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Research Article

Generalized w-Euler Numbers and Polynomials

H. Y. Lee, N. S. Jung, and C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to C. S. Ryoo, ryoocs@hnu.kr

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We generalize the Euler numbers and polynomials by the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$. For the complement theorem, $E_{n,w}(x : a)$ have interesting different properties from the Euler polynomials and we observe an interesting phenomenon of "scattering" of the zeros of the the generalized Euler polynomials $E_{n,w}(x : a)$ in complex plane.

1. Introduction

The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Recently, many mathematicians have studied in the area of the Euler numbers and polynomials (see [1–15]). In [14], we introduced that Euler equation $E_n(x) = 0$ has symmetrical roots for $x = 1/2$ (see [14]). It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the the generalized w-Euler polynomials $E_{n,w}(x : a)$ in complex plane. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.1}$$

Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} g(x)(-1)^x \tag{1.2}$$

(cf. [5–7]).

If we take $g_1(x) = g(x+1)$ in (1.2), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.3}$$

From (1.3), we obtain

$$I_{-1}(g_n) + (-1)^{n-1}I_{-q}(g) = 2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l), \tag{1.4}$$

where $g_n(x) = g(x+n)$ (cf. [1–15]).

As a well-known definition, the Euler polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t,x) &= \frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \end{aligned} \tag{1.5}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers (cf. [1–15]).

Our aim in this paper is to define the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$. We investigate some properties which are related to the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$. In particular, distribution of roots for $E_{n,w}(x : a) = 0$ is different from $E_n(x) = 0'$s. We also derive the existence of a specific interpolation function which interpolate the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$.

2. The Generalized w-Euler Numbers and Polynomials

Our primary goal of this section is to define the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$. We also find generating functions of the generalized w-Euler numbers $E_{n,w}(a)$ and polynomials $E_{n,w}(x : a)$. Let a be strictly positive real number.

The generalized w-Euler numbers and polynomials $E_{n,w}(a)$, $E_{n,w}(x : a)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,w}(a)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-1}(x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} E_{n,w}(x : a)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ay}e^{(ay+x)t}d\mu_{-1}(y), \quad \text{for } t\in\mathbb{R}, w\in\mathbb{C}, \tag{2.2}$$

respectively.

From above definition, we obtain

$$E_{n,w}(a) = \int_{\mathbb{Z}_p} w^{ax}(ax)^n d\mu_{-1}(x),$$
$$E_{n,w}(x:a) = \int_{\mathbb{Z}_p} w^{ay}(ay+x)^n d\mu_{-1}(y). \tag{2.3}$$

Let $g(x) = w^{ax}e^{axt}$. By (1.3) and using p-adic integral on $\mathbb{Z}_p$, we have

$$\begin{aligned} I_{-1}(g_1) + I_{-1}(g) &= \int_{\mathbb{Z}_p} w^{a(x+1)}e^{a(x+1)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-1}(x) \\ &= \left(w^a e^{at}+1\right)\int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-1}(x) \\ &= 2. \end{aligned} \tag{2.4}$$

Hence, by (2.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,w}(a)\ \frac{t^n}{n!} = \frac{2}{w^a e^{at}+1}. \tag{2.5}$$

By (1.3), (2.2) and $g(y) = w^{ay}e^{(ay+x)t}$, we have

$$\sum_{n=0}^{\infty} E_{n,w}(x:a)\frac{t^n}{n!} = \frac{2}{w^a e^{at}+1}e^{xt}. \tag{2.6}$$

After some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} E_{n,w}(x:a)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}(-1)^n w^a e^{ant}e^{xt}. \tag{2.7}$$

From (2.6), we have

$$\begin{aligned} E_{n,w}(x:a) &= \sum_{n=0}^{n}\binom{n}{k}x^{n-k}E_{k,w}(a) \\ &= (x+E_w(a))^n, \end{aligned} \tag{2.8}$$

with the usual convention of replacing $(E_w(a))^n$ by $E_{n,w}(a)$.

3. Basic Properties for the Generalized w-Euler Numbers and Polynomials

By (2.5), we have

$$\begin{aligned}\frac{\partial}{\partial x}\sum_{n=0}^{\infty}E_{n,w}(x:a)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{2}{w^a e^{at}+1}e^{xt}\right)\\ &= t\sum_{n=0}^{\infty}E_{n,w}(x:a)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}nE_{n-1,w}(x:a)\frac{t^n}{n!}.\end{aligned} \tag{3.1}$$

By (3.1), we have the following differential relation.

Theorem 3.1. *For positive integers n, one has*

$$\frac{\partial}{\partial x}E_{n,w}(x:a) = nE_{n-1,w}(x:a). \tag{3.2}$$

By Theorem 3.1, we easily obtain the following corollary.

Corollary 3.2 (Integral formula). *One has*

$$\int_p^q E_{n-1,w}(x:a)dx = \frac{1}{n}\left(E_{n,w}(q:a) - E_{n,w}(p:a)\right). \tag{3.3}$$

By (2.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,w}(x+y:a)\frac{t^n}{n!} &= \frac{2}{w^a e^{at}+1}e^{(x+y)t}\\ &= \sum_{n=0}^{\infty}E_{n,w}(x:a)\frac{t^n}{n!}\sum_{k=0}^{\infty}y^k\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{k,w}(x:a)y^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following addition theorem.

Theorem 3.3 (Addition theorem). *For $n \in \mathbb{Z}_+$,*

$$E_{n,w}(x+y:a) = \sum_{k=0}^{n}\binom{n}{k}E_{k,w}(x:a)y^{n-k}. \tag{3.5}$$

By (2.5), for $m \equiv 1 \pmod 2$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}&\left(m^n\sum_{k=0}^{m-1}(-1)^k w^{ak}E_{n,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{t^n}{n!}\\
&=\sum_{k=0}^{m-1}(-1)^k w^{ak}\left(\sum_{n=0}^{\infty}E_{n,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{(mt)^n}{n!}\\
&=\sum_{k=0}^{m-1}\left((-1)^k w^{ak}\frac{2}{w^{ma}e^{mat}}e^{(x+ak)t}\right)\\
&=\frac{2}{w^a e^{at}+1}e^{xt}\\
&=\sum_{n=0}^{\infty}E_{n,w}(x:a)\frac{t^n}{n!}.
\end{aligned}\tag{3.6}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following multiplication theorem.

Theorem 3.4 (Multiplication theorem). *For $m, n \in \mathbb{N}$*

$$E_{n,w}(x:a)=m^n\sum_{k=0}^{m-1}(-1)^k w^{ak}E_{n,w^m}\left(\frac{x+ak}{m}:a\right).\tag{3.7}$$

From (1.3), we note that

$$\begin{aligned}
2&=\int_{\mathbb{Z}_p}w^{ax+a}e^{(ax+a)t}d\mu_{-1}(x)+\int_{\mathbb{Z}_p}w^{ax}e^{axt}d\mu_{-1}(x)\\
&=\sum_{n=0}^{\infty}\left(w^a\int_{\mathbb{Z}_p}w^{ax}(ax+a)^n d\mu_{-1}(x)+\int_{\mathbb{Z}_p}w^{ax}(ax)^n d\mu_{-1}(x)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(w^aE_{n,w}(a:a)+E_{n,w}(a)\right)\frac{t^n}{n!}.
\end{aligned}\tag{3.8}$$

From the above, we obtain the following theorem.

Theorem 3.5. *For $n \in \mathbb{Z}_+$, one has*

$$w^aE_{n,w}(a:a)+E_{n,w}(a)=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases}\tag{3.9}$$

By (2.8) in the above, we arrive at the following corollary.

Corollary 3.6. *For $n \in \mathbb{Z}_+$, one has*

$$w^a(a+E_w(a))^n+E_{n,w}(a)=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}\tag{3.10}$$

with the usual convention of replacing $(E_w(a))^n$ by $E_{n,w}(a)$.

From (1.4), we note that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(2\sum_{l=0}^{n-1}(-1)^{n-1-l}w^{al}(al)^{m}\right)\frac{t^{n}}{m!}\\
&\quad=\int_{\mathbb{Z}_p} w^{ax+an}e^{(ax+an)t}d\mu_{-1}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-1}(x)\\
&\quad=\sum_{m=0}^{\infty}\left(w^{an}\int_{\mathbb{Z}_p} w^{ax}(ax+an)^{m}d\mu_{-1}(x)+(-1)^{n}\int_{\mathbb{Z}_p} w^{ax}(ax)^{m}d\mu_{-1}(x)\right)\frac{t^{m}}{m!}\\
&\quad=\sum_{m=0}^{\infty}\left(w^{an}E_{m,w}(an:a)+(-1)^{n-1}E_{m,w}(a)\right)\frac{t^{m}}{m!}.
\end{aligned}\tag{3.11}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 3.7. *For $n \in \mathbb{Z}_+$, one has*

$$w^{an}E_{m,w}(na:a)+(-1)^{n-1}E_{m,w}(a)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}w^{al}(al)^{m}.\tag{3.12}$$

4. The Analogue of the Euler Zeta Function

By using the generalized w-Euler numbers and polynomials, the generalized w-Euler zeta function and the generalized Hurwitz w-Euler zeta functions are defined. These functions interpolate the generalized w-Euler numbers and w-Euler polynomials, respectively. Let

$$F_w(x:a)(t)=2\sum_{n=0}^{\infty}(-1)^{n}w^{a}e^{ant}e^{xt}=\sum_{n=0}^{\infty}E_{n,w}(x:a)\frac{t^{n}}{n!}.\tag{4.1}$$

By applying derivative operator, $d^k/dt^k|_{t=0}$ to the above equation, we have

$$\left.\frac{d^k}{dt^k}F_w(x:a)(t)\right|_{t=0}=2\sum_{n=0}^{\infty}(-1)^{n}w^{an}(an+x)^{k},\qquad (k\in\mathbb{N}),\tag{4.2}$$

$$E_{k,w}(x:a)=2\sum_{n=0}^{\infty}(-1)^{n}w^{an}(an+x)^{k}.\tag{4.3}$$

By using the above equation, we are now ready to define the generalized w-Euler zeta functions.

Definition 4.1. For $s \in \mathbb{C}$, one defines

$$\zeta_w^{(a)}(x:s)=2\sum_{n=1}^{\infty}\frac{(-1)^{n}w^{an}}{(an+x)^{s}}.\tag{4.4}$$

Note that $\zeta_w^{(a)}(x,s)$ is a meromorphic function on $\mathbb{C}$. Note that if $w \to 1$ and $a = 1$, then $\zeta_w^{(a)}(x:s) = \zeta(x:s)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_w^{(a)}(x:s)$ and $E_{k,w}(x:a)$ is given by the following theorem.

Theorem 4.2. *For $k \in \mathbb{N}$, one has*

$$\zeta_w^{(a)}(x:-s) = E_{s,w}(x:a). \tag{4.5}$$

Observe that $\zeta_w^{(a)}(x:s)$ function interpolates $E_w(x:s)$ numbers at nonnegative integers.

By using (4.2), we note that

$$\left.\frac{d^k}{dt^k}F_w(0:a)(t)\right|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n w^{an}(an)^k, \quad (k \in \mathbb{N}). \tag{4.6}$$

Hence, we obtain

$$E_{k,w}(a) = 2\sum_{n=0}^{\infty}(-1)^n w^{an}(an)^k. \tag{4.7}$$

By using the above equation, we are now ready to define the generalized Hurwitz w-Euler zeta functions.

Definition 4.3. Let $s \in \mathbb{C}$. One defines

$$\zeta_w^{(a)}(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^n w^{an}}{(an)^s}. \tag{4.8}$$

Note that $\zeta_w^{(a)}(s)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $w \to 1$ and $a = 1$, then $\zeta_w^{(a)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_w^{(a)}(s)$ and $E_{k,w}(s)$ is given by the following theorem.

Theorem 4.4. *For $k \in \mathbb{N}$, one has*

$$\zeta_w^{(a)}(-k) = E_{k,w}(a). \tag{4.9}$$

Observe that $\zeta_w^{(a)}(-k)$ function interpolates $E_{k,w}(a)$ numbers at nonnegative integers.

5. Zeros of the Generalized w-Euler Polynomials $E_{n,w}(x:a)$

In this section, we investigate the reflection symmetry of the zeros of the generalized w-Euler polynomials $E_{n,w}(x:a)$.

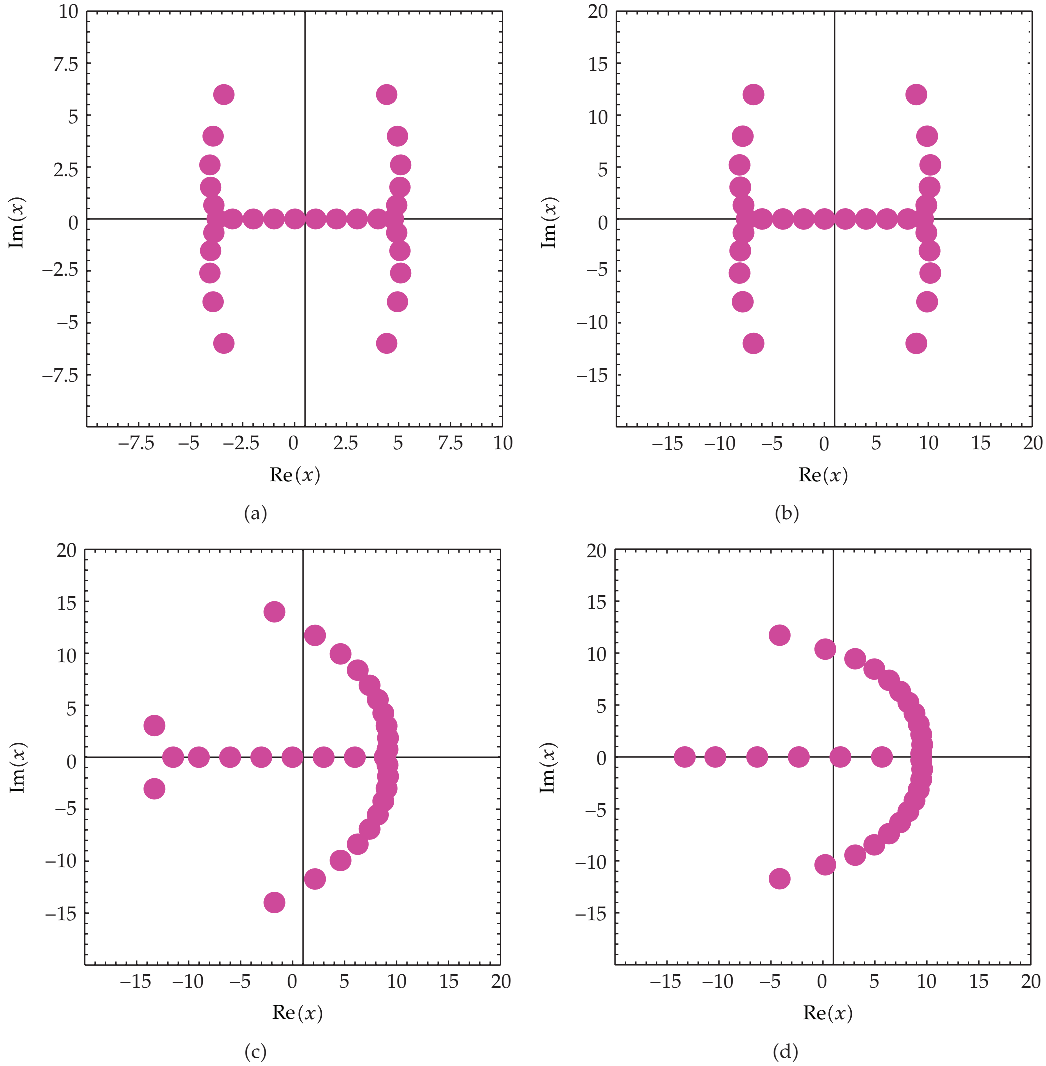

Figure 1: Zeros of $E_{n,w}(x:a)$ for $a=1,2,3,4$.

In the special case, $w=1$, $E_{n,w}(x:a)$ are called generalized Euler polynomials $E_n(x:a)$. Since

$$\begin{aligned}\sum_{n=0}^{\infty} E_n(a-x:a)\frac{(-t)^n}{n!} &= \frac{2}{e^{-at}+1}e^{(a-x)(-t)} \\ &= \frac{2}{e^{at}+1}e^{xt} \\ &= \sum_{n=0}^{\infty} E_n(x:a)\frac{t^n}{n!},\end{aligned} \tag{5.1}$$

we have

$$E_n(x:a)=(-1)^n E_n(a-x:a), \qquad \text{for } n\in\mathbb{N}. \tag{5.2}$$

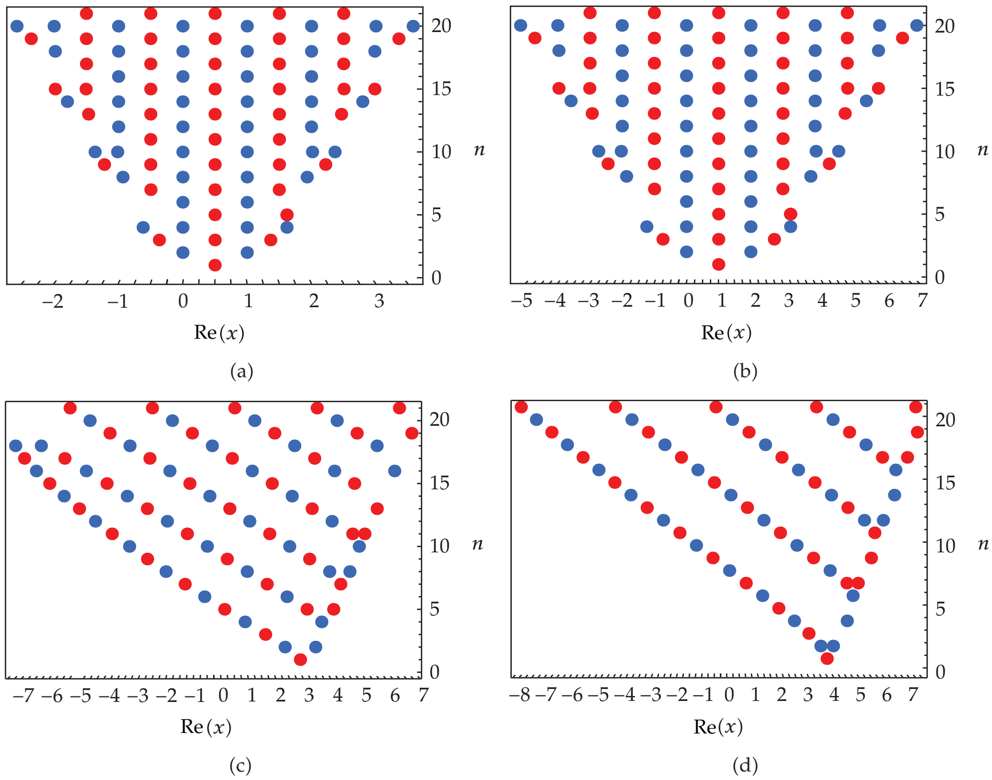

Figure 2: Real zeros of $E_{n,w}(x : a)$ for $1 \leq n \leq 20$.

We observe that $E_n(x : a), x \in \mathbb{C}$ has $\mathrm{Re}(x) = a/2$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions.

Let

$$F_{w,a}(x : t) = \frac{2}{w^a e^{at} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x : a) \frac{t^n}{n!}. \tag{5.3}$$

Then we have

$$\begin{aligned} F_{w^{-1},a}(a - x : -t) &= \frac{2}{w^{-a} e^{-at} + 1} e^{(a-x)(-t)} \\ &= w^a \frac{2}{w^a e^{at} + 1} e^{xt} \\ &= w^a \sum_{n=0}^{\infty} E_{n,w}(x : a) \frac{t^n}{n!}. \end{aligned} \tag{5.4}$$

Hence, we arrive at the following complement theorem.

Theorem 5.1 (Complement theorem). *For $n \in \mathbb{N}$,*

$$E_{n,w^{-1}}(a - x : a) = (-1)^n w^a E_{n,w}(x : a). \tag{5.5}$$

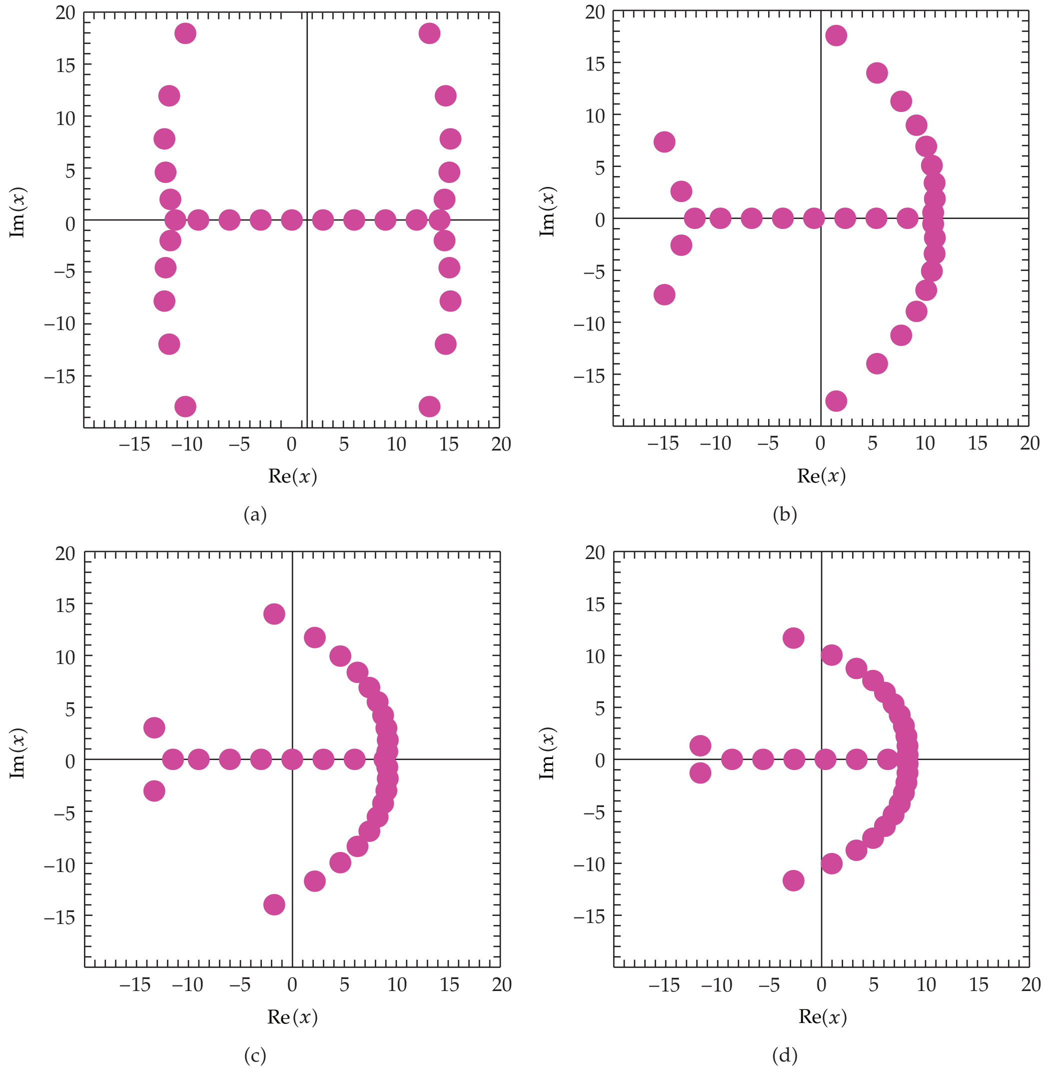

Figure 3: Zeros of $E_{n,w}(x : a)$ for $w = 1, 2, 3, 4$.

Throughout the numerical experiments, we can finally conclude that $E_{n,w}(x : a), x \in \mathbb{C}$ has not $\mathrm{Re}(x) = a/2$ reflection symmetry analytic complex functions. However, we observe that $E_{n,w}(x : a), x \in \mathbb{C}$ has $\mathrm{Im}(x) = 0$ reflection symmetry (see Figures 1, 2, and 3). The obvious corollary is that the zeros of $E_{n,w}(x : a)$ will also inherit these symmetries:

$$\text{if } E_{n,w}(x_0 : a) = 0, \qquad \text{then } E_{n,w}(x_0^* : a) = 0, \tag{5.6}$$

where $*$ denotes complex conjugation (see Figures 1, 2 and 3).

We investigate the beautiful zeros of the generalized w-Euler polynomials $E_{n,w}(x : a)$ by using a computer. We plot the zeros of the generalized Euler polynomials $E_{n,w}(x : a)$ for $n = 30$, $a = 1, 2, 3, 4$, and $x \in \mathbb{C}$ (Figure 1).

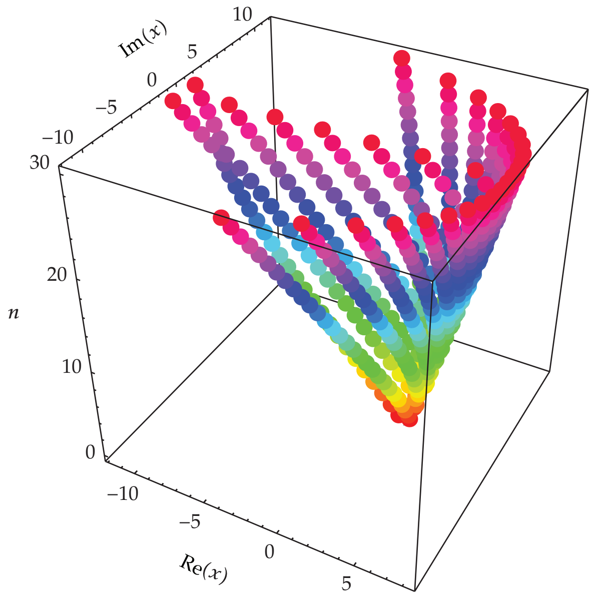

Figure 4: Stacks of zeros of $E_{n,w}(x : a)$ for $1 \le n \le 30$.

In Figure 1(a), we choose $n = 30, w = 1$, and $a = 1$. In Figure 1(b), we choose $n = 30, w = 1$, and $a = 2$. In Figure 1(c), we choose $n = 30, w = 3$, and $a = 3$. In Figure 1(d), we choose $n = 30, w = 4$, and $a = 4$.

Plots of real zeros of $E_{n,w}(x : a)$ for $1 \le n \le 20$ structure are presented (Figure 2).

In Figure 2(a), we choose $w = 1$ and $a = 1$. In Figure 2(b), we choose $w = 1$ and $a = 2$. In Figure 2(c), we choose $w = 3$ and $a = 3$. In Figure 2(d), we choose $w = 4$ and $a = 4$.

We investigate the beautiful zeros of the generalized $E_{n,w}(x : a)$ by using a computer. We plot the zeros of the generalized w-Euler polynomials $E_{n,w}(x : a)$ for $n = 30$ and $x \in \mathbb{C}$ (Figure 3).

In Figure 3(a), we choose $a = 3$ and $w = 1$. In Figure 3(b), we choose $a = 3$ and $w = 2$. In Figure 3(c), we choose $a = 3$ and $w = 3$. In Figure 3(d), we choose $a = 3$ and $w = 4$.

Stacks of zeros of $E_{n,w}(x : a)$ for $1 \le n \le 30, w = 4, a = 3$ from a 3D structure are presented (Figure 4).

Our numerical results for approximate solutions of real zeros of the generalized $E_{n,w}(x : a)$ are displayed (Tables 1 and 2).

We observe a remarkably regular structure of the complex roots of the generalized w-Euler polynomials $E_{n,w}(x : a)$. We hope to verify a remarkably regular structure of the complex roots of the generalized w-Euler polynomials $E_{n,w}(x : a)$ (Table 1). Next, we calculated an approximate solution satisfying $E_{n,w}(x : a), w = 2, a = 2, x \in \mathbb{R}$. The results are given in Table 2.

The plot above shows the generalized w-Euler polynomials $E_{n,w}(x : a)$ for real $1 \le a \le 7$ and $-5 \le x \le 5$, with the zero contour indicated in black (Figure 5). In Figure 5(a), we choose $n = 2$ and $w = 2$. In Figure 5(b), we choose $n = 3$ and $w = 3$. In Figure 5(c), we choose $n = 4$ and $w = 4$. In Figure 5(d), we choose $n = 5$ and $w = 5$.

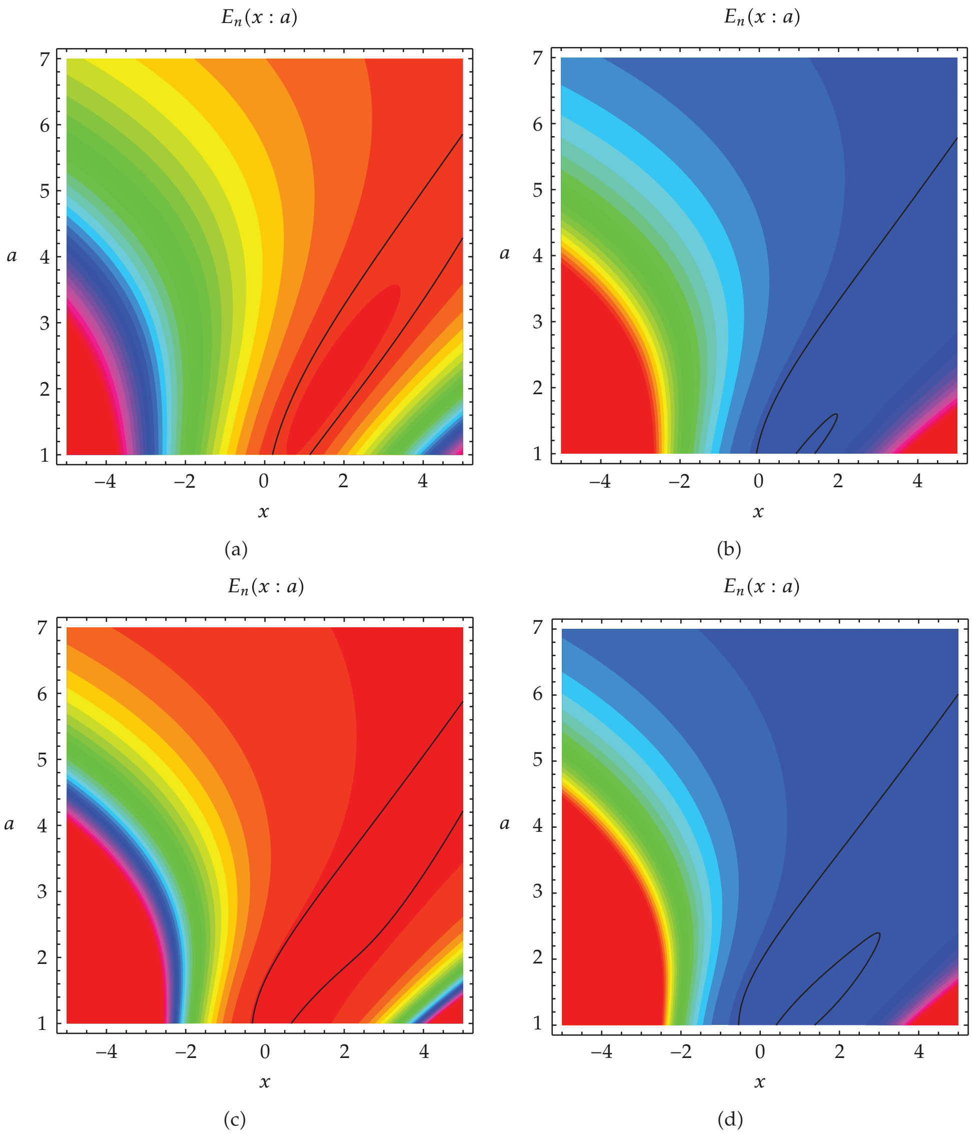

Figure 5: Zero contour of $E_{n,w}(x : a)$.

Finally, we will consider the more general problems. How many roots does $E_{n,w}(x : a)$ have? This is an open problem. Prove or disprove: $E_{n,w}(x : a) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,w}(x:a)}$ of $E_{n,w}(x : a)$, $\mathrm{Im}(x : a) \neq 0$. Since n is the degree of the polynomial $\mathrm{E}_{n,w}(x : a)$, the number of real zeros $R_{E_{n,w}(x:a)}$ lying on the real plane $\mathrm{Im}(x : a) = 0$ is then $R_{E_{n,w}(x:a)} = n - C_{E_{n,w}(x:a)}$, where $C_{E_{n,w}(x:a)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,w}(x:a)}$ and $C_{E_{n,w}(x:a)}$. We plot the zeros of $E_{n,w}(x : a)$, respectively (Figures 1–5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,w}(x : a)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that, generally, are not possible by hand. The authors have no doubt that investigation along this line will lead to

Table 1: Numbers of real and complex zeros of $E_{n,w}(x : a)$.

	$w = 2, a = 2$		$w = 3, a = 3$	
n	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	1	2
4	2	2	2	2
5	3	2	3	2
6	4	2	2	4
7	5	2	3	4
8	4	4	4	4
9	3	6	3	6
10	4	6	4	6
11	3	8	5	6
12	4	8	4	8
13	5	8	5	8

Table 2: Approximate solutions of $E_{n,w}(x : a) = 0, x \in \mathbb{R}$.

n	x
1	1.6000
2	0.8000, 2.400
3	0.04621, 2.046, 2.707
4	−0.6337, 1.326
5	−1.2404, 0.5878, 2.58
6	−1.7731, −0.14859, 1.851, 3.31
7	−2.2231, −0.8856, 1.117, 3.2, 3.4
8	−1.756, 0.244, 2.26, 3.0
9	−0.3544, 1.646, 3.6
10	−1.0898, 0.9102, 2.91, 4.1
11	−1.8231, −2.34, 0.17473, 2.17

a new approach employing numerical method in the field of research of w-Euler polynomials $E_{n,w}(x : a)$ to appear in mathematics and physics.

References

[1] M. Acikgoz and Y. Simsek, "On multiple interpolation functions of the Nörlund-type q-Euler polynomials," *Abstract and Applied Analysis*, Article ID 382574, 14 pages, 2009.

[2] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.

[3] M. Cenkci, M. Can, and V. Kurt, "q-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 203–216, 2004.

[4] L.-C. Jang, "A study on the distribution of twisted q-Genocchi polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 181–189, 2009.

[5] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.

[6] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[7] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.

[8] T. Kim, J. Choi, Y. H. Kim, and C. S. Ryoo, "A note on the weighted p-adic q-Euler measure on $\mathbb{Z}_p$," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 1, pp. 35–40, 2011.

[9] B. A. Kupershmidt, "Reflection symmetries of q-Bernoulli polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. suppl. 1, pp. 412–422, 2005.

[10] E.-J. Moon, S.-H. Rim, J.-H. Jin, and S.-J. Lee, "On the symmetric properties of higher-order twisted q-Euler numbers and polynomials," *Advances in Difference Equations*, Article ID 765259, 8 pages, 2010.

[11] H. Ozden, Y. Simsek, and I. N. Cangul, "Euler polynomials associated with p-adic q-Euler measure," *General Mathematics*, vol. 15, no. 2, pp. 24–37, 2007.

[12] C. S. Ryoo, T. Kim, and L.-C. Jang, "Some relationships between the analogs of Euler numbers and polynomials," *Journal of Inequalities and Applications*, Article ID 86052, 22 pages, 2007.

[13] C. S. Ryoo, "Some relations between twisted q-Euler numbers and Bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 47–54, 2011.

[14] C. S. Ryoo and Y. S. Yoo, "A note on Euler numbers and polynomials," *Journal of Concrete and Applicable Mathematics*, vol. 7, no. 4, pp. 341–348, 2009.

[15] Y. Simsek, O. Yurekli, and V. Kurt, "On interpolation functions of the twisted generalized Frobinuous-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 14, pp. 49–68, 2007.