Research Article

On Some L_k -Finite-Type Euclidean Hypersurfaces

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We study some L_k -finite-type Euclidean hypersurfaces. We classify L_k -1-type Euclidean hypersurfaces and L_k -null-2-type Euclidean hypersurfaces with at most two distinct principal curvatures. We also prove that, under some mild restrictions, there exists no L_k -null-3-type Euclidean hypersurface.

1. Introduction

The study of submanifolds of finite type began in the late seventies with Chen's attempts to find the best possible estimate of the total mean curvature of compact submanifolds of Euclidean space and to find a notion of "degree" for submanifolds of Euclidean space (see [1] for details). Since then the subject has had a rapid development and so many mathematicians contribute to it; see the excellent survey of Chen [2]. By definition, an isometrically immersed submanifold $x : M^n \rightarrow \mathbb{R}^{n+k}$ is said to be of finite type if x has a finite decomposition as $x - x_0 = \sum_{i=1}^p x_i$, for some positive integer $p < +\infty$ and $\Delta x_i = \kappa_i x_i$, $\kappa_i \in \mathbb{R}$, $1 \leq i \leq p$ where x_0 is constant, and x_i , $1 \leq i \leq p$, are nonconstant smooth maps, and Δ is the Laplace operator of M . If all κ_i 's are mutually different, then M^n is said to be of p -type. If, in particular, one of κ_i 's is zero, then M is said to be of null p -type.

As is well known, the Laplace operator of a hypersurface M immersed into $\mathbb{R}^{n+1}$ is an (intrinsic) second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplace operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \dots, L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$ th mean curvature arising from normal variations of the hypersurface (see [3, 4]). These operators are given by $L_k(f) = \text{tr}(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes

the k th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ is the hessian of f (see the next section for details).

In contrast to the operator Δ , which is elliptic, in general the operators L_k are not elliptic but they still share some nice properties with Laplacian of M . Therefore, from this point of view, it seems natural and interesting to generalize the definition of finite-type hypersurface by replacing Δ by L_k and study the properties of such hypersurfaces. Having this idea, for the first time in [5], the second author inspired by the private communication with Alías, introduced such hypersurfaces and called them L_k -finite-type hypersurfaces. In [5], he began to study generalized cylinders, ruled surfaces, and some revolution hypersurfaces from the point of view of L_1 -finiteness type.

In this paper which is a natural continuation of [5], we study some L_k -finite-type hypersurfaces in the Euclidean space. The structure of the paper is as follows.

In Section 2, we give preliminaries.

In Section 3, we classify L_k -1-type Euclidean hypersurfaces and L_k -null-2-type Euclidean hypersurfaces with at most two distinct principal curvatures. We follow the work of Ferrández and Lucas [6], for Euclidean hypersurfaces of null-2-type. They proved that such hypersurfaces with at most two distinct principal curvatures are locally isometric to a generalized cylinder. This is a generalization of Chen's Theorem in [7], stating that null-2-type surfaces are circular cylinders. Here we generalize the result of [6] and prove that any Euclidean hypersurface of L_k -null-2-type, ($k \neq n-1$), with at most two distinct principal curvatures, is locally isometric to a generalized cylinder. One of the important consequences of this theorem is the classification of conformally flat hypersurfaces M^n of L_k -null-2-type for $n > 3$. We also prove that there is no L_{n-1} -null-2-type Euclidean hypersurface M^n with at most two distinct principal curvatures. This is a generalization of Chen's result stating that there is no null-2-type plane curve [2, corollary of Theorem 7.3].

Section 4 is about L_k -3-type Euclidean hypersurfaces. Here we prove that any L_1 -3-type surface has nonconstant Gaussian curvature. The result is an extension of Theorem 1 of [8], stating that any 3-type surface in $\mathbb{R}^3$ has nonconstant mean curvature. Also motivated by the work of Ferrández and Lucas [6], stating that there is no Euclidean hypersurface of null-3-type with constant mean curvature and at most two principal curvatures, we prove that there is no Euclidean hypersurface of L_k -null-3-type with constant H_{k+1} and at most two distinct principal curvatures.

Our main results are the following theorems.

Theorem 1.1. *k -minimal Euclidean hypersurfaces and open parts of hyperspheres are the only L_k -1-type hypersurfaces in $\mathbb{R}^{n+1}$.*

Theorem 1.2. *There is no L_{n-1} -null-2-type hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, with at most two distinct principal curvatures.*

Theorem 1.3. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed Euclidean hypersurface with at most two distinct principal curvatures and multiplicities are greater than one. Then M^n is of L_k -null-2-type ($k \neq n-1$), if and only if M is isoparametric, so locally isometric to $S^m \times \mathbb{R}^{n-m}$, $m \geq k+1$.*

Theorem 1.4. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed Euclidean hypersurface with at most two distinct principal curvatures, one of them is simple. Then M^n is of L_k -null-2-type ($k \neq n-1$) if and only if M is isoparametric, so locally isometric to $\mathbb{R} \times S^{n-1}(r)$ for $0 \leq k < n-1$ or $\mathbb{R}^{n-1} \times S^1(r)$ for $k = 0$.*

In Theorems 1.3 and 1.4, if we assume that M is complete, then M is globally isometric to a generalized cylinder.

Theorem 1.5. *There is no L_1 -3-type surface in $\mathbb{R}^3$ with constant Gaussian curvature.*

Theorem 1.6. *There is no hypersurface of L_k -null-3-type in $\mathbb{R}^{n+1}$ with constant H_{k+1} and at most two distinct principal curvatures.*

2. Preliminaries

In this section we recall some notions and prerequisites about (L_k -finite type) hypersurfaces of the Euclidian space from [9, 10].

Consider an isometrically immersed hypersurface $x : M^n \rightarrow \mathbb{R}^{n+1}$ in the Euclidean space. We choose a local orthonormal frame $\{e_A\}_{1 \leq A \leq n+1}$ in $\mathbb{R}^{n+1}$, with dual coframe $\{\omega_A\}_{1 \leq A \leq n+1}$, such that, at each point of M , $e_1, \dots, e_n$ are tangent to M and e_{n+1} is the positively oriented unit normal vector. We will make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \dots, \leq n+1; \quad 1 \leq i, j, k, \dots, \leq n. \quad (2.1)$$

Then the structure equations of $\mathbb{R}^{n+1}$ are given by

$$\begin{aligned} d\omega_A &= \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, & \omega_{AB} + \omega_{BA} &= 0, \\ d\omega_{AB} &= \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB}. \end{aligned} \quad (2.2)$$

When restricted to M , we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^n \omega_{n+1i} \wedge \omega_i. \quad (2.3)$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{n+1i} = \sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \quad (2.4)$$

This gives the second fundamental form of M , as $B = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}$. The mean curvature H is defined by $H = (1/n) \sum_i h_{ii}$. From (2.2)–(2.4) we obtain the structure equations of M , (see [10]):

$$\begin{aligned} d\omega_i &= \sum_{j=1}^n \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} &= 0, \\ d\omega_{ij} &= \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_k \wedge \omega_l, \end{aligned} \quad (2.5)$$

and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \quad (2.6)$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M .

Let h_{ijk} denote the covariant derivative of h_{ij} . We have

$$\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj}. \quad (2.7)$$

Thus, by exterior differentiation of (2.4), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \quad (2.8)$$

We choose $e_1, \dots, e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}. \quad (2.9)$$

Let H_m be m th mean curvature of M , then we have

$$\binom{n}{m} H_m = \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}. \quad (2.10)$$

And $H_n = \lambda_1 \cdots \lambda_n$ is called the Gauss-Kronecker curvature of M . A hypersurface with zero $(k+1)$ th mean curvature in $\mathbb{R}^{n+1}$ is called k -minimal. To get more information about k -minimal Euclidean hypersurfaces, the reader is referred to [11, 12].

The classical Newton transformations $P_k : \chi(M) \rightarrow \chi(M)$ are defined inductively by the shape operator S as

$$P_0 = I, \quad P_k = \binom{n}{k} H_k I - S \circ P_{k-1}, \quad (2.11)$$

for every $k = 1, \dots, n$, where I denotes the identity transformation in $\chi(M)$. Equivalently,

$$P_k = \sum_{j=0}^k (-1)^j \binom{n}{k-j} H_{k-j} S^j. \quad (2.12)$$

Note that by the Cayley-Hamilton theorem stating that any operator is annihilated by its characteristic polynomial, we have $P_n = 0$.

Since each $P_k(p)$ is also a self-adjoint linear operator on each tangent plane $T_p M$ which commutes with $S(p)$. Indeed, $S(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1, \dots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \dots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_{i,k}(p) = \sum_{i_1 < \dots < i_k, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p), \quad (2.13)$$

for every $1 \leq i \leq n$. We have the following formulae for the Newton transformations, [9]:

$$\text{tr}(P_k) = c_k H_k, \quad (2.14)$$

$$\text{tr}(S \circ P_k) = c_k H_{k+1}, \quad (2.15)$$

$$\text{tr}(S^2 \circ P_k) = \binom{n}{k+1} (n H_1 H_{k+1} - (n-k-1) H_{k+2}), \quad (2.16)$$

where

$$c_k = (n-k) \binom{n}{k} = (k+1) \binom{n}{k+1}. \quad (2.17)$$

Associated to each Newton transformation P_k , we consider the second-order linear differential operator $L_k : C^\infty(M) \rightarrow C^\infty(M)$ given by

$$L_k(f) = \text{tr}(P_k \circ \nabla^2 f). \quad (2.18)$$

Here $\nabla^2 f : \chi(M) \rightarrow \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad (2.19)$$

where $X, Y \in \chi(M)$, ∇f is the gradient of f , and ∇ is the Levi-Civita connections on M .

Now we recall the definition of an L_k -finite-type hypersurface from [5], which is the basic notion of the paper. We also quote Proposition 2.2 from [5].

Definition 2.1. An isometrically immersed hypersurface $x : M^n \rightarrow \mathbb{R}^{n+1}$ is said to be of L_k -finite type if x has a finite decomposition $x = \sum_{i=0}^m x_i$, for some positive integer m satisfying the condition that $L_k x_i = \kappa_i x_i$, $\kappa_i \in \mathbb{R}$, $1 \leq i \leq m$, where $x_i : M^n \rightarrow \mathbb{R}^{n+1}$ are smooth maps, $1 \leq i \leq m$, and x_0 is constant. If all κ_i 's are mutually different, M^n is said to be of L_k - m -type. An L_k - m -type hypersurface is said to be null if some κ_i , $1 \leq i \leq m$, is zero. The polynomial $p(t) = \prod_{i=1}^m (t - \kappa_i)$ is called the minimal polynomial of M for L_k .

We should mention that similar to Proposition 1 of [13], if M is of L_k -finite type, then $p(L_k)(x - x_0) = 0$.

Proposition 2.2 (see [5]). *If the isometrically immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ is a generalized cylinder $S^m(r) \times \mathbb{R}^{n-m}$, then M is of L_k -null-1-type, if $k + 1 > m$, and it is of L_k -null-2-type, if $k + 1 \leq m$.*

3. L_k -2-Type Hypersurfaces

In this section we would like to follow [6] and consider L_k -2-type hypersurfaces. In [6], it was proved that if M is a nonminimal Euclidean hypersurface of 1- or 2-type and with constant mean curvature, then either (a) M is an open piece of S^n or (b) M is of null-2-type. In Theorem 3.2 we obtain an extension of this result for L_k -1 and 2-type Euclidean hypersurfaces. In Theorem 3.5, we prove that there is no L_{n-1} -null-2-type hypersurface in $\mathbb{R}^{n+1}$ with at most two distinct principal curvatures. It is a generalization of Chen's result stating that there is no null-2-type plane curve [2, corollary of Theorem 7.3]. In [7], Chen proved that null-2-type surfaces are circular cylinders. Later, Ferrández and Lucas got a generalization of Chen's theorem and showed that Euclidean hypersurfaces of null-2-type with at most two distinct principal curvatures are locally isometric to a generalized cylinder [6, Theorem 3.10]. In Theorems 3.11 and 3.12, we generalize this theorem and prove that any Euclidean hypersurface of L_k -null-2-type ($k \neq n - 1$), with at most two distinct principal curvatures is locally isometric to a generalized cylinder.

According to Takahashi's theorem [14], we know that minimal Euclidean hypersurfaces and open parts of hyperspheres are the only 1-type Euclidean hypersurfaces. In the next theorem, we show that k -minimal hypersurfaces and open parts of hyperspheres are the only L_k -1-type Euclidean hypersurfaces.

Theorem 3.1. *k -minimal Euclidean hypersurfaces and open parts of hyperspheres are the only L_k -1-type hypersurfaces in $\mathbb{R}^{n+1}$.*

Proof. In Theorem 1 of [9] Alías and Gürbüz have classified hypersurfaces in $\mathbb{R}^{n+1}$ satisfying the general condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+1) \times (n+1)}$ is a matrix and $b \in \mathbb{R}^{n+1}$. Section 3 of [9] and Proposition 2.2 show that k -minimal hypersurfaces and open parts of hyperspheres are the only hypersurfaces satisfying the condition $L_k x = cIx + b$, for some real constant c . $\square$

Now we can state the next theorem.

Theorem 3.2. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed hypersurface in the Euclidean space. If it is of L_k -1-type or L_k -2-type for some $0 \leq k \leq n - 1$ and the $(k + 1)$ th mean curvature of M is a nonzero constant, then one of the following two cases occurs:*

- (a) M is an open piece of S^n ,
- (b) M is of L_k -null-2-type.

Proof. The case $k = 0$ was proved by Chen and Lue [15, Theorem 1] and Ferrández and Lucas [6, Corollary 3.2], so that we may consider the case $k \geq 1$. Here, we will follow the techniques introduced by Chen and Lue [15, Theorem 1] for our context. According to Theorem 3.1, k -minimal hypersurfaces and open parts of hyperspheres are the only L_k -1-type hypersurfaces. If M is of L_k -2-type, the position vector x of M in $\mathbb{R}^{n+1}$ has the following spectral decomposition:

$$x - c = x_1 + x_2, \quad L_k x_1 = \kappa_1 x_1, \quad L_k x_2 = \kappa_2 x_2, \quad \text{for some } c \in \mathbb{R}^{n+1}, \quad (3.1)$$

so

$$L_k^2 x = (\kappa_1 + \kappa_2) L_k x - \kappa_1 \kappa_2 (x - c). \quad (3.2)$$

We also have

$$L_k x = c_k H_{k+1} N, \quad (3.3)$$

$$L_k^2 x = -c_k \binom{n}{k+1} H_{k+1} (n H_1 H_{k+1} - (n - k - 1) H_{k+2}) N. \quad (3.4)$$

The formula (3.4) holds since H_{k+1} is a nonzero constant, see [9]. Therefore, by using (3.2), (3.3), and (3.4), we obtain

$$-c_k \binom{n}{k+1} H_{k+1} (n H_1 H_{k+1} - (n - k - 1) H_{k+2}) N = c_k (\kappa_1 + \kappa_2) H_{k+1} N - \kappa_1 \kappa_2 (x - c). \quad (3.5)$$

From (3.5) we have either $\kappa_1 \kappa_2 = 0$ or $x - c$ is normal to M at every point of M . If $\kappa_1 \kappa_2 = 0$, then M is of L_k -null-2-type. If $x - c$ is normal to M , then $\langle x - c, x - c \rangle$ is a positive constant. In this case, M is an open piece of S^n centered at c ; therefore, M is of L_k -1-type, which is not. $\square$

As a corollary we get the following result.

Corollary 3.3. *If M^n is an L_k -2-type Euclidean hypersurface with constant $(k+1)$ th mean curvature, then M^n is of null L_k -2-type, in particular, if $n = k+1$ then the mean curvature is a nonzero constant.*

Proof. By Theorem 3.2, M is null, and we have the following spectral decomposition for the position vector x of M in $\mathbb{R}^{n+1}$:

$$x = c + x_1 + x_2, \quad L_k x_1 = 0, \quad L_k x_2 = \kappa_2 x_2. \quad (3.6)$$

Since $n = k+1$ and M is null, from formula (3.5) we have

$$-(k+1) H_1 H_{k+1}^2 N = \kappa_2 H_{k+1} N. \quad (3.7)$$

From (3.7) we obtain $\kappa_2 = -(k+1) H_1 H_{k+1}$. Since κ_2 and H_{k+1} are nonzero and constant, so H_1 is also nonzero and constant. $\square$

To express an extension of Chen's result in [2] stating that there is no null-2-type plane curve, we need the next lemma.

Lemma 3.4. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed Euclidean hypersurface which is of L_{n-1} -null-2-type, then the Gauss-Kronecker curvature of M is nonzero and constant.*

Proof. Let us consider the open set $\mathcal{U} = \{p \in M : \nabla H_n^2(p) \neq 0\}$, our objective is to show that $\mathcal{U}$ is empty. From [9], we have

$$L_{n-1}^2 x = -c_{n-1} H_n \nabla H_n - 2c_{n-1} (S \circ P_{n-1}) \nabla H_n - c_{n-1} (nH_1 H_n^2 - L_{n-1} H_n) N. \quad (3.8)$$

From this relation and (3.2) we obtain that

$$-c_{n-1} H_n \nabla H_n - 2c_{n-1} (S \circ P_{n-1}) \nabla H_n - c_{n-1} (nH_1 H_n^2 - L_{n-1} H_n) N = c_{n-1} \kappa_2 H_n N. \quad (3.9)$$

Therefore on $\mathcal{U}$ we get

$$(S \circ P_{n-1}) \nabla H_n = -\frac{1}{2} H_n \nabla H_n. \quad (3.10)$$

But by the Cayley-Hamilton theorem we have $P_n = 0$, so

$$S \circ P_{n-1} = H_n I, \quad (S \circ P_{n-1}) \nabla H_n = H_n \nabla H_n, \quad (3.11)$$

which jointly with (3.10) yields $\nabla H_n^2 = 0$ on $\mathcal{U}$, which is a contradiction. $\square$

Now, we state one of the main results of the section as follows.

Theorem 3.5. *There is no L_{n-1} -null-2-type hypersurface in $\mathbb{R}^{n+1}$, with at most two distinct principal curvatures.*

Proof. If M is an L_{n-1} -null-2-type hypersurface in $\mathbb{R}^{n+1}$ with at most two principal curvatures, by applying Corollary 3.3 and Lemma 3.4 we obtain that the mean and Gauss-Kronecker curvatures are constant. Since M is of L_{n-1} -null-2-type, we conclude that M has exactly two constant principal curvatures. From [16], M is an open piece of $\mathbb{R}^p \times S^{n-p}$ for some $p \geq 1$, so the Gauss-Kronecker curvature of M is zero; therefore, M is of L_{n-1} -null-1-type, which is a contradiction. $\square$

In [7], Chen proved that null 2-type surfaces are circular cylinders. In sharp contrast to this result, in the next corollary, we claim that there is not any L_1 -null-2-type surface at all! It is an obvious consequence of Theorem 3.5.

Corollary 3.6. *There is no L_1 -null-2-type surface in $\mathbb{R}^3$.*

By Theorem 3.5 and the following theorems from [17, 18], we can prove that there is no L_{n-1} -null-2-type compact hypersurface in $\mathbb{R}^{n+1}$ with everywhere nonzero sectional curvature.

Theorem 3.7 (see [17]). *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed compact hypersurface in the Euclidean space, then the following conditions are equivalent:*

- (i) $H_r = \text{const} \neq 0$ for some $r \in \{2, \dots, n\}$ and $K \geq 0$,
- (ii) $H = \text{const}$ and $K \geq 0$,
- (iii) M has parallel second fundamental form (i.e., M is isoparametric with at most two principle curvatures),

where K is the sectional curvature of M .

The next theorem shows that there is no compact isometrically immersed hypersurface M^n in $\mathbb{R}^{n+1}$ with everywhere nonpositive sectional curvature.

Theorem 3.8 (see [18]). *Let M be a compact n -dimensional Riemannian manifold, and let $\bar{M}$ be a complete simply connected Riemannian manifold of dimension less than 2. If the sectional curvatures K and $\bar{K}$ of M and $\bar{M}$ satisfy $K \leq \bar{K} \leq 0$, then M cannot be immersed in $\bar{M}$.*

Now we can express our result.

Proposition 3.9. *There is no L_{n-1} -null-2-type compact hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ with everywhere nonzero sectional curvature.*

Proof. By Theorem 3.8, we assume that $K > 0$. By lemma 4.3, H_n is nonzero and constant. So we get from Theorem 3.7 that M is isoparametric with at most two distinct principle curvatures, and this is a contradiction with Theorem 3.5. $\square$

By considering two different cases we prove that L_k -null-2-type Euclidean hypersurfaces ($k \neq n-1$) with at most two distinct principal curvatures are circular cylinders. Case I: The multiplicities are greater than one. Case II: one of the principal curvatures is simple.

We use the following Lemma from [19], for the proof of claim.

Lemma 3.10 (see [19], Theorem 2 and its corollary). *Let M be an n -dimensional hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ such that multiplicities of principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than one, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

Theorem 3.11. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed hypersurface with at most two distinct principal curvatures and multiplicities are greater than one. Then M^n is of L_k -null-2-type ($k \neq n-1$) if and only if M is isoparametric so locally isometric to $S^m \times \mathbb{R}^{n-m}$, $m \geq k+1$.*

Proof. Let M^n is of L_k -null-2-type. If M^n is totally umbilical, then M^n is a piece of $\mathbb{R}^n$ or S^n . By using Theorem 3.1, $\mathbb{R}^n$ and S^n are of L_k -1-type, so M is not totally umbilical. Therefore M has two distinct principal curvatures of multiplicities q and $n-q$, ($q, n-q > 1$).

Let us consider the open set

$$\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}. \quad (3.12)$$

Our objective is to show that $\mathcal{U}$ is empty.

Consider $\{e_1, \dots, e_n\}$ to be a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for every $i = 1, \dots, n$. We assume that

$$\lambda_1 = \lambda_2 = \dots = \lambda_q = \lambda, \quad \lambda_{q+1} = \dots = \lambda_n = \mu. \quad (3.13)$$

Therefore from (2.13) we have

$$P_{k+1}e_i = \mu_{i,k+1}e_i, \quad (3.14)$$

with

$$\mu_{i,k+1} = \sum_{i_1 < \dots < i_{k+1}, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_{k+1}}. \quad (3.15)$$

So we get

$$\begin{aligned} \mu_{1,k+1} = \dots = \mu_{q,k+1} &= \sum_s \binom{q-1}{s} \binom{n-q}{k+1-s} \lambda^s \mu^{k+1-s}, \\ \mu_{q+1,k+1} = \dots = \mu_{n,k+1} &= \sum_s \binom{q}{s} \binom{n-q-1}{k+1-s} \lambda^s \mu^{k+1-s}. \end{aligned} \quad (3.16)$$

We obtain from (2.10) that

$$\binom{n}{k+1} H_{k+1} = \sum_s \binom{q}{s} \binom{n-q}{k+1-s} \lambda^s \mu^{k+1-s}. \quad (3.17)$$

Since M is of L_k -null-2-type, the position vector field x satisfies the following equation for some constant $m \neq 0$,

$$L_k^2 x = m L_k x. \quad (3.18)$$

So by using the formulae of $L_k x$ and $L_k^2 x$ from [9], we can write

$$\begin{aligned} -c_k \binom{n}{k+1} H_{k+1} \nabla H_{k+1} - 2c_k (S \circ P_k) (\nabla H_{k+1}) \\ - c_k \left(\binom{n}{k+1} H_{k+1} (n H_1 H_{k+1} - (n-k-1) H_{k+2}) - L_k H_{k+1} \right) N = m c_k H_{k+1} N. \end{aligned} \quad (3.19)$$

From (3.19) we get

$$(S \circ P_k) (\nabla H_{k+1}^2) = -\frac{1}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}^2. \quad (3.20)$$

Since $k \neq n - 1$, it follows from the inductive definition of P_{k+1} that (3.20) is equivalent to

$$P_{k+1}(\nabla H_{k+1}^2) = \frac{3}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}^2 \quad \text{on } \mathcal{U}. \quad (3.21)$$

Therefore, writing

$$\nabla H_{k+1}^2 = \sum_{i=1}^n \langle \nabla H_{k+1}^2, e_i \rangle e_i, \quad (3.22)$$

we see that (3.21) is equivalent to

$$\langle \nabla H_{k+1}^2, e_i \rangle \left(\mu_{i,k+1} - \frac{3}{2} \binom{n}{k+1} H_{k+1} \right) = 0 \quad \text{on } \mathcal{U} \quad (3.23)$$

for every $i = 1, \dots, n$. So there is no loss of generality, assuming that

$$\mu_{1,k+1} = \dots = \mu_{q,k+1} = \frac{3}{2} \binom{n}{k+1} H_{k+1}. \quad (3.24)$$

Let us denote the integral submanifolds through $x \in \mathcal{U}$ corresponding to λ and μ by $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$, respectively. From Lemma 3.10, we know that λ is constant on $\mathcal{U}_1^q(x)$. Equations (3.16), (3.17), and (3.24) imply that μ is constant on $\mathcal{U}_1^q(x)$. Again by Lemma 3.10, we get that μ is constant on $\mathcal{U}_1^{n-q}(x)$. It now follows from [20, page 182, I] that $\mathcal{U}$ is locally isometric to the Riemannian product of the maximal integral manifolds $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$. Therefore, μ is constant on $\mathcal{U}$. By the same assertion, we know that λ is constant on $\mathcal{U}$, so H_{k+1} is constant on $\mathcal{U}$, which is a contradiction. Hence H_{k+1} is constant and nonzero on M . From (3.19), we obtain that $\text{tr}(S^2 \circ P_k) = m$; therefore, $\text{tr}(S^2 \circ P_k)$ is constant. By the fact that M has two principal curvatures and H_{k+1} , $\text{tr}(S^2 \circ P_k)$ are constant, we get that the principal curvatures are constant. So M is isoparametric.

A classical result of Segre [16] states that isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are locally isometric to $\mathbb{R}^n$, $S^n(r)$ and circular cylinder. On the other hand, since M^n is of L_k -null-2-type, by using Proposition 2.2, we conclude that M^n is locally isometric to $S^m \times \mathbb{R}^{n-m}$, $m \geq k + 1$. $\square$

Theorem 3.12. *Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometrically immersed Euclidean hypersurface with at most two distinct principal curvatures, one of them is simple. Then M^n is of L_k -null-2-type, ($k \neq n-1$), if and only if M is isoparametric, so locally isometric to $\mathbb{R} \times S^{n-1}(r)$ $0 \leq k < n-1$ or $\mathbb{R}^{n-1} \times S^1(r)$ for $k = 0$.*

Proof. Let M^n be of L_k -null-2-type ($k \neq n-1$). If M^n is totally umbilic, then M^n is a piece of $\mathbb{R}^n$ or S^n . By using Theorem 3.1, $\mathbb{R}^n$ and S^n are of L_{k-1} -type, so M cannot be totally umbilic. Therefore suppose M has two distinct principal curvatures of multiplicities 1 and $n-1$.

Let us consider the open set.

$$\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}. \quad (3.25)$$

We want to prove that $\mathcal{U}$ is empty. If $\mathcal{U} \neq \emptyset$, then we express H_{k+1} as a polynomial in λ (the nonsimple principal curvature of M) with constant coefficients, after that we express λ as a constant multiple of the simple principal curvature of M . By using Otsuki's Lemma (Lemma 3.10), the structure equations of M , and the fact that M is of L_k -null-2-type, we get that λ satisfies a polynomial with constant coefficients. So λ is constant; hence, H_{k+1} is constant, a contradiction with $\mathcal{U} \neq \emptyset$. So $\mathcal{U}$ is empty.

Here is the detailed treatment of the proof.

With the assumption that $\mathcal{U} \neq \emptyset$, consider $\{e_1, \dots, e_n\}$ to be a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for every $i = 1, \dots, n$. We assume

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu. \quad (3.26)$$

Therefore from (2.10) we have

$$P_{k+1}e_i = \mu_{i,k+1}e_i, \quad (3.27)$$

with

$$\mu_{i,k+1} = \sum_{i_1 < \dots < i_{k+1}, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_{k+1}}. \quad (3.28)$$

So we get

$$\begin{aligned} \mu_{1,k+1} = \dots = \mu_{n-1,k+1} &= \binom{n-2}{k+1} \lambda^{k+1} + \binom{n-2}{k} \lambda^k \mu, \\ \mu_{n,k+1} &= \binom{n-1}{k+1} \lambda^{k+1}. \end{aligned} \quad (3.29)$$

We obtain from (2.13) that

$$\binom{n}{k+1} H_{k+1} = \binom{n-1}{k+1} \lambda^{k+1} + \binom{n-1}{k} \lambda^k \mu. \quad (3.30)$$

Since M is of L_k -null-2-type, the position vector field x satisfies the following equation for some constant $m \neq 0$,

$$L_k^2 x = mL_k x. \quad (3.31)$$

So by using the formulae of $L_k x$ and $L_k^2 x$ from [9], we can write

$$\begin{aligned} &-c_k \binom{n}{k+1} H_{k+1} \nabla H_{k+1} - 2c_k (S \circ P_k) (\nabla H_{k+1}) \\ &-c_k \left(\binom{n}{k+1} H_{k+1} (nH_1 H_{k+1} - (n-k-1)H_{k+2}) - L_k H_{k+1} \right) N = mc_k H_{k+1} N. \end{aligned} \quad (3.32)$$

From (3.32) we get

$$(S \circ P_k)(\nabla H_{k+1}^2) = -\frac{1}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}^2. \quad (3.33)$$

Since $k \neq n-1$, it follows from the inductive definition of P_{k+1} that (3.33) is equivalent to

$$P_{k+1}(\nabla H_{k+1}^2) = \frac{3}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}^2 \quad \text{on } \mathcal{U}. \quad (3.34)$$

Therefore, by the formula

$$\nabla H_{k+1}^2 = \sum_{i=1}^n \langle \nabla H_{k+1}^2, e_i \rangle e_i, \quad (3.35)$$

we see that (3.34) is equivalent to

$$\langle \nabla H_{k+1}^2, e_i \rangle \left(\mu_{i,k+1} - \frac{3}{2} \binom{n}{k+1} H_{k+1} \right) = 0 \quad \text{on } \mathcal{U} \quad (3.36)$$

for every $i = 1, \dots, n$. Hence, for every i such that $\langle \nabla H_{k+1}^2, e_i \rangle \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,k+1} = \frac{3}{2} \binom{n}{k+1} H_{k+1}. \quad (3.37)$$

So for the expression ∇H_{k+1}^2 in (3.35) we consider two cases.

Case 1. $\langle \nabla H_{k+1}^2, e_n \rangle \neq 0$, by (4.31), we obtain that

$$H_{k+1} = \frac{2}{3} \frac{(n-k-1)}{n} \lambda^{k+1}. \quad (3.38)$$

Case 2. $\langle \nabla H_{k+1}^2, e_n \rangle = 0$, so on $\mathcal{U}$ we have $\langle \nabla H_{k+1}^2, e_j \rangle \neq 0$ for some $j = 1, \dots, n-1$. By (4.31) and using the fact that $\mu_{i,k}$ are the eigenvalues of P_k and the formula of $\text{tr}(P_{k+1})$, we obtain that

$$H_{k+1} = \frac{(n-k-1)}{n(-(1/2)n-k+1/2)} \lambda^{k+1}. \quad (3.39)$$

Both cases require the same calculation, so we consider just Case I.

By Lemma 3.10, let us denote the maximal integral submanifold through $x \in \mathcal{U}$, corresponding to λ by $\mathcal{M}_1^{n-1}(x)$. We write

$$d\lambda = \sum_i \lambda_{,i} \omega_i \quad d\mu = \sum_j \mu_{,j} \omega_j. \quad (3.40)$$

Then Lemma 3.10 implies that $\lambda_{,1} = \dots = \lambda_{,n-1} = 0$. We can assume that $\lambda > 0$ on $\mathcal{U}$, then (3.30) and (3.38) yield

$$\mu = \frac{k+1-n}{3k+3}\lambda. \quad (3.41)$$

By means of (2.7) and (2.9), we obtain that

$$\sum_k h_{ijk}\omega_k = \delta_{ij}d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \quad (3.42)$$

We adopt the notational convention that $1 \leq a, b, c, \dots \leq n-1$.

From (3.41) and (3.42), we have

$$\begin{aligned} h_{ijk} &= 0, & \text{if } i \neq j, \\ h_{aab} &= 0, & h_{aan} = \lambda_{,n}, \\ h_{nna} &= 0, & h_{nnn} = \mu_{,n}. \end{aligned} \quad (3.43)$$

Combining this with (2.8) and the formula

$$\sum_i h_{ami}\omega_i = dh_{an} + \sum_i h_{in}\omega_{ia} + \sum_i h_{ai}\omega_{in} = (\lambda - \mu)\omega_{an}, \quad (3.44)$$

we obtain from (3.41)

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a = \frac{(3k+3)\lambda_{,n}}{(2k+2+n)\lambda}\omega_a. \quad (3.45)$$

Therefore we have

$$d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0. \quad (3.46)$$

Notice that we may consider λ to be locally a function of the parameter s , where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ . We may put $\omega_n = ds$.

Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_{,n}ds, \quad \lambda_{,n} = \lambda'(s), \quad (3.47)$$

so from (3.45), we get

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a = \frac{(3k+3)\lambda'(s)}{(2k+2+n)\lambda}\omega_a. \quad (3.48)$$

According to the structure equations of $\mathbb{R}^{n+1}$ and (3.48), we may compute

$$\begin{aligned} d\omega_{an} &= \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_{bn} + \omega_{an+1} \wedge \omega_{n+1n} \\ &= \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right) \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b - \lambda\mu\omega_a \wedge ds, \end{aligned} \quad (3.49)$$

$$\begin{aligned} d\omega_{an} &= d \left\{ \frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \omega_a \right\} \\ &= \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right) ds \wedge \omega_a + \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right) d\omega_a \\ &= \left\{ \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)' + \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)^2 \right\} \omega_a \wedge ds + \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right) \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b. \end{aligned} \quad (3.50)$$

Then we obtain from the two equalities above that

$$\left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)' - \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)^2 - \lambda\mu = 0. \quad (3.51)$$

Combining (3.51) with (3.41), we have

$$\left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)' - \left(\frac{(3k+3)\lambda'}{(2k+2+n)\lambda} \right)^2 - \left(\frac{k+1-n}{3k+3} \right) \lambda^2 = 0. \quad (3.52)$$

Let us define a function $\beta(s)$, $s \in (-\infty, +\infty)$ by $\beta = (1/\lambda)^{(3k+3)/(2k+2+n)}$, then (3.52) reduces to

$$\beta'' = \left(\frac{n-k-1}{3k+3} \right) \beta^{(-7k-7-2n)/(3k+3)}. \quad (3.53)$$

Integrating (3.53), we obtain

$$(\beta')^2 = \left(\frac{k+1-n}{2k+2+n} \right) \beta^{(-4k-4-2n)/(3k+3)} + C, \quad (3.54)$$

where C is the constant of integration.

Equation (3.54) is equivalent to

$$(\lambda')^2 = \left(\frac{(n+2k+2)(k+1-n)}{(3k+3)^2} \right) \lambda^{(14k+4n+14)/(2k+2+n)} + C \left(\frac{2+2k+n}{3k+3} \right)^2 \lambda^{(10k+10+2n)/(2k+2+n)}. \quad (3.55)$$

Now we use the definition of $L_k H_{k+1} = \text{tr}(P_k \circ \nabla^2 H_{k+1})$ to compute $L_k H_{k+1}$. So we need to compute $\nabla_{e_a} \nabla H_{k+1}$, $\nabla_{e_n} \nabla H_{k+1}$, $P_k(e_a)$, and $P_k(e_n)$.

From (3.51) we have

$$\nabla H_{k+1} = \frac{2(k+1)(n-k-1)}{3n} \lambda^k \lambda' e_n. \quad (3.56)$$

By using (3.48) and (3.56) we obtain

$$\begin{aligned} \nabla_{e_a} \nabla H_{k+1} &= \frac{2(k+1)(n-k-1)}{3n} \lambda^k \lambda' \nabla_{e_a} e_n = \frac{2(k+1)(n-k-1)}{3n} \lambda^k \lambda' \sum_b \omega_{nb}(e_a) e_b \\ &= -\frac{2(n-k-1)(k+1)^2}{n(2k+2+n)} \lambda^{k-1} \lambda'^2 e_a \\ \nabla_{e_n} \nabla H_{k+1} &= \frac{2(k+1)(n-k-1)}{3n} \nabla_{e_n} \lambda^k \lambda' e_n \\ &= \frac{2k(k+1)(n-k-1)}{3n} \lambda^{k-1} \lambda'^2 e_n + \frac{2(k+1)(n-k-1)}{3n} \lambda^k \lambda'' e_n. \end{aligned} \quad (3.57)$$

Now we compute $P_k(e_a)$ and $P_k(e_n)$:

$$\begin{aligned} P_k(e_a) &= \mu_{a,k} e_a = \left(\sum_{i_1 < \dots < i_k, i_j \neq a} \lambda_{i_1} \cdots \lambda_{i_k} \right) e_a = \binom{n-2}{k} \frac{2k+3}{3k+3} \lambda^k e_a, \\ P_k(e_n) &= \binom{n-1}{k} \lambda^k e_n. \end{aligned} \quad (3.58)$$

From (3.57) and (3.58), we get

$$L_k H_{k+1} = c_k H_{k+1} \left(\frac{(-2k-3)(k+1)(n-k-1)}{n(2k+2+n)} \lambda^{k-2} \lambda'^2 + \frac{k(k+1)}{n} \lambda^{k-2} \lambda'^2 + \frac{k+1}{n} \lambda^{k-1} \lambda'' \right). \quad (3.59)$$

Since M^n is of L_k -null-2-type, hence from (3.32), we get

$$L_k H_{k+1} = c_k H_{k+1} \left(m - \text{tr}(S^2 \circ P_k) \right) = c_k H_{k+1} \left(m - \binom{n-1}{k} \frac{2nk+3n-2k-2k^2}{3k+3} \lambda^{k+2} \right). \quad (3.60)$$

Combining (3.58) and (3.60), we have

$$\lambda \lambda'' + \left(k + \frac{(-2k-3)(n-k-1)}{2k+2+n} \right) \lambda'^2 + \binom{n-1}{k} \frac{n(2nk+3n-2k-2k^2)}{(k+1)(3k+3)} \lambda^4 - \frac{mn}{k+1} \lambda^{2-k} = 0. \quad (3.61)$$

Equation (3.52) is equivalent to

$$\lambda\lambda'' = \frac{5k+5+n}{2k+2+n}\lambda'^2 + \frac{(2k+2+n)(k+1-n)}{(3k+3)^2}\lambda^4. \quad (3.62)$$

Thus, putting together (3.61) and (3.62) one has

$$\begin{aligned} & \frac{4k^2 + 12k - kn - 2n + 8}{2k+2+n}\lambda'^2 + \frac{(2k+2+n)(k+1-n) + 3\binom{n-1}{k}n(2nk+3n-2k-2k^2)}{(3k+3)^2}\lambda^4 \\ & - \frac{mn}{k+1}\lambda^{2-k} = 0. \end{aligned} \quad (3.63)$$

We deduce, using (3.55), (3.63), and (3.38), that H_{k+1} is locally constant on $\mathcal{M}$, which is a contradiction with the definition of $\mathcal{M}$. Hence H_{k+1} is constant and nonzero on M . From the discussion as in the last part of the proof of the Theorem 3.11, we get the result. $\square$

An important consequence of the theorem is the classification of conformally flat hypersurfaces M^n of L_k -null-2-type for $n > 3$.

Definition 3.13 (see [21]). A Riemannian manifold (M^n, g) is called conformally flat if it is locally conformally equivalent to a Euclidean space $\mathbb{R}^n$, that is, if every point of M has a neighborhood which is conformal to an open set in the Euclidean space $\mathbb{R}^n$. A submanifold N of the Euclidean space $\mathbb{R}^n$ is said to be conformally flat if N with the induced metric for $\mathbb{R}^n$ is conformally flat.

The dimension of the hypersurface plays an important role in the study of conformally flat Euclidean hypersurfaces. For $n = 2$, the existence of isothermal coordinates means that any Riemannian surface is conformally flat. For $n > 3$, the result of Cartan-Schouten states that a conformally flat hypersurface is characterized with two principal curvatures that one multiplicity at least $n - 1$ (see [21] for more details). This significant fact is crucial in our classification of L_k -null-2-type conformally flat Euclidean hypersurfaces M^n for $n > 3$.

As a simple consequence of Theorem 3.12, we obtain the following nice corollary which is an extension of the result of [22].

Corollary 3.14. *Let M^n be a conformally flat hypersurface of $\mathbb{R}^{n+1}$, $n > 3$. Then M^n is of L_k -null-2-type, if only if M is locally isometric to the cylinder $\mathbb{R} \times S^{n-1}(r)$ ($0 < k < n - 1$).*

4. L_{n-1} -3-Type Hypersurfaces in $\mathbb{R}^{n+1}$

In Theorem 1 of [8], Hasanis and Vlachos proved that there is no 3-type surface in $\mathbb{R}^3$ with constant mean curvature. Later Ferrández and Lucas proved that there is no Euclidean hypersurface of null 3-type with constant mean curvature with at most two distinct principal curvatures [6, Theorem 3.5]. Here we follow Hasanis and Vlachos's work to consider L_1 -3-type surfaces. Also we get a generalization of Theorem 3.5 of [6] and prove that there is no Euclidean hypersurface of L_k -null-3-type with constant H_{k+1} and at most two distinct principal curvatures.

Theorem 4.1. *There is no L_1 -3-type surface in $\mathbb{R}^3$ with constant Gaussian curvature.*

Proof. Let $x : M^2 \rightarrow \mathbb{R}^3$ be the position vector of an L_1 -3-type surface M , and set $|x| = r$ for the corresponding distance function. Let N be the unit normal vector field of M . We decompose the position vector x of M as follows

$$x = x_t + \langle x, N \rangle N, \quad N \perp x_t, \quad (4.1)$$

where x_t is the tangential component, then the gradient of $\langle x, x \rangle = r^2$ on M is given by $\nabla r^2 = 2x_t$. By taking covariant derivative of (4.1) and using the Gauss and Weingarten formula, we have

$$\nabla_X \nabla r^2 = \nabla_X 2x_t = 2X + 2\langle x, N \rangle SX, \quad (4.2)$$

for every tangent vector field $X \in \chi(M)$. Therefore by using (2.14) and (2.15) we have

$$L_1 r^2 = \text{tr}(P_1 \circ \nabla^2 r^2) = 4H_1 + 4H_2 \langle x, N \rangle. \quad (4.3)$$

If M has constant Gaussian curvature H_2 , it is nonzero. In fact, if $H_2 = 0$, then according to Theorem 3.1, M must be of L_1 -1-type. So we assume that M has nonzero constant Gaussian curvature H_2 and is of L_1 -3-type. By computing $L_1^i x$, $i = 1, 2, 3$, one finds that

$$L_1 x = 2H_2 N, \quad L_1^2 x = -4H_1 H_2^2 N, \quad (4.4)$$

$$L_1^3 x = 8H_2^2 (S \circ P_1) \nabla H_1 - 4H_2^2 (L_1 H_1 - 2H_1^2 H_2) N. \quad (4.5)$$

Since M is of L_1 -3-type, we have

$$L_1^3 x = (\kappa_1 + \kappa_2 + \kappa_3) L_1^2 x - (\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_1 \kappa_3) L_1 x + \kappa_1 \kappa_2 \kappa_3 x, \quad (4.6)$$

where $x = x_1 + x_2 + x_3$, $L_1 x_i = \kappa_i x_i$, $1 \leq i \leq 3$. Taking into account (4.4), and comparing the tangential and normal components of $L_1^3 x$ in (4.5), (4.6) we obtain the following useful equations:

$$(S \circ P_1) \nabla H_1 = \frac{\kappa_1 \kappa_2 \kappa_3}{8H_2^2} x_t, \quad (4.7)$$

$$L_1 H_1 = 2H_2 H_1^2 + H_1 (\kappa_1 + \kappa_2 + \kappa_3) + \frac{1}{2H_2} (\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_1 \kappa_3) - \frac{\kappa_1 \kappa_2 \kappa_3}{4H_2^2} \langle x, N \rangle. \quad (4.8)$$

For brevity we set $\kappa = \kappa_1 \kappa_2 \kappa_3 / 8H_2^2$. We distinguish the following two cases.

Case I ($\kappa = 0$). That is M is of L_1 -null-3-type, then, by (4.7), $(S \circ P_1) \nabla H_1 = 0$. By the Cayley-Hamilton theorem, we have $P_2 = 0$, and from the inductive definition of P_2 we get that

$$S \circ P_1 = H_2 I. \quad (4.9)$$

So

$$(S \circ P_1)\nabla H_1 = H_2\nabla H_1 = 0. \quad (4.10)$$

Since $H_2 \neq 0$ we conclude that H_1 is constant, so M is isoparametric, then by Theorem 3.1, M has exactly two constant principal curvatures, therefore from Lemma 2.B of [17] we know that one of the principal curvatures has to be zero, so $H_2 = 0$, this is a contradiction.
Case II ($\kappa \neq 0$). Then from (4.7) and (4.9) we obtain that

$$H_2\nabla H_1 = \kappa x_t, \quad (4.11)$$

so we have $\nabla(H_2H_1 - (\kappa/2)r^2) = 0$; hence, $H_2H_1 - (\kappa/2)r^2$ is constant, therefore

$$L_1H_1 = \frac{\kappa}{2H_2}L_1r^2, \quad (4.12)$$

from which, by using (4.3), one finds that

$$L_1H_1 = \frac{2\kappa}{H_2}H_1 + 2\kappa\langle x, N \rangle, \quad (4.13)$$

from which, by using (4.8) we obtain

$$\alpha\nabla H_1 = 4\kappa\nabla\langle x, N \rangle, \quad (4.14)$$

where we set

$$\alpha = \left(-\frac{2\kappa}{H_2}H_1 + 4H_2H_1 + (\kappa_1 + \kappa_2 + \kappa_3) \right). \quad (4.15)$$

Since $\nabla\langle x, N \rangle = -Sx_t$, by (4.11), (4.14) we have $Sx_t = -(\alpha/4H_2)x_t$. If x_t is identically zero on an open subset $V \subseteq M$, then V should be a sphere, thus of L_1 -1-type, a contradiction. Hence x_t is a principal direction with corresponding principal curvature $-\alpha/4H_2$. Therefore, the other principal curvature is $2H_1 + \alpha/4H_2$, thus the Gaussian curvature has to be $H_2 = -(\alpha/4H_2)(2H_1 + \alpha/4H_2)$ or

$$16H_2^3 + 8H_1H_2\alpha + \alpha^2 = 0. \quad (4.16)$$

Substituting α from (4.15) in (4.16), we get the following polynomial equation for H_1 :

$$\left(-48H_2^2 \right) H_1^2 + (24\kappa - 12H_2(\kappa_1 + \kappa_2 + \kappa_3))H_1 + \left(\frac{-4\kappa^2}{H_2^2} + \frac{2\kappa}{H_2}(\kappa_1 + \kappa_2 + \kappa_3) - 16H_2^3 \right) = 0. \quad (4.17)$$

This shows that H_1 is constant, a contradiction. So H_2 is nonconstant. $\square$

We use the following theorem from [10] to prove of Theorem 4.3.

Theorem 4.2 (see [10]). *Let M be an n -dimensional hypersurface in a Euclidean space $\mathbb{R}^{n+1}$, with constant m th mean curvature H_m and two distinct principal curvatures. If the multiplicities are greater than one, then M is locally isometric to $\mathbb{R}^k \times S^{n-k}(c)$, $2 \leq k \leq n-2$.*

The next theorem will generalize the Ferrández and Lucas result in [6].

Theorem 4.3. *There is no hypersurface of L_k -null-3-type in $\mathbb{R}^{n+1}$ with constant H_{k+1} and at most two distinct principal curvatures.*

Proof. Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be the position vector of an L_k -null-3-type hypersurface M , and let N be the unit normal vector field of M . If M has constant H_{k+1} , it is nonzero. In fact, if $H_{k+1} = 0$, then according to Theorem 3.1, M must be of L_k -1-type. So, we assume that M has nonzero constant H_{k+1} and is of L_k -null-3-type.

By computing $L_k^i x$, $i = 1, 2, 3$ one finds that

$$L_k x = c_k H_{k+1} N, \quad L_k^2 x = -c_k \binom{n}{k+1} H_{k+1} (nH_1 H_{k+1} - (n-k-1)H_{k+2}) N, \quad (4.18)$$

$$\begin{aligned} L_k^3 x = & 2c_k \binom{n}{k+1} H_{k+1} (S \circ P_k) \nabla (nH_1 H_{k+1} - (n-k-1)H_{k+2}) \\ & - c_k \binom{n}{k+1} H_{k+1} \left(L_k (nH_1 H_{k+1} - (n-k-1)H_{k+2}) - (nH_1 H_{k+1} - (n-k-1)H_{k+2})^2 \right) N. \end{aligned} \quad (4.19)$$

Since M is of L_k -null-3-type, we have

$$L_k^3 x = (\kappa_1 + \kappa_2 + \kappa_3) L_k^2 x - (\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_1 \kappa_3) L_k x, \quad (4.20)$$

where $x = x_1 + x_2 + x_3$, $L_k x_i = \kappa_i x_i$, $1 \leq i \leq 3$.

We prove the theorem in three steps.

Step I ($k = n-1$). From (4.18) and comparing the tangential and normal components of $L_{n-1}^3 x$ in (4.19), (4.20) and by the Cayley-Hamilton theorem we obtain the following useful equation:

$$(S \circ P_{n-1}) \nabla H_1 = H_n \nabla H_1 = 0. \quad (4.21)$$

Since $H_n \neq 0$, we conclude that H_1 is constant; Therefore, M has at most two constant principal curvatures. From [16] and Theorem 3.1, M is an open piece of $\mathbb{R}^p \times S^{n-p}$ for some $p \geq 1$, so Gauss-Kronecker curvature of M is zero, therefore M is of L_{n-1} -null-1-type, which is a contradiction.

Step II ($k \neq n-1$). M has at most two distinct principal curvatures with multiplicities greater than one.

The conclusion is directly obtained from Theorem 4.2 and Proposition 2.2.

Step III ($k \neq n-1$). M has two distinct principal curvatures, one of them is simple.

By (4.18) and comparing the tangential and normal components of $L_k^3 x$ in (4.19), (4.20) and using the definition of P_{k+1} , we obtain that

$$\begin{aligned} S \circ P_k \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}) &= \binom{n}{k+1} H_{k+1} \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}) \\ &\quad - P_{k+1} \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}) = 0. \end{aligned} \quad (4.22)$$

It gives

$$P_{k+1} \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}) = \binom{n}{k+1} H_{k+1} \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}). \quad (4.23)$$

Consider $\{e_1, \dots, e_n\}$ to be a local orthonormal frame of principal directions of S on M such that $Se_i = \lambda_i e_i$ for every $i = 1, \dots, n$. We assume that

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu. \quad (4.24)$$

Therefore from (2.13) we have

$$P_{k+1} e_i = \mu_{i,k+1} e_i, \quad (4.25)$$

with

$$\mu_{i,k+1} = \sum_{i_1 < \dots < i_{k+1}, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_{k+1}}. \quad (4.26)$$

So we get

$$\begin{aligned} \mu_{1,k+1} = \dots = \mu_{n-1,k+1} &= \binom{n-2}{k+1} \lambda^{k+1} + \binom{n-2}{k} \lambda^k \mu, \\ \mu_{n,k+1} &= \binom{n-1}{k+1} \lambda^{k+1}. \end{aligned} \quad (4.27)$$

We obtain from (2.10) that

$$\binom{n}{k+1} H_{k+1} = \binom{n-1}{k+1} \lambda^{k+1} + \binom{n-1}{k} \lambda^k \mu. \quad (4.28)$$

We can write

$$\nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}) = \sum_{i=1}^n \langle \nabla(nH_1 H_{k+1} - (n-k-1)H_{k+2}), e_i \rangle e_i, \quad (4.29)$$

we see that (4.29) is equivalent to

$$\langle \nabla(nH_1H_{k+1} - (n-k-1)H_{k+2}), e_i \rangle \left(\mu_{i,k+1} - \binom{n}{k+1} H_{k+1} \right) = 0, \quad (4.30)$$

for every $i = 1, \dots, n$.

If $(nH_1H_{k+1} - (n-k-1)H_{k+2})$ is constant, by the fact that M has two principal curvature and H_{k+1} , $(nH_1H_{k+1} - (n-k-1)H_{k+2})$ are constant, we conclude that M is isoparametric, by using [16] and Proposition 2.2, this is a contradiction. Therefore, for every i such that $\langle \nabla(nH_1H_{k+1} - (n-k-1)H_{k+2}), e_i \rangle \neq 0$ we get

$$\mu_{i,k+1} = \binom{n}{k+1} H_{k+1}. \quad (4.31)$$

So by using (4.27) and (4.28), we conclude that M is isoparametric and this is a contradiction. $\square$

From Theorem 4.3, we can easily obtain the following corollary.

Corollary 4.4. *There is no L_k -null-3-type conformally flat Euclidean hypersurface with constant H_{k+1} .*

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