $a_{ij} = 0$ as $(i < j)$ and a_{ii} are equal. Also, in the above table of coefficients, s is the number of stages. We use the four-stage A-stable DIRKN method [16], with algebraic fourth order, which is defined by the following table of coefficients for solving all experiments.

Using Table 1, we can calculate Q_i , $i = 1, \dots, 4$, separately from (3.14) with Newton-Raphson iteration method. We iterate until the following criterion is satisfied:

$$\left\| (Q_i^{k+1} - Q_i^k) \right\|_{\infty} < 10^{-12}, \tag{3.17}$$

where Q_i^k is the value of Q_i at the k th iteration of the Newton-Raphson method.

Table 2: The approximate solution for $N = 16$ of Experiment 1.

x	$T = 0.1$	$T = 0.2$	$T = 0.3$	$T = 0.4$	$T = 0.5$
0.0	0.3986746100	0.7895822394	1.1658271779	1.5220255084	1.8545904360
0.2	0.3908822100	0.7744385442	1.1441349052	1.4948183070	1.8229915644
0.4	0.3689530692	0.7311732829	1.0827641117	1.4175301638	1.7328129513
0.6	0.3366233389	0.6685450323	0.9914459040	1.3016971244	1.5965446689
0.8	0.2985244887	0.5937600941	0.8826302804	1.1624500568	1.4310481369
1.0	0.2588597329	0.5155692281	0.7680835230	1.0145556439	1.2534069046

4. Numerical Experiments

In this section, we present the results of numerical experiments using the method introduced in Section 3. The L_2 and L_∞ error norms are defined as follows:

$$L_2 = \|u - U^N\| = \left[\sum_{i=0}^N (u_i - U_i^N)^2 \right]^{1/2}, \quad (4.1)$$

$$L_\infty = \|u - U^N\| = \max_{0 \leq i \leq N} |u_i - U_i^N|.$$

We choose $\Delta t = 1/1000$, for solving all experiments.

The computations associated with the experiments discussed above were performed in Maple 13 on a PC with a CPU of 2.4 GHZ.

Experiment 1. In this experiment $[3, 6, 10]$, we consider (1.1) with $t_0 = 0$ and the initial conditions

$$\begin{aligned} f_0(x) &= 0, \\ f_1(x) &= 4\operatorname{sech}(x), \end{aligned} \quad (4.2)$$

with the boundary conditions (1.3), which can be obtained from the following exact solution:

$$u(x, t) = 4 \arctan(\operatorname{sech}(x)t). \quad (4.3)$$

In Table 2, the approximate solution for several final times (T) with different locations is shown. Figure 1 shows two type plots at $t = 0.5$ of Experiment 1.

Experiment 2. In this experiment $[3, 8]$, we consider (1.1) with $t_0 = 0$ and the initial conditions

$$f_0(x) = 0, \quad (4.4)$$

$$f_1(x) = \frac{4}{\sqrt{1+c^2}} \operatorname{sech}\left(\frac{x}{\sqrt{1+c^2}}\right), \quad (4.5)$$

Figure 1: (a) The exact and approximate solutions for $N = 16$ are shown. (b) The L_∞ error norm for $N = 16$ is shown.

Table 3: The L_2 error norm for different values of N of Experiment 2.

T	$N = 4$	$N = 8$	$N = 16$
0.1	5.02×10^{-6}	9.11×10^{-9}	9.56×10^{-15}
0.2	5.37×10^{-5}	1.00×10^{-8}	1.35×10^{-13}
0.3	2.05×10^{-4}	3.67×10^{-7}	2.22×10^{-13}
0.4	5.00×10^{-4}	8.37×10^{-6}	9.10×10^{-13}
0.5	9.46×10^{-4}	1.29×10^{-6}	3.43×10^{-12}
0.6	1.50×10^{-3}	1.52×10^{-6}	1.75×10^{-11}
0.7	2.10×10^{-3}	1.58×10^{-6}	1.29×10^{-10}
0.8	2.65×10^{-3}	1.65×10^{-6}	8.08×10^{-10}
0.9	3.08×10^{-3}	1.94×10^{-6}	4.94×10^{-9}
1.0	3.37×10^{-3}	2.13×10^{-6}	2.26×10^{-8}

with the boundary conditions (1.3), which can be obtained from the following exact solution:

$$u(x, t) = 4 \arctan \left(c^{-1} \sin \left(\frac{ct}{\sqrt{1+c^2}} \right) \operatorname{sech} \left(\frac{x}{\sqrt{1+c^2}} \right) \right), \quad c = 1. \quad (4.6)$$

In Tables 3 and 4, we show the L_2 and L_∞ error norms, for several final times (T). As we see from Tables 3 and 4, for the given values of N , the L_∞ error norm is less than the L_2 error norm. Figure 2 shows four type plots at $t = 0.5$ of Experiment 2.

Figure 2: (a) The exact and approximate solutions for $N = 16$ are shown. (b) The L_∞ error norm for $N = 16$ is shown. (c) The L_2 error norm for different values of N is shown. (d) The L_∞ error norm for different values of N is shown.

Table 4: The L_∞ error norm for different values of N of Experiment 2.

T	$N = 4$	$N = 8$	$N = 16$
0.1	1.30×10^{-6}	2.37×10^{-9}	2.68×10^{-15}
0.2	9.79×10^{-6}	1.83×10^{-8}	2.23×10^{-14}
0.3	2.98×10^{-5}	5.36×10^{-8}	2.97×10^{-14}
0.4	6.10×10^{-5}	9.14×10^{-8}	2.31×10^{-13}
0.5	9.82×10^{-5}	1.00×10^{-7}	7.94×10^{-13}
0.6	1.33×10^{-4}	1.00×10^{-7}	2.38×10^{-12}
0.7	1.57×10^{-4}	1.00×10^{-7}	1.29×10^{-11}
0.8	1.63×10^{-4}	1.00×10^{-7}	1.14×10^{-10}
0.9	1.63×10^{-4}	1.16×10^{-7}	8.03×10^{-10}
1.0	1.80×10^{-4}	1.16×10^{-7}	4.59×10^{-9}

Table 5: Error norms for different values of N of Experiment 3.

N	L_2 error norm	L_∞ error norm
4	9.13×10^{-3}	7.36×10^{-4}
6	5.29×10^{-4}	3.77×10^{-5}
8	4.13×10^{-5}	2.94×10^{-6}
10	9.31×10^{-7}	7.30×10^{-8}
12	2.00×10^{-7}	3.82×10^{-8}
14	3.23×10^{-8}	7.21×10^{-9}
16	1.87×10^{-8}	3.94×10^{-9}

Experiment 3. In experiment [3, 9], we consider (1.1) with $t_0 = 0$ and the initial conditions

$$f_0(x) = 4 \arctan \left(c \sinh \left(\frac{x}{\sqrt{1-c^2}} \right) \right), \quad (4.7)$$

$$f_1(x) = 0, \quad (4.8)$$

with the boundary conditions (1.3), which can be obtained from the following exact solution:

$$u(x, t) = 4 \arctan \left(\frac{c \sinh \left(x / \sqrt{1-c^2} \right)}{\cosh \left(ct / \sqrt{1-c^2} \right)} \right), \quad c = 0.5. \quad (4.9)$$

In Table 5, we represent the L_2 and L_∞ error norms for different values of N at $t = 0.5$. With increasing N , we obtain better results, and these error norms decrease five orders in magnitude. Figure 3 shows four-type plots at $t = 0.5$ of Experiment 3.

5. Conclusion

In this paper, we have presented the spectral method for solving one-dimensional nonlinear sine-Gordon equation using a new orthogonal polynomial. Numerical experiments show that the spectral method is an efficient one, and the results for value of $N = 16$ is more accurate

Figure 3: (a) The exact and approximate solutions for $N = 16$ are shown. (b) The L_∞ error norm for $N = 16$ is shown. (c) The L_2 error norm for different values of N is shown. (d) The L_∞ error norm for different values of N is shown.

than the other values of N . The proposed method has higher order of accuracy even for small values of N (i.e., $N = 4$). We obtain better results when we choose small value of Δt , in comparison to those obtained in the literature.

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