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Research Article

Hybrid Projection Algorithm for a New General System of Variational Inequalities in Hilbert Spaces

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A new general system of variational inequalities in a real Hilbert space is introduced and studied. The solution of this system is shown to be a fixed point of a nonexpansive mapping. We also introduce a hybrid projection algorithm for finding a common element of the set of solutions of a new general system of variational inequalities, the set of solutions of a mixed equilibrium problem, and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Several strong convergence theorems of the proposed hybrid projection algorithm are established by using the demiclosedness principle. Our results extend and improve recent results announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C be a nonempty closed convex subset of H. Recall that $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Let $A : C \to H$ be a nonlinear mapping. Then A is called

(i) *α-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C; \tag{1.1}$$

(ii) *L-Lipschitz continuous* (or Lipschitzian), if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C; \tag{1.2}$$

(iii) *relaxed (c,d)-cocoercive*, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C, \tag{1.3}$$

for $c = 0$, A is d-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings.

Next, we consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.4}$$

The set of solutions of the variational inequality (1.4) is denoted by $\mathrm{VI}(C, A)$. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied in the literature, see, Piri [1], Qin et al. [2], Shehu [3], Wangkeeree and Preechasilp [4], Yao et al. [5], Yao et al. [6], and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous and $\mathrm{VI}(C, A)$ is nonempty, Korpelevič [7] introduced the following called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda A x_n), \\ x_{n+1} &= P_C(x_n - \lambda A y_n), \end{aligned} \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, 1/k)$ and P_C is the projection of $\mathbb{R}^n$ onto C. He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in \mathrm{VI}(C, A)$. Later on, the idea of Korpelevich was generalized and extended by many authors, see for example, [1–5, 8, 9] for finding a common element of the set of fixed points and the set of solutions of the variational inequality.

We recall the following well-known result which is called the best approximation result or the projection lemma.

Lemma 1.1. *For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C, \text{ iff } u = P_C z, \tag{1.6}$$

where P_C is the projection of H onto a closed convex set C.

Lemma 1.2. *$x^* \in C$ is a solution of the variational inequality if and only if $x^* \in C$ satisfies the relation*

$$x^* = P_C(x^* - \lambda A x^*), \tag{1.7}$$

where P_C is the projection of H onto a closed convex set C and $\lambda > 0$ is a constant.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_i : C \to H$ for all $i = 1, 2, 3$ be three mappings. In this paper, we focus on the problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned}
&\langle \lambda_1 A_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle \lambda_2 A_2 z^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle \lambda_3 A_3 x^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C,
\end{aligned} \tag{1.8}$$

which is called *a new general system of variational inequalities*, where $\lambda_i > 0$ for all $i = 1, 2, 3$. In particular, if $A_3 = 0$ and $z^* = x^*$, then problem (1.8) reduces to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned}
&\langle \lambda_1 A_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle \lambda_2 A_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C,
\end{aligned} \tag{1.9}$$

which is called *a general system of variational inequalities* and defined by Ceng et al. [10]. If we add up the requirement that $A_1 = A_2 := A$, then problem (1.9) reduces to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned}
&\langle \lambda_1 A_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle \lambda_2 A_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C,
\end{aligned} \tag{1.10}$$

which is defined by Verma [11], and is called *the new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$ and $\lambda_1 = \lambda_2 = 1$, then problem (1.10) reduces to the classical variational inequality $\mathrm{VI}(C, A)$. Ceng et al. [10] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of problem (1.9) for the α and β-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Some related works, we refer to see [9, 12–16].

Recently, in 2012, Ceng et al. [12] considered an iterative method for the system of problem (1.9) and obtained a strong convergence theorem for the two different systems of problem (1.9) and the set of fixed points of a strict pseudocontraction mapping in a real Hilbert space.

Let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [17] considered the following mixed equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.11}$$

The set of solutions of problem (1.11) is denoted by $\mathrm{MEP}(F,\varphi)$. It is easy to see that x is a solution of problem (1.11) implies that $x \in \mathrm{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the problem (1.11) becomes the following equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.12}$$

The set of solution of (1.12) is denoted by $\mathrm{EP}(F)$.

If $F = 0$, then the problem (1.11) reduces to the convex minimization problem:

$$\text{Find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.13}$$

If $\varphi = 0$ and $F(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then problem (1.11) reduces to the classical variational inequality and $\mathrm{EP}(F) = \mathrm{VI}(C,A)$. For solving problem (1.11), Ceng and Yao [17] introduced a hybrid iterative scheme for finding a common element of the set $\mathrm{MEP}(F,\varphi)$ and the set of common fixed points of finite many nonexpansive mappings in a Hilbert space. Some related works, we refer to see [3, 5, 9, 15].

Recently, in 2012, Kumam and Katchang [14] introduced an iterative algorithm for finding a common element of the set of solutions of a system of mixed equilibrium problems, the set of solutions of a general system of variational inequalities for Lipschitz continuous and relaxed cocoercive mappings, the set of common fixed points for nonexpansive semigroups, and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in Hilbert spaces.

In this paper, motivated and inspired by the idea of Kumam and Katchang [14], we introduce a hybrid projection algorithm for finding a common element of the set of solutions of a new general system of variational inequalities, the set of solutions of a mixed equilibrium problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Starting with an arbitrary $v \in C$ and let $x_1 \in C$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} F(u_n,y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,& \\ z_n = P_C(u_n - \lambda_3 A_3 u_n),& \\ y_n = P_C(z_n - \lambda_2 A_2 z_n),& \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1,& \end{aligned} \tag{1.14}$$

where $\lambda_i > 0$ for all $i = 1,2,3$, $\{r_n\} \subset (0,\infty)$ and $\{a_n\}, \{b_n\} \subset [0,1]$. Using the demiclosedness principle for nonexpansive mappings, we show that the sequence $\{x_n\}$ converges strongly to a common element of those three sets under some control conditions. Our results extend and improve recent results announced by many others.

2. Preliminaries

In this section, we recall the well-known results and give some useful lemmas that are used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered} \langle x - P_C x, y - P_C x \rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{gathered} \tag{2.4}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [18] for more details.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x \rangle < \varphi(z). \tag{2.5}$$

(B2) C is a bounded set.

In the sequel we will need to use the following lemma.

Lemma 2.1 (see [19]). *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C \right\} \tag{2.6}$$

for all $x \in H$. Then the following conclusions hold:

(1) *for each $x \in H$, $T_r(x) \neq \emptyset$;*

(2) *T_r is single-valued;*

(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.7}$$

(4) *$F(T_r) = \mathrm{MEP}(F, \varphi)$;*

(5) *$\mathrm{MEP}(F, \varphi)$ is closed and convex.*

We also need the following lemmas.

Lemma 2.2 (see [20]). *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ - \alpha\gamma\|x - y\|^2 - \beta\gamma\|y - z\|^2.\end{aligned} \tag{2.8}$$

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.9}$$

Lemma 2.4 (see [21]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \tag{2.10}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (see [22]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.6 (see [18]). *Demiclosedness principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \to y$), it follows that $(I - T)x = y$. Here I is the identity operator of H.*

In 2009, Kangtunyakarn and Suantai [23] introduced a new mapping called the S-mapping. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself. For each

$j = 1,2,\ldots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. They defined the new mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{2.11}$$

This mapping is called S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Nonexpansivity of each T_i ensures the nonexpansivity of S.

Lemma 2.7 (see [23]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1,2,\ldots,N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$, $\alpha_1^N \in (0,1]$ and $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

3. Main Results

In this section, we prove strong convergence theorems of the iterative scheme (1.14) to a common element of the set of solutions of a new general system of variational inequalities for relaxed (c,d)-cocoercive mappings, the set of fixed points of a nonexpansive mapping, and the set of solutions of a mixed equilibrium problem in a real Hilbert space.

The next lemmas are crucial for proving the main theorems.

Lemma 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $A_i : C \to H$ be L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mappings for $i = 1,2,3$. Let $G : C \to C$ defined by*

$$\begin{aligned}
G(x) = P_C[&P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x))\\
&-\lambda_1 A_1 P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x))], \quad \forall x \in C.
\end{aligned} \tag{3.1}$$

If $0 < \lambda_i \leq 2(d_i - c_i L_i^2)/L_i^2$, for $i = 1,2,3$, then $G : C \to C$ is a nonexpansive mapping.

Proof. For all $x, y \in C$, we have

$$
\begin{aligned}
\|G(x) - G(y)\| &= \|P_C[P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x)) \\
&\qquad -\lambda_1 A_1 P_C(P_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 P_C(x - \lambda_3 A_3 x))] \\
&\quad - P_C\left[P_C\left(P_C\left(y - \lambda_3 A_3 y\right) - \lambda_2 A_2 P_C\left(y - \lambda_3 A_3 y\right)\right)\right. \\
&\qquad \left.-\lambda_1 A_1 P_C\left(P_C\left(y - \lambda_3 A_3 y\right) - \lambda_2 A_2 P_C\left(y - \lambda_3 A_3 y\right)\right)\right]\| \\
&= \|P_C[P_C(P_C(I - \lambda_3 A_3) x - \lambda_2 A_2 P_C(I - \lambda_3 A_3) x) \\
&\qquad -\lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3) x - \lambda_2 A_2 P_C(I - \lambda_3 A_3) x)] \\
&\quad - P_C\left[P_C\left(P_C(I - \lambda_3 A_3) y - \lambda_2 A_2 P_C(I - \lambda_3 A_3) y\right)\right. \\
&\qquad \left.-\lambda_1 A_1 P_C\left(P_C(I - \lambda_3 A_3) y - \lambda_2 A_2 P_C(I - \lambda_3 A_3) y\right)\right]\| \\
&\le \|P_C(P_C(I - \lambda_3 A_3) x - \lambda_2 A_2 P_C(I - \lambda_3 A_3) x) \\
&\qquad - \lambda_1 A_1 P_C(P_C(I - \lambda_3 A_3) x - \lambda_2 A_2 P_C(I - \lambda_3 A_3) x) \\
&\qquad - \left[P_C\left(P_C(I - \lambda_3 A_3) y - \lambda_2 A_2 P_C(I - \lambda_3 A_3) y\right)\right. \\
&\qquad \left.-\lambda_1 A_1 P_C\left(P_C(I - \lambda_3 A_3) y - \lambda_2 A_2 P_C(I - \lambda_3 A_3) y\right)\right]\| \\
&= \|(I - \lambda_1 A_1) P_C(I - \lambda_2 A_2) P_C(I - \lambda_3 A_3) x \\
&\qquad -(I - \lambda_1 A_1) P_C(I - \lambda_2 A_2) P_C(I - \lambda_3 A_3) y\|.
\end{aligned}
\tag{3.2}
$$

It is well known that if $A : C \to H$ is L-Lipschitzian and relaxed (c, d)-cocoercive, then $I - \lambda A$ is nonexpansive for all $0 < \lambda \le 2(d - cL^2)/L^2$. By our assumption, we obtain $I - \lambda_i A_i$ is nonexpansive for all $i = 1, 2, 3$. It follows that $(I - \lambda_1 A_1) P_C (I - \lambda_2 A_2) P_C (I - \lambda_3 A_3)$ is nonexpansive. Therefore, from (3.2), we obtain immediately that the mapping G is nonexpansive. □

Lemma 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be three possibly nonlinear mappings. For given $x^*, y^*, z^* \in C$, (x^*, y^*, z^*) is a solution of problem* (1.8) *if and only if $x^* \in F(G)$, $y^* = P_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = P_C(x^* - \lambda_3 A_3 x^*)$, where G is the mapping defined as in Lemma 3.1.*

Proof. Note that we can rewrite (1.8) as

$$
\begin{aligned}
&\left\langle x^* - \left(y^* - \lambda_1 A_1 y^*\right), x - x^*\right\rangle \ge 0, \quad \forall x \in C, \\
&\left\langle y^* - (z^* - \lambda_2 A_2 z^*), x - y^*\right\rangle \ge 0, \quad \forall x \in C, \\
&\left\langle z^* - (x^* - \lambda_3 A_3 x^*), x - z^*\right\rangle \ge 0, \quad \forall x \in C.
\end{aligned}
\tag{3.3}
$$

From Lemma 1.1, we can deduce that (3.3) is equivalent to

$$
\begin{aligned}
x^* &= P_C(y^* - \lambda_1 A_1 y^*),\\
y^* &= P_C(z^* - \lambda_2 A_2 z^*),\\
z^* &= P_C(x^* - \lambda_3 A_3 x^*).
\end{aligned}
\tag{3.4}
$$

It is easy to see that (3.4) is equivalent to $x^* \in F(G)$, $y^* = P_C(z^* - \lambda_2 A_2 z^*)$, and $z^* = P_C(x^* - \lambda_3 A_3 x^*)$. □

Throughout this paper, the set of fixed points of the mapping G is denoted by $G\,\mathrm{VI}(C, A_1, A_2, A_3)$.

Now we prove the strong convergence theorems of the algorithm (1.14) for solving problem (1.8), fixed point problem of nonexpansive mapping and mixed equilibrium problem.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $A_i : C \to H$ be L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive for $i = 1, 2, 3$ and T be a nonexpansive self-mapping of C such that $\Omega = F(T) \bigcap G\,\mathrm{VI}(C, A_1, A_2, A_3) \bigcap \mathrm{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by*

$$
\begin{aligned}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, &\quad \forall y \in C,\\
z_n = P_C(u_n - \lambda_3 A_3 u_n),&\\
y_n = P_C(z_n - \lambda_2 A_2 z_n),&\\
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), &\quad n \geq 1,
\end{aligned}
\tag{3.5}
$$

where $\{r_n\} \subset (0, \infty)$ and $0 < \lambda_i < 2(d_i - c_i L_i^2)/L_i^2$, for $i = 1, 2, 3$ and $\{a_n\}$, $\{b_n\}$ are two sequences in $[0, 1]$ such that

- (C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
- (C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$;
- (C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.8), where $\overline{y} = P_C(\overline{z} - \lambda_2 A_2 \overline{z})$ and $\overline{z} = P_C(\overline{x} - \lambda_3 A_3 \overline{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 3.2 that

$$
\begin{aligned}
x^* = P_C[&P_C(P_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 P_C(x^* - \lambda_3 A_3 x^*))\\
&- \lambda_1 A_1 P_C(P_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 P_C(x^* - \lambda_3 A_3 x^*))].
\end{aligned}
\tag{3.6}
$$

Put $y^* = P_C(z^* - \lambda_2 A_2 z^*)$, $z^* = P_C(x^* - \lambda_3 A_3 x^*)$ and $t_n = P_C(y_n - \lambda_1 A_1 y_n)$. Then $x^* = P_C(y^* - \lambda_1 A_1 y^*)$, and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T t_n. \tag{3.7}$$

By nonexpansiveness of $I - \lambda_i A_i$ $(i = 1, 2, 3)$, we have

$$\begin{aligned} \|t_n - x^*\| &= \left\|P_C(I - \lambda_1 A_1) y_n - P_C(I - \lambda_1 A_1) y^*\right\| \\ &\le \left\|y_n - y^*\right\| = \|P_C(I - \lambda_2 A_2) z_n - P_C(I - \lambda_2 A_2) z^*\| \\ &\le \|z_n - z^*\| = \|P_C(I - \lambda_3 A_3) u_n - P_C(I - \lambda_3 A_3) x^*\| \\ &\le \|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \le \|x_n - x^*\|, \end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n v + b_n x_n + (1 - a_n - b_n) T t_n - x^*\| \\ &\le a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|t_n - x^*\| \\ &\le a_n \|v - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n) \|x_n - x^*\| \\ &\le \max\{\|v - x^*\|, \|x_1 - x^*\|\}. \end{aligned} \tag{3.9}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1 y_n\}$, $\{A_2 z_n\}$, $\{A_3 u_n\}$, and $\{T t_n\}$ are also bounded. Also, observe that

$$\begin{aligned} \|t_{n+1} - t_n\| &= \left\|P_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - P_C(y_n - \lambda_1 A_1 y_n)\right\| \\ &\le \left\|y_{n+1} - y_n\right\| \\ &= \|P_C(z_{n+1} - \lambda_2 A_2 z_{n+1}) - P_C(z_n - \lambda_2 A_2 z_n)\| \\ &\le \|z_{n+1} - z_n\| \\ &= \|P_C(u_{n+1} - \lambda_3 A_3 u_{n+1}) - P_C(u_n - \lambda_3 A_3 u_n)\| \\ &\le \|u_{n+1} - u_n\|. \end{aligned} \tag{3.10}$$

On the other hand, from $u_n = T_{r_n} x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}} x_{n+1} \in \operatorname{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.11}$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C. \tag{3.12}$$

Putting $y = u_{n+1}$ in (3.11) and $y = u_n$ in (3.12), we have

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$
$$F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.13}$$

From the monotonicity of F, we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.14}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.15}$$

Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.16}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.17}$$

It follows from (3.10) and (3.17) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.18}$$

Let $x_{n+1} = b_n x_n + (1 - b_n) w_n$ for all $n \geq 1$. Then, we obtain

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1}x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1}v + (1 - a_{n+1} - b_{n+1})Tt_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n)Tt_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}}(v - Tt_{n+1}) + \frac{a_n}{1 - b_n}(Tt_n - v) + Tt_{n+1} - Tt_n. \end{aligned} \tag{3.19}$$

By (3.18) and (3.19), we have

$$\begin{aligned}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\le \frac{a_{n+1}}{1-b_{n+1}}\|v-Tt_{n+1}\|+\frac{a_n}{1-b_n}\|Tt_n-v\| \\ &\quad+\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\| \\ &\le \frac{a_{n+1}}{1-b_{n+1}}\|v-Tt_{n+1}\|+\frac{a_n}{1-b_n}\|Tt_n-v\| \\ &\quad+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|.\end{aligned} \tag{3.20}$$

This together with (C1)–(C3), we obtain that

$$\limsup_{n\to\infty}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\le 0. \tag{3.21}$$

Hence, by Lemma 2.5, we get $\|x_n-w_n\|\to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-b_n)\|w_n-x_n\|=0. \tag{3.22}$$

Since

$$x_{n+1}-x_n=a_n(v-x_n)+(1-a_n-b_n)(Tt_n-x_n), \tag{3.23}$$

we have that

$$\|Tt_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.24}$$

Next, we prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$. From Lemma 2.1 (3), we have

$$\begin{aligned}\|u_n-x^*\|^2&=\|T_{r_n}x_n-T_{r_n}x^*\|^2\le\langle T_{r_n}x_n-T_{r_n}x^*,x_n-x^*\rangle \\ &=\langle u_n-x^*,x_n-x^*\rangle=\frac{1}{2}\left\{\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-u_n\|^2\right\}.\end{aligned} \tag{3.25}$$

Hence

$$\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2. \tag{3.26}$$

From Lemma 2.2, (3.8) and (3.26), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 + a_n - b_n)\|t_n - x^*\|^2 \\ &\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 + a_n - b_n)\|u_n - x^*\|^2 \\ &\le a_n\|v - x^*\|^2 b_n\|x_n - x^*\|^2 \\ &\quad + (1 + a_n - b_n)\left[\|x_n - x^*\|^2 - \|x_n - u_n\|^2\right] \\ &\le a_n\|v - x^*\|^2\|x_n - x^*\|^2 - (1 + a_n - b_n)\|x_n - u_n\|^2.\end{aligned} \tag{3.27}$$

It follows that

$$\begin{aligned}(1 - a_n - b_n)\|x_n - u_n\|^2 &\le a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\le a_n\|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.\end{aligned} \tag{3.28}$$

From the conditions (C1), (C2) and (3.22), we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.29}$$

By (3.24) and (3.29), we have

$$\|Tt_n - u_n\| \le \|Tt_n - x_n\| + \|x_n - u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Next, we show that $\|A_1y_n - A_1y^*\| \to 0$, $\|A_2z_n - A_2z^*\| \to 0$, and $\|A_3u_n - A_3u^*\| \to 0$, as $n \to \infty$.

From (3.8) and the fact that A_1 is L_1-Lipschitzian and relaxed (c_1, d_1)-cocoercive, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \\ &= a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\left\|P_C(y_n - \lambda_1A_1y_n) - P_C(y^* - \lambda_1A_1y^*)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\left\|(y_n-\lambda_1A_1y_n)-(y^*-\lambda_1A_1y^*)\right\|^2\\
&= a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\Big[\left\|y_n-y^*\right\|^2+\lambda_1^2\left\|A_1y_n-A_1y^*\right\|^2\\
&\qquad\qquad -2\lambda_1\langle y_n-y^*,A_1y_n-A_1y^*\rangle\Big]\\
&\leq a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\Big[\left\|y_n-y^*\right\|^2+\lambda_1^2\left\|A_1y_n-A_1y^*\right\|^2\\
&\qquad\qquad -2\lambda_1\Big(-c_1\left\|A_1y_n-A_1y^*\right\|^2+d_1\left\|y_n-y^*\right\|^2\Big)\Big]\\
&\leq a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\left[\|x_n-x^*\|^2+\left(2\lambda_1c_1+\lambda_1^2-\frac{2\lambda_1d_1}{L_1^2}\right)\left\|A_1y_n-A_1y^*\right\|^2\right]\\
&\leq a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\left(2\lambda_1c_1+\lambda_1^2-\frac{2\lambda_1d_1}{L_1^2}\right)\left\|A_1y_n-A_1y^*\right\|^2.
\end{aligned}
\tag{3.31}
$$

Similarly, since A_i is L_i-Lipschitzian and relaxed (c_i,d_i)-cocoercive mappings for $i = 2,3$, $\|t_n-x^*\| \leq \|y_n-y^*\|$ and $\|y_n-y^*\| \leq \|z_n-z^*\|$, we can show that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\left(2\lambda_2c_2+\lambda_2^2-\frac{2\lambda_2d_2}{L_2^2}\right)\|A_2z_n-A_2z^*\|^2,\\
\|x_{n+1}-x^*\|^2 &\leq a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\left(2\lambda_3c_3+\lambda_3^2-\frac{2\lambda_3d_3}{L_3^2}\right)\|A_3u_n-A_3x^*\|^2.
\end{aligned}
\tag{3.32}
$$

From (3.31) and (3.32), we have

$$
\begin{aligned}
-(1-a_n-b_n)\left(2\lambda_1c_1+\lambda_1^2-\frac{2\lambda_1d_1}{L_1^2}\right)\left\|A_1y_n-A_1y^*\right\|^2 &\leq a_n\|v-x^*\|^2\\
+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&-(1-a_n-b_n)\left(2\lambda_2 c_2+\lambda_2^2-\frac{2\lambda_2 d_2}{L_2^2}\right)\|A_2 z_n-A_2 z^*\|^2 \le a_n\|v-x^*\|^2\\
&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\
&-(1-a_n-b_n)\left(2\lambda_3 c_3+\lambda_3^2-\frac{2\lambda_3 d_3}{L_3^2}\right)\|A_3 u_n-A_3 x^*\|^2 \le a_n\|v-x^*\|^2\\
&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.33}
$$

This together with (C1), (C2) and (3.2), we obtain that

$$
\lim_{n\to\infty}\|A_1 y_n-A_1 y^*\| = \lim_{n\to\infty}\|A_2 z_n-A_2 z^*\| = \lim_{n\to\infty}\|A_3 u_n-A_3 x^*\| = 0. \tag{3.34}
$$

Next, we prove that $\|Tt_n-t_n\| \to 0$ as $n \to \infty$. From (2.2), (3.8) and nonexpansiveness of $I-\lambda_2 A_2$ and $I-\lambda_3 A_3$, we get

$$
\begin{aligned}
\|y_n-y^*\|^2 &= \|P_C(z_n-\lambda_2 A_2 z_n)-P_C(z^*-\lambda_2 A_2 z^*)\|^2\\
&\le \langle (z_n-\lambda_2 A_2 z_n)-(z^*-\lambda_2 A_2 z^*), y_n-y^*\rangle\\
&= \frac{1}{2}\Big[\|(z_n-\lambda_2 A_2 z_n)-(z^*-\lambda_2 A_2 z^*)\|^2+\|y_n-y^*\|^2\\
&\qquad -\|(z_n-\lambda_2 A_2 z_n)-(z^*-\lambda_2 A_2 z^*)-(y_n-y^*)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|z_n-z^*\|^2+\|y_n-y^*\|^2\\
&\qquad\quad -\|(z_n-y_n)-(z^*-y^*)-\lambda_2(A_2 z_n-A_2 z^*)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|x_n-x^*\|^2+\|y_n-y^*\|^2-\|(z_n-y_n)-(z^*-y^*)\|^2\\
&\qquad +2\lambda_2\langle (z_n-y_n)-(z^*-y^*), A_2 z_n-A_2 z^*\rangle-\lambda_2^2\|A_2 z_n-A_2 z^*\|^2\Big],\\
\|z_n-z^*\|^2 &= \|P_C(u_n-\lambda_3 A_3 u_n)-P_C(x^*-\lambda_3 A_3 x^*)\|^2\\
&\le \langle (u_n-\lambda_3 A_3 u_n)-(x^*-\lambda_3 A_3 x^*), z_n-z^*\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2}\Big[\|(u_n - \lambda_3 A_3 u_n) - (x^* - \lambda_3 A_3 x^*)\|^2 + \|z_n - z^*\|^2 \\
&\qquad -\|(u_n - \lambda_3 A_3 u_n) - (x^* - \lambda_3 A_3 x^*) - (z_n - z^*)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|z_n - z^*\|^2 \\
&\qquad -\|(u_n - z_n) - (x^* - z^*) - \lambda_3(A_3 u_n - A_3 x^*)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - z^*\|^2 - \|(u_n - z_n) - (x^* - z^*)\|^2 \\
&\qquad +2\lambda_3\langle (u_n - z_n) - (x^* - z^*), A_3 u_n - A_3 x^*\rangle - \lambda_3^2\|A_3 u_n - A_3 x^*\|^2\Big].
\end{aligned} \tag{3.35}$$

Therefore

$$\begin{aligned}
\|y_n - y^*\|^2 &\le \|x_n - x^*\|^2 - \|(z_n - y_n) - (z^* - y^*)\|^2 \\
&\quad + 2\lambda_2\langle (z_n - y_n) - (z^* - y^*), A_2 z_n - A_2 z^*\rangle, \\
\|z_n - z^*\|^2 &\le \|x_n - x^*\|^2 - \|(u_n - z_n) - (x^* - z^*)\|^2 \\
&\quad + 2\lambda_3\langle (u_n - z_n) - (x^* - z^*), A_3 u_n - A_3 x^*\rangle.
\end{aligned} \tag{3.36}$$

From (3.36), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|y_n - y^*\|^2 \\
&\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + (1 - a_n - b_n)\Big[\|x_n - x^*\|^2 - \|(z_n - y_n) - (z^* - y^*)\|^2 \\
&\qquad +2\lambda_2\langle (z_n - y_n) - (z^* - y^*), A_2 z_n - A_2 z^*\rangle\Big] \\
&\le a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - (1 - a_n - b_n)\|(z_n - y_n) - (z^* - y^*)\|^2 \\
&\quad + (1 - a_n - b_n)2\lambda_2\|(z_n - y_n) - (z^* - y^*)\|\|A_2 z_n - A_2 z^*\|, \\
\|x_{n+1} - x^*\|^2 &\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|z_n - z^*\|^2 \\
&\le a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2
\end{aligned} \tag{3.37}$$

$$
\begin{aligned}
&+ (1-a_n-b_n)\Big[\|x_n-x^*\|^2 - \|(u_n-z_n)-(x^*-z^*)\|^2 \\
&\qquad +2\lambda_3\langle (u_n-z_n)-(x^*-z^*), A_3u_n - A_3x^*\rangle\Big] \\
&\le a_n\|v-x^*\|^2 + \|x_n-x^*\|^2 - (1-a_n-b_n)\|(u_n-z_n)-(x^*-z^*)\|^2 \\
&\quad + (1-a_n-b_n)2\lambda_3\|(u_n-z_n)-(x^*-z^*)\|\|A_3u_n-A_3x^*\|.
\end{aligned}
\tag{3.38}
$$

Hence

$$
\begin{aligned}
&(1-a_n-b_n)\big\|(z_n-y_n)-(z^*-y^*)\big\|^2 \\
&\quad \le a_n\|v-x^*\|^2 + (1-a_n-b_n)2\lambda_2\big\|(z_n-y_n)-(z^*-y^*)\big\|\|A_2z_n-A_2z^*\| \\
&\qquad + (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|, \\
&(1-a_n-b_n)\|(u_n-z_n)-(x^*-z^*)\|^2 \\
&\quad \le a_n\|v-x^*\|^2 + (1-a_n-b_n)2\lambda_3\|(u_n-z_n)-(x^*-z^*)\|\|A_3u_n-A_3x^*\| \\
&\qquad + (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.39}
$$

This together with (C1), (C2), (21), and (3.34), we obtain

$$
\lim_{n\to\infty}\big\|(z_n-y_n)-(z^*-y^*)\big\| = \lim_{n\to\infty}\|(u_n-z_n)-(x^*-z^*)\| = 0. \tag{3.40}
$$

Therefore

$$
\begin{aligned}
\big\|(u_n-y_n)-(x^*-y^*)\big\| &\le \big\|(z_n-y_n)-(z^*-y^*)\big\| \\
&\quad + \|(u_n-z_n)-(x^*-z^*)\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned}
\tag{3.41}
$$

From Lemma 2.3 and (2.3), it follows that

$$
\begin{aligned}
&\big\|(y_n-t_n)+(x^*-y^*)\big\|^2 \\
&\quad = \big\|(y_n-\lambda_1A_1y_n)-(y^*-\lambda_1A_1y^*) \\
&\qquad -\big[P_C(y_n-\lambda_1A_1y_n)-P_C(y^*-\lambda_1A_1y^*)\big] + \lambda_1(A_1y_n-A_1y^*)\big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) - \left[P_C(y_n - \lambda_1 A_1 y_n) - P_C(y^* - \lambda_1 A_1 y^*) \right] \right\|^2 \\
&\quad + 2\lambda_1 \langle A_1 y_n - A_1 y^*, (y_n - t_n) + (x^* - y^*) \rangle \\
&\leq \left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) \right\|^2 - \left\| P_C(y_n - \lambda_1 A_1 y_n) - P_C(y^* - \lambda_1 A_1 y^*) \right\|^2 \\
&\quad + 2\lambda_1 \left\| A_1 y_n - A_1 y^* \right\| \left\| (y_n - t_n) + (x^* - y^*) \right\| \\
&\leq \left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) \right\|^2 - \left\| TP_C(y_n - \lambda_1 A_1 y_n) - TP_C(y^* - \lambda_1 A_1 y^*) \right\|^2 \\
&\quad + 2\lambda_1 \left\| A_1 y_n - A_1 y^* \right\| \left\| (y_n - t_n) + (x^* - y^*) \right\| \\
&\leq \left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) - (Tt_n - x^*) \right\| \\
&\quad \times \left[\left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) \right\| + \| Tt_n - x^* \| \right] \\
&\quad + 2\lambda_1 \left\| A_1 y_n - A_1 y^* \right\| \left\| (y_n - t_n) + (x^* - y^*) \right\| \\
&= \left\| u_n - Tt_n + x^* - y^* - (u_n - y_n) - \lambda_1 (A_1 y_n - A_1 y^*) \right\| \\
&\qquad \times \left[\left\| (y_n - \lambda_1 A_1 y_n) - (y^* - \lambda_1 A_1 y^*) \right\| + \| Tt_n - x^* \| \right] \\
&\quad + 2\lambda_1 \left\| A_1 y_n - A_1 y^* \right\| \left\| (y_n - t_n) + (x^* - y^*) \right\|.
\end{aligned}
\tag{3.42}
$$

This together with (3.30), (3.34), and (3.41), we obtain $\|(y_n - t_n) + (x^* - y^*)\| \to 0$ as $n \to \infty$. This together with (3.30) and (3.40), we obtain that

$$
\begin{aligned}
\|Tt_n - t_n\| \leq \|Tt_n - u_n\| + \|(u_n - z_n) - (x^* - z^*)\| + \left\| (z_n - y_n) - (z^* - y^*) \right\| \\
+ \left\| (y_n - t_n) + (x^* - y^*) \right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.43}
$$

Next, we show that

$$
\limsup_{n \to \infty} \langle v - \overline{x}, x_n - \overline{x} \rangle \leq 0, \tag{3.44}
$$

where $\overline{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{Tt_n\}$ are two bounded sequences in C, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$
\limsup_{n \to \infty} \langle v - \overline{x}, Tt_n - \overline{x} \rangle = \lim_{i \to \infty} \langle v - \overline{x}, Tt_{n_i} - \overline{x} \rangle. \tag{3.45}
$$

Since $\lim_{n \to \infty} \|Tt_n - t_n\| = 0$, we obtain that $Tt_{n_i} \rightharpoonup z$ as $i \to \infty$.

Next, we show that $z \in \Omega$.

Since $t_{n_i} \rightharpoonup z$ and $\|Tt_n - t_n\| \to 0$, we obtain by Lemma 2.6 that $z \in F(T)$.

From (3.43) and (3.24), we obtain

$$
\|t_n - x_n\| \leq \|Tt_n - t_n\| + \|Tt_n - x_n\| \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.46}
$$

Furthermore, by Lemma 3.1, we have $G : C \to C$ is nonexpansive. Then, we have

$$\begin{aligned}
\|t_n - G(t_n)\| &= \left\| P_C\left(y_n - \lambda_1 A_1 y_n\right) - G(t_n)\right\| \\
&= \|P_C[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)] - G(t_n)\| \\
&= \|P_C[P_C(P_C(u_n - \lambda_3 A_3 u_n) - \lambda_2 A_2 P_C(u_n - \lambda_3 A_3 u_n)) \\
&\qquad - \lambda_1 A_1 P_C(P_C(u_n - \lambda_3 A_3 u_n) - \lambda_2 A_2 P_C(u_n - \lambda_3 A_3 u_n))] - G(t_n)\| \\
&= \|G(u_n) - G(t_n)\| \le \|u_n - t_n\| \\
&\le \|u_n - x_n\| + \|x_n - t_n\|,
\end{aligned} \tag{3.47}$$

hence $\lim_{n\to\infty}\|t_n - G(t_n)\| = 0$. Again by Lemma 2.6, we have $z \in G\,\mathrm{VI}(C, A_1, A_2, A_3)$.

Since $t_{n_i} \rightharpoonup z$ and $\|x_n - t_n\| \to 0$, we obtain that $x_{n_i} \rightharpoonup z$. From $\|u_n - x_n\| \to 0$, we also obtain that $u_{n_i} \rightharpoonup z$. By using the same argument as that in the proof of [19, Theorem 3.1, pp. 1825], we can show that $z \in \mathrm{MEP}(F, \varphi)$. Therefore $z \in \Omega$.

On the other hand, it follows from (2.4), (3.24), and $Tt_{n_i} \rightharpoonup z$ as $i \to \infty$ that

$$\begin{aligned}
\limsup_{n\to\infty}\langle v - \overline{x}, x_n - \overline{x}\rangle = \limsup_{n\to\infty}\langle v - \overline{x}, Tt_n - \overline{x}\rangle &= \lim_{i\to\infty}\langle v - \overline{x}, Tt_{n_i} - \overline{x}\rangle \\
&= \langle v - \overline{x}, z - \overline{x}\rangle \le 0.
\end{aligned} \tag{3.48}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - \overline{x}\|^2 &= \langle a_n v + b_n x_n + (1 - a_n - b_n)Tt_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\
&= a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + b_n\langle x_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\
&\quad + (1 - a_n - b_n)\langle Tt_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\
&\le a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2}b_n\left(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right) \\
&\quad + \frac{1}{2}(1 - a_n - b_n)\left(\|t_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right) \\
&\le a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2}b_n\left(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right) \\
&\quad + \frac{1}{2}(1 - a_n - b_n)\left(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right) \\
&= a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2}(1 - a_n)\left(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right),
\end{aligned} \tag{3.49}$$

which implies that

$$\|x_{n+1} - \overline{x}\|^2 \le (1 - a_n)\|x_n - \overline{x}\|^2 + 2a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle. \tag{3.50}$$

This together with (C1) and (3.48), we have by Lemma 2.4 that $\{x_n\}$ converges strongly to $\overline{x}$. This completes the proof. □

Theorem 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $A_i : C \to H$ be L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive for $i = 1,2,3$ and $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i) \bigcap \mathrm{GVI}(C, A_1, A_2, A_3) \bigcap \mathrm{MEP}(F,\varphi) \neq \emptyset$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1,2,\ldots,N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$, $\alpha_1^N \in (0,1]$ and $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by*

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_3 A_3 u_n), \\ y_n = P_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1. \end{gathered} \tag{3.51}$$

If $0 < \lambda_i < 2(d_i - c_i L_i^2)/L_i^2$, for $i = 1,2,3$ and $\{r_n\}$, $\{a_n\}$, $\{b_n\}$ are as in Theorem 3.3. Then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.8), *where $\overline{y} = P_C(\overline{z} - \lambda_2 A_2 \overline{z})$ and $\overline{z} = P_C(\overline{x} - \lambda_3 A_3 \overline{x})$.*

Proof. By Lemma 2.7, we obtain that S is nonexpansive and $F(S) = \bigcap_{i=1}^N F(T_i)$. Hence, the result is obtained directly from Theorem 3.3. □

If $\varphi = 0$ in Theorem 3.3, then, we obtain the following result.

Corollary 3.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $A_i : C \to H$ be L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive for $i = 1,2,3$ and T be a nonexpansive self-mapping of C such that $\Omega = F(T) \bigcap \mathrm{GVI}(C, A_1, A_2, A_3) \bigcap \mathrm{EP}(F) \neq \emptyset$.*

Let $v, x_1 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(x_n - \lambda_3 A_3 x_n), \\ y_n = P_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) T P_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1. \end{gathered} \tag{3.52}$$

If $0 < \lambda_i < 2(d_i - c_i L_i^2)/L_i^2$, for $i = 1,2,3$ and the sequences $\{a_n\}$, $\{b_n\}$, and $\{r_n\}$ are as in Theorem 3.3, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.8), *where $\overline{y} = P_C(\overline{z} - \lambda_2 A_2 \overline{z})$ and $\overline{z} = P_C(\overline{x} - \lambda_3 A_3 \overline{x})$.*

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