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*Research Article*

# Unsteady Rotational Motion of a Slip Spherical Particle in a Viscous Fluid

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The unsteady rotational motion of a slip spherical particle with a nonuniform angular velocity in an incompressible viscous fluid flow is discussed. The technique of Laplace transform is used. The slip boundary condition is applied at the surface of the sphere. A general formula for the resultant torque acting on the surface of the sphere is deduced. Special fluid flows are considered and their results are represented graphically.

## 1. Introduction

The unsteady incompressible viscous fluid flows have been discussed extensively in the literature both analytically and numerically and have been reported in numerous studies. Basset [1] gave the force exerted on a sphere in an arbitrary axisymmetric time-dependent motion by integrating the result of Stokes [2]. Mazur and Bedeaux used the Faxen theorem to obtain the force on a moving sphere when moving with time-dependent velocity in [3].

Sano [4] utilized an asymptotic expansion to solve the problem of the low-Reynolds number unsteady fluid flow past a sphere when a constant rectilinear velocity is suddenly imparted to the sphere. Maxey and Riley [5] discussed the forces acting on a small rigid sphere in a non-uniform viscous fluid flow. In [6], Lawrence and Weinbaum used the linearized Navier-Stokes equations to obtain the force acting on an arbitrary axisymmetric body in oscillatory motion. Lawrence and Weinbaum [7] presented a more general analysis of the unsteady Stokes equations for the axisymmetric flow past a spheroidal body. Mei et al. examined in [8] the dependence of the unsteady drag on the frequency of the fluctuations at various Reynolds numbers. Lovalenti and Brady calculated the hydrodynamic force acting on a rigid spherical particle translating with arbitrary time-dependent motion in a time-dependent flowing fluid for small but finite values of the Reynolds number based on the particle's slip velocity relative to the uniform flow in [9]. Feng and Joseph used in [10]

a numerical simulation to examine the validity of the quasisteady method for solving unsteady viscous fluid flows. Ranger [11] studied time-dependent decay of the motion of a sphere translating and rotating in a viscous liquid. In [12], Coimbra and Rangel used a fractional-differential operator to obtain the general solution of the particle momentum equation for unsteady Stokes flows analytically. Asmolov [13] discussed the flow induced by a sphere undergoing unsteady motion in a Newtonian fluid at small Reynolds number. Venkatalaxmi et al. discussed a simple method of solution for the problem of an arbitrary unsteady Stokes flow in the presence of a sphere with no-slip condition in [14]. Avudainayagam and Ramakrisnna [15] used Saffman's technique of integral identities to obtain the total couple acting on a sphere rotating in a viscous fluid with classical no-slip boundary condition.

The slip condition introduced by Navier [16] assumes that the surface friction force determined by the tangential velocity of fluid relative to the boundary at a point on its surface is proportional to the tangential stress acting at that point. The constant of proportionality between the fluid and the boundary is called the coefficient of sliding friction. It depends only on the nature of the fluid and the surface of the boundary [17]. While slip conditions are of actual interest for gases, recently it has been found that these conditions are also of interest for liquids as well [18–24]. Albano et al. [25] have obtained an expression to the drag force exerted on a sphere moving with a time-dependent velocity through an incompressible fluid in an unsteady inhomogeneous flow for the case of arbitrary slip on the surface of the sphere. Datta and Deo [26] investigated the forces experienced by randomly and homogeneously distributed parallel circular cylinders or spheres in uniform viscous flow with slip boundary condition under Stokes approximation using particle-in-cell model technique. Senchenko and Keh [27] have obtained analytical approximations for the resistance relations for a rigid, slightly deformed slip sphere in an unbounded Stokesian steady flow. The problem of slow steady rotation of two axisymmetric particles (separated by a certain distance) about their common axis of symmetry in an infinite viscous fluid with slip boundary conditions at their surfaces has been studied numerically by Tekasakul et al. [28]. Keh and Lee [29] discussed the quasisteady creeping flow caused by a spherical fluid or solid particle with a slip surface translating in a viscous fluid within a spherical cavity along the line connecting their centers in the limit of small Reynolds number.

In the present work, we discuss the unsteady slow motion of a rigid sphere with interfacial slip condition in a viscous fluid flow. A simple formula for the torque acting on the surface of the sphere is deduced. The case of steady flow and the case of classical no-slip condition are obtained as special cases of this work.

## 2. Formulation and Solution of the Problem

Assume that a rigid sphere of radius "$a$" rotates suddenly with a nonconstant angular velocity "$\Omega(t)$" about an instantaneous axis of revolution in a viscous fluid which is otherwise at rest. For the creeping motion of a viscous fluid, the equation of motion takes the following form:

$$\rho \frac{\partial \mathbf{q}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{q}, \tag{2.1}$$

where $\rho$ is the density of the fluid, $\mathbf{q}$ represents the velocity vector of the fluid, $p$ is the pressure, and $\mu$ denotes the dynamical viscosity coefficient. Moreover, $\nabla$ is the usual Hamilton "nabla" operator.

The stress tensor can also be represented as follows:

$$t_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \quad \text{with } e_{ij} = \frac{1}{2}\left(q_{i,j} + q_{j,i}\right). \tag{2.2}$$

Working with the spherical polar coordinates $(r, \theta, \phi)$, with origin at the centre of the sphere, the only non vanishing velocity component is the azimuthal $\phi$-component denoted by $q_\phi$. Introducing the Laplace transform (denoted by an over bar) defined by

$$\overline{f}(r, \theta, s) = \int_0^\infty f(r, \theta, t) e^{-st} dt, \tag{2.3}$$

the equation of motion then reduces to the following form:

$$\left(E^2 - \ell^2\right)\left\{r \sin\theta \, \overline{q}_\phi(r, \theta, s)\right\} = 0, \tag{2.4}$$

where $\ell^2 = s/\nu$ and $\nu$ represents the kinematical viscosity coefficient. Moreover the operator "$E$" is defined by

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{2.5}$$

The initial condition is taking the following form:

$$q_\phi(r, \theta, 0^+) = \begin{cases} \Omega(0) a \sin\theta, & r = a, \\ 0, & r > a. \end{cases} \tag{2.6}$$

Moreover, the following slip boundary condition is applied on the surface of the rigid sphere:

$$\beta\left\{q_\phi(a, \theta, t) - \Omega(t) a \sin\theta\right\} = t_{r\phi}\big|_{r=a}, \quad t > 0, \tag{2.7}$$

where $\beta$ is the slip parameter. It is a measure of the degree of tangential slip existing between the fluid and the boundary. This coefficient depends only on the nature of the fluid and solid surface. The classical case of no-slip can be recovered as a special case when $\beta \to \infty$.

Applying the boundary conditions (2.6)-(2.7), in the Laplace transform domain, to the solution of the differential equation (2.4), we obtain the velocity $\overline{q}_\phi$ in the following form:

$$\overline{q}_\phi = \left(\frac{\overline{\Omega}(s)\alpha\ell a^3}{\ell^2 a^2 + (\ell a + 1)(\alpha + 3)}\right)\left(\frac{1}{r} + \frac{1}{\ell r^2}\right) e^{-\ell(r-a)} \sin\theta, \tag{2.8}$$

where

$$\alpha = \frac{a\beta}{\mu}. \tag{2.9}$$

This later dimensionless parameter $\alpha$ is known as Navier number. Moreover, the ratio of $\mu/\beta$ is called the slip length.

Also, the shear stress has the following form:

$$\bar{t}_{r\phi} = -\left(\frac{\overline{\Omega}(s)\alpha a^3\mu}{\ell^2 a^2 + (\ell a + 1)(\alpha + 3)}\right)\left(\frac{\ell^2}{r} + \frac{3\ell}{r^2} + \frac{3}{r^3}\right)e^{-\ell(r-a)}\sin\theta. \tag{2.10}$$

The torque acting on the surface of the rigid sphere can be obtained, in the Laplace transform domain, by

$$\overline{T}_z = 2\pi a^3 \int_0^{\pi} \bar{t}_{r\phi}\Big|_{r=a} \sin^2\theta \, d\theta. \tag{2.11}$$

This can be simplified to the following form:

$$\overline{T}_z = \frac{-8\pi a^3\mu\alpha}{(\alpha+3)}\left(\overline{\Omega}(s) + \frac{\alpha s\overline{\Omega}(s)}{3\left(s + ((\alpha+3)/a)\sqrt{s\nu} + (\alpha+3)\nu/a^2\right)}\right). \tag{2.12}$$

When $\alpha \to \infty$, we recover the results obtained for the case of no-slip given by [15]

$$\overline{T}_z = -8\pi a^3\mu\left(\overline{\Omega}(s) + \frac{a^2 s\overline{\Omega}(s)}{3\left[a\sqrt{s\nu} + \nu\right]}\right). \tag{2.13}$$

Taking the inverse Laplace transform of (2.12), we arrive at

$$T_z(t) = \frac{-8\pi a^3\mu\alpha}{(\alpha+3)}\left(\Omega(t) + \frac{\alpha}{3}L^{-1}\left\{\left(s\overline{\Omega}(s)\right)\left(s + \frac{(\alpha+3)}{a}\sqrt{s\nu} + \frac{(\alpha+3)\nu}{a^2}\right)^{-1}\right\}\right), \tag{2.14}$$

where $L^{-1}$ represents the inverse Laplace operator.

By using the convolution theorem together with the complex Laplace inversion formula with the aid of contour integration, we can obtain the inverse Laplace transform of the formula (2.14) in the following form:

$$T_z(t) = \frac{-8\pi a^3\mu\alpha}{(\alpha+3)}\left(\Omega(t) + \int_0^t \left(\frac{d\Omega}{d\tau} + \Omega(0)\right)F(t-\tau)d\tau\right), \tag{2.15}$$

where

$$F(t) = \frac{a^3}{3\pi\nu\sqrt{\nu}} \int_0^\infty \frac{\sqrt{x}e^{-xt}}{(1 - xa^2/\nu(\alpha+3))^2 + xa^2/\nu} dx. \tag{2.16}$$

The general formula (2.15) can be employed to obtain the total torque acting on a sphere rotating in a Stokes viscous fluid flow with a nonuniform angular velocity.

The case of no-slip can be recovered in the limiting case of $\alpha \to \infty$ to take the following form:

$$T_z(t) = -8\pi a^3 \mu \left( \Omega(t) + \frac{a^3}{3\pi\nu\sqrt{\nu}} \int_0^t \left( \frac{d\Omega(\tau)}{d\tau} + \Omega(0) \right) \int_0^\infty \frac{\sqrt{x}e^{-x(t-\tau)}}{1 + xa^2/\nu} dx\, d\tau \right). \tag{2.17}$$

To recover the result of Basset [1] for the torque acting on a rigid sphere, of radius $a$, rotating steadily in a viscous fluid with a slip-flow boundary condition, we put $\Omega(t) = \Omega_o$(constant) and $t \to \infty$ into (2.15) to get

$$T_z(t) = \frac{-8\pi a^3 \mu \alpha \Omega_o}{(\alpha+3)}. \tag{2.18}$$

### 2.1. Some Special Flows

In this section we will employ the obtained general formula (2.15) to some non-uniform viscous fluid flows.

*Case 1* (Damping oscillation). Here we assume that the rigid sphere starts to move with the angular velocity:

$$\Omega(t) = \Omega_o e^{-\omega t} \sin(\omega t). \tag{2.19}$$

In this case, the torque acting on the sphere is given by

$$T_z(t) = \frac{-8\pi a^3 \mu \alpha \Omega_o}{(\alpha+3)} \left\{ e^{-wt} \sin(\omega t) + \omega \int_0^t e^{-\omega\tau} (\cos(\omega\tau) - \sin(\omega\tau)) F(t-\tau) d\tau \right\}, \tag{2.20}$$

where $\Omega_o$ is the characteristic angular velocity and $\omega$ is the oscillation angular velocity.

*Case 2* (Accelerating velocity). In the second special case we assume that the sphere starts to rotate with the angular velocity of the following form:

$$\Omega(t) = \Omega_o \{\omega t + c\}, \tag{2.21}$$

where $c$ is an arbitrary constant.

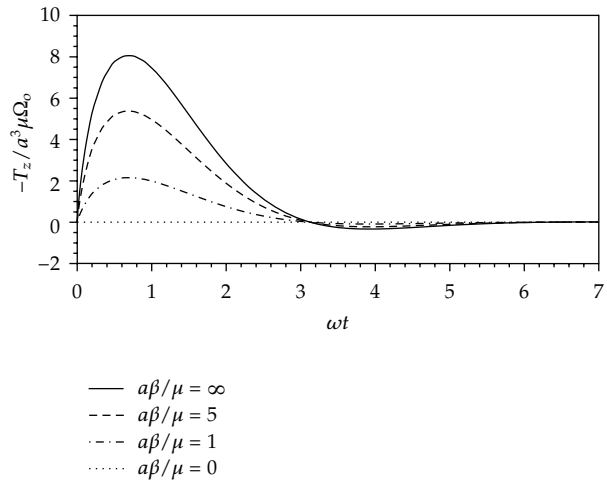


**Figure 1:** Torque on the sphere for Case 1.

The torque in this case takes the following form:

$$T_z(t) = \frac{-8\pi a^3 \mu\alpha\Omega_o}{(\alpha+3)}\left\{(\omega t + c) + \frac{\omega a^3}{3\,\pi\nu\sqrt{\nu}}\int_0^t\int_0^\infty \frac{\sqrt{x}e^{-x(t-\tau)}}{(1 - xa^2/\nu\ (\alpha+3))^2 + xa^2/\nu}\,dx\,d\tau\right\}. \tag{2.22}$$

*Case 3* (Impulsive rotation). Assume that the sphere rotates impulsively by the angular velocity:

$$\Omega(t) = \Omega_o H(t), \tag{2.23}$$

where $H(t)$ is the Heaviside step function.

The total Torque is found to be

$$T_z(t) = \frac{-8\pi a^3 \mu\alpha\Omega_o}{(\alpha+3)}\left\{H(t) + \frac{a^3}{3\pi\nu\sqrt{\nu}}\int_0^\infty \frac{\sqrt{x}e^{-xt}}{(1 - xa^2/\nu(\alpha+3))^2 + xa^2/\nu}\,dx\right\}. \tag{2.24}$$

## 3. Conclusion

In this paper, we have applied the slip condition and obtained a general formula that can be used to evaluate the torque on a sphere rotating with a non-uniform angular velocity in a viscous fluid flow. The classical no-slip case can be recovered as a special case of our work in the limiting case of $\alpha \to \infty$. Moreover, the couple acting on a sphere moving steadily in a viscous fluid flow can be obtained from Case 3 of this paper when $t \to \infty$. From Figure 1, we conclude that the torque in the case of damping oscillation goes to zero after short time. Also, the torque in the case of accelerating velocity represented in Figure 2 is increasing linearly with the time. Figure 3 shows that after very short time the torque tends to the steady state as expected. From Figures 1, 2, and 3 we note that the torque is identically zero in the case of

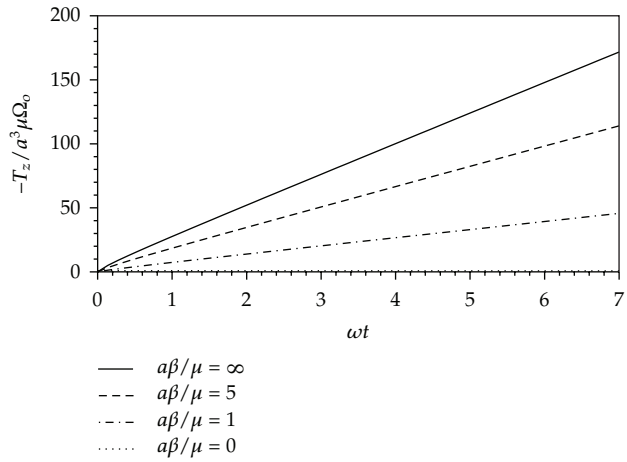


**Figure 2:** Torque on the sphere for Case 2.

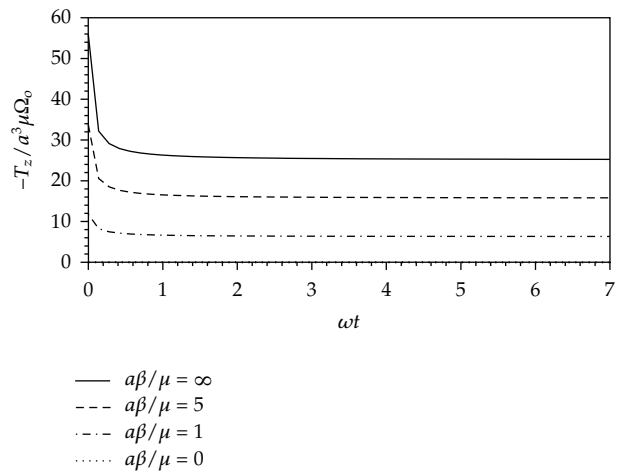


**Figure 3:** Torque on the sphere for Case 3.

perfect slip, that is, when $\alpha = 0$. Also, we observe that the values of the total couple increase with the increase of the slip parameter.

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