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Research Article

Median Sets and Median Number of a Graph

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A *profile* is a finite sequence of vertices of a graph. The set of all vertices of the graph which minimises the sum of the distances to the vertices of the profile is the *median* of the profile. Any subset of the vertex set such that it is the *median* of some profile is called a *median set*. The number of *median sets* of a graph is defined to be the *median number* of the graph. In this paper, we identify the median sets of various classes of graphs such as $K_p - e$, $K_{p,q}$ for $P > 2$, and wheel graph and so forth. The median numbers of these graphs and hypercubes are found out, and an upper bound for the median number of even cycles is established. We also express the median number of a product graph in terms of the median number of their factors.

1. Introduction

We consider only nonempty finite simple undirected connected graphs. For the graph G, $V(G)$, and $E(G)$ denote its vertex set and edge set, respectively. When the underlying graph is obvious, we will use V and E for $V(G)$ and $E(G)$, respectively. A finite sequence of vertices $\pi = (v_1, \ldots, v_k) \in V^k$ is called a *profile*. For the profile $\pi = (v_1, \ldots v_k)$ and $x \in V$, the *remoteness* $D(x, \pi)$ is $\sum_{1 \leq i \leq n} d(x, v_i)$ [1]. The set of all vertices x for which $D(x, \pi)$ is minimum is the *median* of π in G and is denoted by $M(\pi)$. A set S such that $S = M(\pi)$ for some profile π is called a *Median set* of G. The *Interval* $I(u, v)$ between vertices u and v of G consists of all vertices which lie in some shortest path between u and v. The number of intervals of a graph G is denoted by in (G). The *Hypercube* Q_n is the graph with vertex set $\{0, 1\}^n$, two vertices being adjacent if they differ exactly in one coordinate. A *Subcube* of the hypercube Q_n is an induced subgraph of Q_n, isomorphic to Q_m for some $m \leq n$. A graph G is a *Median graph* if, for every $(x, y, z) \in V^3$, $I(x, y) \cap I(y, z) \cap I(x, z)$ contains a unique vertex, see Avann [2]. Trees, hypercubes, and grid graphs are all median graphs. For further references concerning median graphs, see Mulder and Schrijver [3], Mulder [4], and Bandelt and Hedlíková [5]. The graph on p vertices formed by joining all the vertices of a $(p - 1)$-*cycle* to a vertex is a *wheel graph* and is denoted by W_p. The vertex which is joined to all the vertices of the $(p-1)$-*cycle* is the *universal vertex*. The *Cartesian product* $G\square H$ of two graphs G and H has vertex set $V(G)\times V(H)$, two vertices (u,v) and (x,y) being adjacent if either $u=x$ and $vy\in E(H)$ or $ux\in E(G)$ and $v=y$. The *eccentricity* $e(u)$ of a vertex u is $\max_{v\in V(G)} d(u,v)$. A vertex v is an *eccentric vertex* of u if $e(u)=d(u,v)$. Graph products form a well-studied area in graph theory, see [6, 7]. The problem of finding the median of a profile is very much significant in location theory, particularly in efficiency-oriented model [8], as it corresponds to finding the most desirable service point for a group of customers when the minimisation of the total cost is a primary objective. Here we use profiles of vertices instead of a set of vertices which gives us the liberty to repeat a vertex any number of times. This makes the problem more relevant in location theory as it enables us to assign different weights to different customers. The problem of finding the median of a profile has been thoroughly investigated by many authors; see [9–14]. The objective of this paper is to identify median sets of some classes of graphs and to enumerate them.

2. Median Number

On the set of all profiles on V, $V^*=\bigcup_{k\in Z} V^k$, we define a relation M by $\pi_1 M \pi_2$ if $M(\pi_1)=M(\pi_2)$. This is an equivalence relation. This equivalence relation gives a partition of the whole set of profiles. The number of equivalence classes in this partition is defined as the *Median number* of graph G and is denoted by $mn(G)$. That is, it is the number of distinct median sets in G.

Proposition 2.1. *For any graph $G=(V,E)$ on p vertices, $in(G)\le mn(G)\le 2^p-1$.*

Proof. The upper bound is obvious as it is the number of nonempty subsets of the vertex set. For every $v\in V$, v is a median set of the profile (v). For every $u,v\in V$, the set $I(u,v)$ is the median set of the profile (u,v). Therefore $in(G)\le mn(G)\le 2^p-1$. □

Proposition 2.2. $mn(K_p)=2^p-1$, *where K_p is the complete graph on p vertices.*

Proof. In K_p, each nonempty subset of the vertex set is a median set, namely, of the profile formed by taking all the elements of the set exactly once. Therefore the number of distinct median sets is the number of nonempty subsets of V which is 2^p-1. □

Proposition 2.3. *If e is an edge of K_p, $p\ge 3$, $mn(K_p-e)=2^p-2^{p-2}$.*

Proof. Let $e=(u,v)\in E$. For every vertex set S such that $\{u,v\}\not\subseteq S$, there exists a profile which has S as its median set, namely, the profile formed by taking the vertices of S exactly once. Let π be a profile which does not simultaneously contain u and v. Then $M(\pi)$ is a subset of the set of vertices corresponding to the profile π and hence does not contain u and v. Now, let π be profile which contains both u and v. Then if u or v is repeated more than the other in the profile, then $D(u,\pi)\ne D(v,\pi)$ and so they cannot appear together in the $M(\pi)$. Assume that π contains both u and v where both are repeated the same number of times. Let the profile be $(x_1,\dots,x_k,\underbrace{u,\dots,u}_{m\text{ times}},\underbrace{v,\dots,v}_{m\text{ times}})$, $m\ge 1$. For x_i, $1\le i\le k$, $D(x_i,\pi)\le k-1+2m$. Also, $D(u,\pi)=k+2m$ and $D(v,\pi)=k+2m$. Therefore $M(\pi)$ does not contains both u and v. Now the profile (u,v) has V as its median set. Hence V is the only median set which contains both

u and v. Therefore the class of all median sets of the graph consists of V and all subsets of V which do not simultaneously contain u and v. Hence,

$$\begin{aligned} mn(K_p - e) &= \binom{p}{1} + \binom{p}{2} - \binom{p-2}{0} + \binom{p}{3} - \binom{p-2}{1} \\ &\quad + \cdots + \binom{p}{p-1} - \binom{p-2}{p-3} + \binom{p}{p} \\ &= 2^p - 1 - \left(2^{p-2} - 1\right) \\ &= 2^p - 2^{p-2}. \end{aligned} \tag{2.1}$$

□

Proposition 2.4 (Bandelt and Barthélémy [10]). *Let $G = (V, E)$ be a median graph. For any profile π in G the median set is an interval $I(u, v)$ in G.*

Proposition 2.5. *The median number of a tree T on p vertices is $\binom{p}{2} + p$.*

Proof. Since T is a median graph, by the above proposition all median sets are intervals. As observed in the proof of Proposition 2.1, all intervals are median sets. Therefore, class of median sets of T is precisely the class of intervals of T which is the class of all paths in T. Hence the median number is the number of distinct paths in T which is $\binom{p}{2} + p$. □

Proposition 2.6 (Imrich and Klavžar [6]). *Let Q_r be a hypercube. Then, for any pair of vertices $u, v \in Q_r$ the subgraph induced by the interval $I(u, v)$ is a hypercube of dimension $d(u, v)$.*

Theorem 2.7. *For the hypercube Q_r, $mn(Q_r) = 3^r$*

Proof. Since Q_r is a median graph, by Propositions 2.4 and 2.6, every median set of Q_r is a subcube. Also in any graph G, $I(u, v)$ is the median set of the profile (u, v), where u, $v \in V(G)$. Thus in a hypercube every subcube is a median set. Therefore, the median sets of Q_r are precisely the induced subcubes. So the Median number of Q_r is the number of subcubes of Q_r. Every vertex of Q_r contains r coordinates where each coordinate is either 0 or 1. Keeping k co-ordinates fixed and varying 0 and 1 over the other $r - k$ positions, we get a subcube of dimension $r - k$. By varying 0's and 1's over these k positions we get 2^k such subcubes. The k positions, to be fixed can be chosen in $\binom{r}{k}$ ways. So the total number of subcubes of dimension $r - k$ is $2^k \times \binom{r}{k}$. Therefore the total number of subcubes of Q_r is $\sum_{0 \le k \le r} \binom{r}{k} \times 2^k = 3^r$. □

Theorem 2.8. *For the wheel graph W_p, $p \ge 6$, $mn(W_p) = (p^2 + 3p - 2)/2$.*

Proof. Let $\{v_1, v_2, \dots, v_{p-1}, v_p\}$ be the vertex set of W_p with v_p as the universal vertex, and let C_{p-1} be the cycle $v_1, v_2, \dots, v_{p-1}, v_1$. Each singleton set $\{v_i\}$, $1 \le i \le p - 1$, is a median set. The sets $\{v_i, v_j\}$, where v_i and v_j are adjacent, are also median sets. The profile $(v_i, v_{i+_{(p-1)}1}, v_p)$, $1 \le i \le p - 1$ ($+_{p-1}$ means addition modulo $p - 1$) has $\{v_i, v_{i+_{(p-1)}1}, v_p\}$ as median set. The set $\{v_i, v_{i+_{(p-1)}1}, v_{i+_{p-1}2}, v_p\}$ is the median set of the profile $(v_i, v_{i+_{(p-1)}2})$. Let $\pi = (x_1, x_2, \dots, x_k)$ be a profile of W_p which contains the universal vertex v_p. Then since π contains the vertex v_p, $D(v_p, \pi) \le k - 1$. If some v_i, $1 \le i \le p - 1$, belongs to $M(\pi)$, then $D(v_i, \pi) \le k - 1$ and this implies $x_j = v_i$ at least for some j. Also, the number of x_j's with $d(v_i, x_j) = 2$ is less than the number of repetitions of v_i in π. Let v_k be such that $d(v_k, v_i) = 2$. Then v_k belongs to $M(\pi)$

that implies number of repetitions of v_k is greater than the number of repetitions of v_i in the profile π. But these two statements are contradictory. Thus for a profile which contains the universal vertex the median set cannot contain two vertices which are at distance 2. Hence the only possible median sets for such a profile are

(i) sets of type $\{v_i, v_{i+_{(p-1)}1}\}$,

(ii) sets of type $\{v_i, v_p\}$,

(iii) sets of type $\{v_i, v_{i+_{(p-1)}1}, v_p\}$.

Now, let $\pi = (x_1, \ldots, x_k)$ be a profile which does not contain v_p. Then $D(v_p, \pi) = k$. If some v_i, $1 \le i \le p-1$, belongs to $M(\pi)$, then $D(v_i, \pi) \le k$. Let v_j be such that $v_j \in M(\pi)$ and $d(v_i, v_j) = 2$. Then $D(v_j, \pi) \le k$. since $D(v_i, \pi) \le k$, the number of zeroes in $\{d(v_i, x_1), \ldots, d(v_i, x_k\} \ge$ number of twos in $\{d(v_i, x_1), \ldots, d(v_i, x_k\}$. Similarly, number of zeroes in $\{d(v_j, x_1), \ldots, d(v_j, x_k\} \ge$ number of twos in $\{d(v_j, x_1), \ldots, d(v_j, x_k\}$.

Thus, number of repetitions of v_i in π = number of repetitions of v_j in π.

Now, let $d_{C_{p-1}}(v_i, v_j) = 2$. Without loss of generality, we may assume that $j = i+_{(p-1)}2$. If some vertex other than v_i, $v_{i+_{(p-1)}1}$, $v_{i+_{(p-1)}2}$ belongs to π, then $D(v_i, \pi) = D(v_j, \pi) > D(v_p, \pi)$. If $v_{i+_{(p-1)}1} \in \pi$, then $D(v_{i+_{(p-1)}1}, \pi) < D(v_i, \pi)$. Therefore π can only be $(v_i, \ldots, v_i, v_j, \ldots, v_j)$, where v_i and v_j are repeated the same number of times. Since $j = i+_{(p-1)}2$, we have $D(v_i, \pi) = D(v_j, \pi) = D(v_p, \pi) = D(v_{i+_{(p-1)}1}, \pi)$ and for all other $x \in V$, $D(x, \pi) > k$. Hence $M(\pi) = \{v_i, v_{i+_{(p-1)}1}, v_{i+_{(p-1)}2}, v_p\}$. If $d_{C_{p-1}}(v_i, v_j) \neq 2$ then some vertex other than v_i and v_j that belong to π will contradict the fact that v_i and v_j belong to $M(\pi)$. Therefore, in this case also $\pi = (v_i, \ldots, v_i, v_j, \ldots, v_j)$, where v_i and v_j are repeated the same number of times. Here $D(v_i, \pi) = k$, $D(v_j, \pi) = k$, $D(v_p, \pi) = k$ and for all other $x \in V$, $D(x, \pi) > k$. In other words, $M(\pi) = \{v_i, v_j, v_p\}$.

Hence the only median sets are

(1) $\{v_i\}$, $1 \le i \le p$,

(2) $\{v_i, v_{i+_{(p-1)}1}\}$, $1 \le i \le p-1$,

(3) $\{v_i, v_p\}$, $1 \le i \le p-1$,

(4) $\{v_i, v_p, v_{i+_{(p-1)}1}\}, 1 \le i \le p-1$,

(5) $\{v_i, v_j, v_p\} 1 \le i, j \le p-1, (i-j) \bmod (p-1) \ge 3$ (since $p \ge 6$, such vertices do exist),

(6) $\{v_i, v_{i+_{(p-1)}1}, v_{i+_{(p-1)}2}, v_p\}$, $1 \le i \le p-1$.

Thus, $mn(W_p) = p + p - 1 + p - 1 + p - 1 + ((p-1)(p-6))/2 + p - 1 = (p^2 + 3p - 2)/2$. □

Theorem 2.9. *$mn(K_{p,q}) = 2^{p+q} - 1$, where $K_{p,q}$ is the complete bipartite graph with $p \le q$, $p > 2$.*

Proof. Let (X, Y) be a bipartition of $K_{p,q}$ with $|X| = p$ and $|Y| = q$. Let $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$. Let A be a k-element subset of X with $k \le p$. Without loss of generality, we may assume that $A = \{x_1, \ldots, x_k\}$.

If $k < q$, take $\pi = (x_1, \ldots, x_k, y_1, \ldots, y_q)$. For each x_i, $1 \le i \le k$, $D(x_i, \pi) = 2(k-1) + q$. For each x_i, $k+1 \le i \le p$, $D(x_i, \pi) = 2k + q$. For each y_i, $1 \le i \le q$, $D(y_i, \pi) = 2(q-1) + k$. Therefore, $A = \{x_1, \ldots, x_k\} = M(\pi)$.

If $k = q$, then $\pi = (y_1, \ldots, y_q)$ has median set $A = \{x_1, \ldots, x_k\}$. Therefore, every subset of X is a median set.

Now, let $B \subseteq Y$ with $B = \{y_1, \ldots, y_k\}$.

If $k < p$ then as in the previous case $\pi = (x_1, \ldots, x_p, y_1, \ldots, y_k)$ has median set B.

Now, let $k \geq p$ and let π be the profile $(x_1, \ldots x_1, \ldots, x_p, \ldots x_p, y_1, \ldots, y_k)$, where each x_i is repeated the same number of times, (say) r.

For each y_i, $1 \leq i \leq k$, $D(y_i, \pi) = 2(k-1)+pr$, for each y_i, $k+1 \leq i \leq q$, $D(y_i, \pi) = 2k+pr$, and for each x_i, $1 \leq i \leq q$, $D(x_i, \pi) = 2r(p-1) + k$. Moreover, $2(k-1) + pr < 2r(p-1) + k \Leftrightarrow k-2 < (p-2)r \Leftrightarrow r > ((k-2)/(p-2))$ $(p > 2)$. That is, if each x_i is repeated r times where $r > (k-2)/(p-2)$, then $M(\pi) = B$.

Now, let $C = \{x_1, \ldots, x_k, y_1, \ldots, y_r\}$, $1 \leq k \leq p$, $1 \leq r \leq q$.

Take $\pi = (x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k, y_1, \ldots, y_1, \ldots, y_r, \ldots, y_r)$, where each x_i is repeated s_x times and y_i is repeated s_y times.

For each x_i, $1 \leq i \leq k$, $D(x_i, \pi) = 2(k-1)s_x + rs_y$, for each y_i, $1 \leq i \leq r$, $D(y_i, \pi) = 2(r-1)s_y + ks_x$, for each x_i, $k+1 \leq i \leq p$, $D(x_i, \pi) = 2ks_x + rs_y$, and for each y_i, $r+1 \leq i \leq q$, $D(y_i, \pi) = 2rs_y + ks_x$. Any x_i, $k+1 \leq i \leq p$ or y_i, $r+1 \leq i \leq q$ cannot be in $M(\pi)$.

Now $2(k-1)s_x + rs_y = 2(r-1)s_y + ks_x \Leftrightarrow (k-2)s_x = (r-2)s_y$. Hence for any s_x and s_y such that $(k-2)s_x = (r-2)s_y$, the profile $(x_1, \ldots x_1, \ldots, x_k, \ldots, x_k, y_1, \ldots, y_1, \ldots, y_r, \ldots, y_r)$, where each x_i is repeated s_x times and y_i is repeated s_y times, has C as its median. Therefore, every nonempty subset of $X \cup Y$ is a median set. Hence, $mn(K_{p,q}) = 2^{p+q} - 1$. □

Lemma 2.10. *Let π_1 and π_2 be profiles in the graphs G_1 and G_2, respectively. If $M(\pi_1) = M_1$ and $M(\pi_2) = M_2$, then $\pi_1 \times \pi_2$ is a profile in the graph $G_1 \Box G_2$ with $M(\pi_1 \times \pi_2) = M_1 \times M_2$.*

Proof. Let $\pi_1 = (u_1, u_2, \ldots, u_m)$, $\pi_2 = (v_1, v_2, \ldots, v_n)$, and $M = M(\pi_1 \times \pi_2)$.

Consider $\pi_1 \times \pi_2 = ((u_1, v_1), \ldots, (u_1, v_n), \ldots, (u_m, v_1), \ldots, (u_m, v_n))$.

If $(x_1, y_1), (x_2, y_2) \in V(G_1 \Box G_2)$, then $d_{G_1 \Box G_2}((x_1, y_1), (x_2, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$, see [9]. For any $(x, y) \in V(G_1 \Box G_2)$, $D((x, y), \pi_1 \times \pi_2) = n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i)$.

Let $(a, b) \in M_1 \times M_2$, that is, $a \in M_1$ and $b \in M_2$. Then,

$$\begin{aligned} \sum_{1 \leq i \leq m} d(a, u_i) &\leq \sum_{1 \leq i \leq m} d(x, u_i), \quad \forall x \in V(G_1), \\ \sum_{1 \leq i \leq n} d(b, v_i) &\leq \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall y \in V(G_2). \end{aligned} \tag{2.2}$$

Therefore,

$$n \sum_{1 \leq i \leq m} d(a, u_i) + m \sum_{21 \leq i \leq n} d(b, v_i) \leq n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall (x, y) \in V(G_1 \Box G_2). \tag{2.3}$$

Hence, $D((a, b), \pi_1 \times \pi_2) \leq D((x, y), \pi_1 \times \pi_2)$, for all $(x, y) \in V(G_1 \Box G_2)$.

Thus, $(a, b) \in M_1 \times M_2 \Rightarrow (a, b) \in M$ or $M_1 \times M_2 \subseteq M$.

Now, let $(a, b) \in M$

$$\begin{aligned} D((a, b), \pi_1 \times \pi_2) &= n \sum_{1 \leq i \leq m} d(a, u_i) + m \sum_{1 \leq i \leq n} d(b, v_i) \\ &\leq n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall (x, y) \in V(G_1 \times G_2). \end{aligned} \tag{2.4}$$

If for some $x' \in V(G_1)$,

$$\sum_{1\le i\le m} d(x',u_i) < \sum_{1\le i\le m} d(a,u_i), \text{ then}$$
$$n\sum_{1\le i\le m} d(x',u_i) + m\sum_{1\le i\le n} d(b,v_i) < n\sum_{1\le i\le m} d(a,u_i) + m\sum_{1\le i\le n} d(b,v_i). \tag{2.5}$$

This contradicts $(a,b) \in M = M(\pi_1 \times \pi_2)$.

Therefore

$$\sum_{1\le i\le m} d(a,u_i) \le \sum_{1\le i\le m} d(x,u_i), \quad \forall x \in V(G_1),$$
$$\sum_{1\le i\le m} d(b,v_i) \le \sum_{1\le i\le m} d(y,u_i), \quad \forall y \in V(G_2). \tag{2.6}$$

Hence, $a \in M_1$ and $b \in M_2$ or $(a,b) \in M_1 \times M_2$. That is, $M = M_1 \times M_2$. □

Theorem 2.11. $mn(G_1\square G_2) = mn(G_1) \times mn(G_2)$.

Proof. By the above lemma the product of median sets of G_1 and G_2 is again a median set of $G_1\square\, G_2$. Now, let M be a median set of $G_1\square G_2$, with $M = M(\pi)$, where $\pi = ((u_1,v_1),\dots,(u_k,v_k))$. Let $\pi_1 = (u_1,\dots,u_k)$, $\pi_2 = (v_1,\dots,v_k)$, $M_1 = M(\pi_1)$, and $M_2 = M(\pi_2)$. Let $(a,b) \in M$.

We have

$$\sum_{1\le i\le k} d(a,u_i) + \sum_{1\le i\le k} d(b,v_i) \le \sum_{1\le i\le k} d(x,u_i) + \sum_{1\le i\le k} d(y,v_i), \ \forall x \in V(G_1), \quad \forall y \in V(G_2),$$
$$\therefore k\sum_{1\le i\le k} d(a,u_i) + k\sum_{1\le i\le m} d(b,v_i) \le k\sum_{1\le i\le k} d(x,u_i) + k\sum_{1\le i\le m} d(y,v_i), \quad \forall (x,y) \in V(G_1\square G_2). \tag{2.7}$$

In other words, $D((a,b),\pi_1\times\pi_2) \le D((x,y),\pi_1\times\pi_2)$, for all $(x,y) \in V(G_1\square G_2)$.

$\therefore (a,b) \in M(\pi_1\times\pi_2)$ or $M \subseteq M(\pi_1\times\pi_2)$.

Let $(a,b) \in M(\pi_1\times\pi_2)$. Then $D((a,b),\pi_1\times\pi_2) \le D((x,y),\pi_1\times\pi_2)$, for all $x \in V(G_1)$, for all $y \in V(G_2)$.

That is

$$k\sum_{1\le i\le k} d(a,u_i) + k\sum_{1\le i\le k} d(b,v_i) \le k\sum_{1\le i\le k} d(x,u_i) + k\sum_{1\le i\le k} d(y,v_i),$$
$$\forall x \in V(G_1),\ \forall y \in V(G_2),$$
$$\sum_{1\le i\le k} d(a,u_i) + \sum_{1\le i\le k} d(b,v_i) \le \sum_{1\le i\le k} d(x,u_i) + \sum_{1\le i\le k} d(y,v_i), \tag{2.8}$$
$$\forall x \in V(G_1), \quad \forall y \in V(G_2),$$
$$\therefore \sum_{1\le i\le k} d((a,b),(u_i,v_i)) \le \sum_{1\le i\le k} d((x,y),(u_i,v_i)), \quad \forall (x,y) \in V(G_1\square G_2).$$

Therefore, $(a,b) \in M$ which implies $M(\pi_1 \times \pi_2) \subseteq M$ or $M = M(\pi_1 \times \pi_2) = M(\pi_1) \times M(\pi_2)$. Thus, the class of all median sets of $G_1 \square G_2$ is the same as the class of all Cartesian products of median sets of G_1 and G_2.

Hence, $mn(G_1 \square G_2) = mn(G_1) \times mn(G_2)$. □

The above theorem can be used to find the median number of various classes of graphs. The first of the following corollaries gives another technique to find out the Median number of a hypercube.

Corollary 2.12. *If $G_1, \ldots, G_k$ are k graphs, then $mn(G_1 \square \cdots \square G_k) = mn(G_1) \times \cdots \times mn(G_k)$.*

Corollary 2.13. *For the hypercube Q_r, $mn(Q_r) = 3^r$.*

Proof. Since $Q_r = \underbrace{K_2 \square \cdots \square K_2}_{r \text{ times}}$, $mn(Q_r) = \underbrace{mn(K_2) \times \cdots \times mn(K_2)}_{r \text{ times}} = \underbrace{3 \times \cdots \times 3}_{r \text{ times}} = 3^r$. □

Corollary 2.14. *If G is the Grid graph $P_r \square P_s$, $mn(G) = (\binom{r}{2} + r) \times (\binom{s}{2} + s)$.*

Corollary 2.15. *If G is the Hamming graph $K_{p_1} \square K_{p_2} \square \cdots \square K_{p_r}$, $mn(G) = (2^{p_1} - 1) \times (2^{p_2} - 1) \times \cdots \times (2^{p_r} - 1)$.*

Lemma 2.16. *The only median set of the cycle C_{2r} which contains a vertex and its eccentric vertex is V.*

Proof. Let a and b be two eccentric vertices of the cycle C_{2r} which belongs to $M(\pi)$ where $\pi = (x_1, \ldots, x_k)$. Let $D(a,\pi) = D(b,\pi) = s$. Then

$$\begin{aligned} D(a,\pi) + D(b,\pi) &= d(a,x_1) + \cdots + d(a,x_k) + d(b,x_1) + \cdots + d(b,x_k) \\ &= d(a,x_1) + d(b,x_1) + \cdots + d(a,x_k) + d(b,x_k) \\ &= \underbrace{d(a,b) + \cdots + d(a,b)}_{k \text{ times}} \\ &= kr. \end{aligned} \tag{2.9}$$

Hence $2s = kr$. Now, suppose $M(\pi) \neq V$. Then there exists an $x \in V$ such that $D(x,\pi) > s$. That is, $d(x,v_1) + \cdots + d(x,v_k) > s$. Let y be the eccentric vertex of x. $d(y,v_1) + \cdots + d(y,v_k) \geq s$. Therefore, $d(x,v_1) + \cdots + d(x,v_k) + d(y,v_1) + \cdots + d(y,v_k) > 2s$. That is, $d(x,y) + \cdots + d(x,y)$ (k times) $> 2s$ or $kr > 2s$, a contradiction. Therefore, any set distinct from V which is a median set cannot contain two eccentric vertices. Also, $M((a,b)) = V$, since a and b are diametrical and $I(a,b) = V$. Hence the only median set of C_{2r} which contains a vertex and its eccentric vertex is V. □

Theorem 2.17. *For the even cycle C_{2r}, $mn(C_{2r}) \leq 3^r$.*

Proof. Let V be the vertex set of C_{2r} with $V = \{v_1, \ldots, v_{2r}\}$. Let $A = \{S : S \subseteq V$ and S does not contain any pair of eccentric vertices$\}$. By the above lemma, the set of all median sets is a subset of $A \cup \{V\}$. Hence $mn(C_{2r}) \leq |A| + 1$. Let $B_i = \{v_i, v_{i+r}\}$, $1 \leq i \leq r$. Now A consists of all subsets of V which does not simultaneously contain both the elements from the same B_i, $1 \leq i \leq r$. The number of ways of choosing a k-element subset of V so that it belongs to A is the product of the number of ways of choosing kB_i's from the rB_i's and the number of ways of choosing one element from each of these chosen B_i's. That is, $\binom{r}{k} \times 2^k$. Therefore $|A| = \sum_{k=1}^{r} \binom{r}{k} \times 2^k$. Hence $mn(C_{2r}) \leq (\sum_{1 \leq k \leq r} \binom{r}{k} \times 2^k) + 1 = \sum_{0 \leq k \leq r} \binom{r}{k} \times 2^k = 3^r$. □

3. Conclusion

In this paper, we could identify the median set of certain classes of graphs and evaluate their median numbers. Identifying the median sets of more classes of graphs and finding their median numbers will be interesting and challenging. Another possibility is the realisation problem; given a number of finding a graph whose median number is that number.

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