## Research Article

# A q-Analogue of Rucinski-Voigt Numbers 

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Received 1 August 2012; Accepted 19 September 2012
Academic Editors: L. Ji and W. F. Klostermeyer
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A $q$-analogue of Rucinski-Voigt numbers is defined by means of a recurrence relation, and some properties including the orthogonality and inverse relations with the $q$-analogue of the limit of the differences of the generalized factorial are obtained.

## 1. Introduction

Rucinski and Voigt [1] defined the numbers $S_{k}^{n}(\mathbf{a})$ satisfying the relation

$$
\begin{equation*}
x^{n}=\sum_{k=0}^{n} S_{k}^{n}(\mathbf{a}) p_{k}^{\mathbf{a}}(x) \tag{1.1}
\end{equation*}
$$

where $\mathbf{a}$ is the sequence $(a, a+r, a+2 r, \ldots)$ and $p_{k}^{\mathbf{a}}(x)=\prod_{i=0}^{k-1}(x-(a+i r))$ and proved that these numbers are asymptotically normal. We call these numbers Rucinski-Voigt numbers. Note that the classical Stirling numbers of the second kind $S(n, k)$ in [2-4] and the $r$-Stirling numbers of the second kind $\left[\begin{array}{l}n \\ k\end{array}\right]$ of Broder [5] can be expressed in terms of $S_{k}^{n}(\mathbf{a})$ as follows:

$$
\begin{align*}
& S(n, k)=S_{k}^{n}(\mathbf{d}), \\
& \widehat{\left[\begin{array}{l}
n+r \\
k+r
\end{array}\right]_{r}=S_{k}^{n}(\mathbf{e}),} \tag{1.2}
\end{align*}
$$

where $\mathbf{d}$ and $\mathbf{e}$ are the sequences $(0,1,2, \ldots)$ and $(r, r+1, r+2, \ldots)$, respectively. With these observations, $S_{k}^{n}(\mathbf{a})$ may be considered as certain generalization of the second kind Stirlingtype numbers.

Several properties of Rucinski-Voigt numbers can easily be established parallel to those in the classical Stirling numbers of the second kind. To mention a few, we have the triangular recurrence relation

$$
\begin{equation*}
S_{k}^{n+1}(\mathbf{a})=S_{k-1}^{n}(\mathbf{a})+(k r+a) S_{k}^{n}(\mathbf{a}) \tag{P1}
\end{equation*}
$$

the exponential and rational generating function

$$
\begin{gather*}
\sum_{n \geq 0} S_{k}^{n}(\mathbf{a}) \frac{x^{n}}{n!}=\frac{1}{r^{k} k!} e^{a x}\left(e^{r x}-1\right)^{k}  \tag{P2}\\
\sum_{n \geq 0} S_{k}^{n}(\mathbf{a}) x^{n}=\frac{x^{k}}{\prod_{j=0}^{k}(1-(r j+a) x)} \tag{P3}
\end{gather*}
$$

and explicit formulas

$$
\begin{gather*}
S_{k}^{n}(\mathbf{a})=\frac{1}{r^{k} k!} \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(r j+a)^{n},  \tag{P4}\\
S_{k}^{n}(\mathbf{a})=\sum_{c_{0}+c_{1}+\cdots+c_{k}=n-k} \prod_{j=0}^{k}(r j+a)^{c_{j}} . \tag{P5}
\end{gather*}
$$

The explicit formula in (P4) can be used to interpret $r^{k} k!S_{k}^{n}(\mathbf{a})$ as the number of ways to distribute $n$ distinct balls into the $k+1$ cells ( one ball at a time ), the first $k$ of which has $r$ distinct compartments and the last cell with $a$ distinct compartments, such that
(i) the capacity of each compartment is unlimited;
(ii) the first $k$ cells are nonempty.

The other explicit formula (P5) can also be used to interpret $S_{k}^{n}(\mathbf{a})$ as the number of ways of assigning $n$ people to $k+1$ groups of tables where all groups are occupied such that the first group contains $a$ distinct tables and the rest of the group each contains $r$ distinct tables.

The Rucinski-Voigt numbers are nothing else but the $r$-Whitney numbers of the second kind, denoted by $W_{m, r}(n, k)$, in Mező [6]. That is, $S_{k}^{n}(\mathbf{a})=W_{r, a}(n, k)$. It is worth-mentioning that the $r$-Whitney numbers of the second kind are generalization of Whitney numbers of the second kind in Benoumhani's papers [7-9].

On the other hand, the limit of the differences of the generalized factorial [10]

$$
\begin{equation*}
F_{\alpha, \gamma}(n, k)=\lim _{\beta \rightarrow 0} \frac{\left[\Delta_{t}^{k}(\beta t+\gamma \mid \alpha)_{n}\right]_{t=0}}{k!\beta^{k}}, \quad(\beta t+\gamma \mid \alpha)_{n}=\prod_{j=0}^{n-1}(\beta t+\gamma-j \alpha) \tag{1.3}
\end{equation*}
$$

was also known as a generalization of the Stirling numbers of the first kind. That is, all the first kind Stirling-type numbers may also be expressed in terms of $F_{\alpha, \gamma}(n, k)$ by a special choice of the values of $\alpha$ and $\gamma$. It was shown in [10] that

$$
\begin{equation*}
\sum_{k=0}^{n} F_{\alpha, \gamma}(n, k) t^{k}=p_{n}^{\mathbf{b}}(t) \tag{1.4}
\end{equation*}
$$

where $\mathbf{b}$ is the sequence $(-\gamma,-\gamma+\alpha,-\gamma+2 \alpha, \ldots)$. Recently, $q$-analogue and $(p, q)$-analogue of $F_{\alpha, \gamma}(n, k)$, denoted by $\phi_{\alpha, \gamma}[n, k]_{q}$ and $\phi_{\alpha, \gamma}[n, k]_{p q}$, respectively, were established by Corcino and Hererra in [10] and obtained several properties including the horizontal generating function for $\phi_{\alpha, \gamma}[n, k]_{q}$

$$
\begin{equation*}
\sum_{k=0}^{n} \phi_{\alpha, \gamma}[n, k]_{q} t^{k}=\left\langle t+[\gamma]_{q} \mid[\alpha]_{q}\right\rangle_{n^{\prime}}^{q} \tag{1.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\langle t+[\gamma]_{q} \mid[\alpha]_{q}\right\rangle_{n}^{q}=\prod_{j=0}^{n-1}\left(t+[\gamma]_{q}-[j \alpha]_{q}\right), \quad\left\langle t+[\gamma]_{q} \mid[\alpha]_{q}\right\rangle_{0}^{q}=1 . \tag{1.6}
\end{equation*}
$$

The numbers $F_{\alpha,-\gamma}(n, k)$ are equivalent to the $r$-Whitney numbers of the first kind, denoted by $w_{m, r}(n, k)$, in [6]. More precisely, $F_{\alpha,-\gamma}(n, k)=w_{\alpha, \gamma}(n, k)$. These numbers are generalization of Whitney numbers of the first kind in Benoumhani's papers [7-9].

In this paper, we establish a $q$-analogue of $S_{k}^{n}(\mathbf{a})$ and obtain some properties including recurrence relations, explicit formulas, generating functions, and the orthogonality and inverse relations.

## 2. Definition and Some Recurrence Relations

It is known that a given polynomial $a_{k}(q)$ is a $q$-analogue of an integer $a_{k}$ if

$$
\begin{equation*}
\lim _{q \rightarrow 1} a_{k}(q)=a_{k} \tag{2.1}
\end{equation*}
$$

For example, the polynomials

$$
[n]_{q}=\frac{q^{n}-1}{q-1}, \quad[n]_{q}!=\prod_{i=1}^{n}[i]_{q}, \quad\left[\begin{array}{l}
n  \tag{2.2}\\
k
\end{array}\right]_{q}=\prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^{i}-1}
$$

are the $q$-analogues of the integers $n, n!$, and $\binom{n}{k}$, respectively, since

$$
\lim _{q \rightarrow 1}[n]_{q}=n, \quad \lim _{q \rightarrow 1}[n]_{q}!=n!, \quad \lim _{q \rightarrow 1}\left[\begin{array}{l}
n  \tag{2.3}\\
k
\end{array}\right]_{q}=\binom{n}{k} .
$$

The last two polynomials in (2.2) are called the $q$-factorial and $q$-binomial coefficients, respectively. With these in mind, it is interesting also that, for a given property of an integer $a_{k}$, we can find an analogous property for the polynomial $a_{k}(q)$. For example, the binomial coefficients $\binom{n}{k}$ satisfy the known inversion formula

$$
\begin{equation*}
f_{n}=\sum_{k=0}^{n}\binom{n}{k} g_{k} \Longleftrightarrow g_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f_{k} \tag{2.4}
\end{equation*}
$$

and Vandermondes identity

$$
\begin{equation*}
\binom{m+n}{k}=\sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r} \tag{2.5}
\end{equation*}
$$

while the $q$-binomial coefficients $\left[\begin{array}{c}n \\ k\end{array}\right]_{q}$ satisfy the $q$-binomial inversion formula [3]

$$
\begin{align*}
& f_{n}=\sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} g_{k} \Longleftrightarrow g_{n}=\sum_{k=0}^{n}(-1)^{n-k} q^{\binom{n-k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} f_{k},  \tag{2.6}\\
& f_{n}=\sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q^{\beta}} g_{k} \Longleftrightarrow g_{n}=\sum_{k=0}^{n}(-1)^{n-k} q^{\beta}\binom{n-k}{2}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q^{\beta}} f_{k}
\end{align*}
$$

and $q$-Vandermondes identity [11]

$$
\left[\begin{array}{c}
m+n  \tag{2.7}\\
k
\end{array}\right]_{q}=\sum_{r=0}^{k} q^{r(m-k+r)}\left[\begin{array}{c}
m \\
r
\end{array}\right]_{q}\left[\begin{array}{c}
n \\
k-r
\end{array}\right]_{q}
$$

Carlitz [12] defined a $q$-Stirling number of the second kind in terms of a recurrence relation

$$
\begin{equation*}
S_{q}[n, k]=S_{q}[n-1, k-1]+[k]_{q} S_{q}[n-1, k] \tag{2.8}
\end{equation*}
$$

in connection with a problem in abelian groups, such that when $q \rightarrow 1$, this gives the triangular recurrence relation for the classical Stirling numbers of the second kind $S(n, k)$

$$
\begin{equation*}
S(n, k)=S(n-1, k-1)+k S(n-1, k) \tag{2.9}
\end{equation*}
$$

This motivates the authors to define a $q$-analogue of the $S_{k}^{n}(\mathbf{a})$ as follows.
Definition 2.1. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, a $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ of $S_{k}^{n}(\mathbf{c})$ is defined by

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sigma[n-1, k-1]_{q}^{\beta, r}+\left([k \beta]_{q}+[r]_{q}\right) \sigma[n-1, k]_{q}^{\beta, r} \tag{2.10}
\end{equation*}
$$

where $\mathbf{c}$ is the sequence $(r, r+\beta, r+2 \beta, \ldots), \sigma[0,0]_{q}^{\beta, r}=1$, and $\sigma[n, k]_{q}^{\beta, r}=0$ for $n<k$ or $n, k<0$.

The numbers $\sigma[n, k]_{q}^{\beta, r}$ may be considered as a $q$-analogue of $S_{k}^{n}(\mathbf{c})$ since, when $q \rightarrow 1$,

$$
\begin{equation*}
[k \beta]_{q}+[r]_{q} \rightarrow k \beta+r \tag{2.11}
\end{equation*}
$$

and, hence, the recurrence relation in (2.10) will give the recurrence relation in ( $P 1$ ) for $S_{k}^{n}(\mathbf{c})$ where $\mathbf{c}$ is the sequence $(r, r+\beta, r+2 \beta, \ldots)$. This fact will also be verified in Section 3 (Remark 3.4).

The above triangular recurrence relation for the $q$-Stirling numbers of the second kind can easily be deduced from (2.10) by taking $\beta=1$ and $r=0$.

Clearly, using the initial conditions of $\sigma[n, k]_{q}^{\beta, r}$, we can have

$$
\begin{gather*}
\sigma[n, 0]_{q}^{\beta, r}=[r]_{q}^{n} \quad \forall n \geq 0,  \tag{2.12}\\
\sigma[n, n]_{q}^{\beta, r}=1, \quad \forall n \geq 0 .
\end{gather*}
$$

By repeated application of (2.10), we obtain the following theorem.
Theorem 2.2. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ satisfies the following vertical recurrence relation:

$$
\begin{equation*}
\sigma[n+1, k+1]_{q}^{\beta, r}=\sum_{j=k}^{n}\left([(k+1) \beta]_{q}+[r]_{q}\right)^{n-j} \sigma[j, k]_{q}^{\beta, r} \tag{2.13}
\end{equation*}
$$

with initial conditions $\sigma[0,0]_{q}^{\beta, r}=1$ and $\sigma[n, n]_{q}^{\beta, r}=1, \sigma[n, 0]_{q}^{\beta, r}=[r]_{q}^{n}$ for all $n \geq 0$.
Using the following notation

$$
\begin{equation*}
\left\{[r]_{q} \mid[\beta]_{q}\right\}_{k}=\prod_{j=0}^{n-1}\left([r]_{q}+[j \beta]_{q}\right), \quad\left\{[r]_{q} \mid[\beta]_{q}\right\}_{0}=1, \tag{2.14}
\end{equation*}
$$

we can now state the horizontal recurrence relation for $\sigma[n, k]_{q}^{\beta, r}$.
Theorem 2.3. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ satisfies the following horizontal recurrence relation:

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{j=0}^{n-k}(-1)^{j} \frac{\left\{[r]_{q} \mid[\beta]_{q}\right\}_{k+j+1}}{\left\{[r]_{q} \mid[\beta]_{q}\right\}_{k+1}} \sigma[n+1, k+j+1]_{q}^{\beta, r}, \tag{2.15}
\end{equation*}
$$

with initial condition $\sigma[0,0]_{q}^{\beta, r}=1$ and $\sigma[n, n]_{q}^{\beta, r}=1, \sigma[n, 0]_{q}^{\beta, r}=[r]_{q}^{n}$ for all $n \geq 0$.

Proof. To prove (2.15), we simply evaluate its right-hand side using (2.10) and obtain $\sigma[n, k]_{q}^{\beta, r}$.

It will be shown in Section 3 that

$$
\left[\begin{array}{l}
n  \tag{2.16}\\
k
\end{array}\right]_{q}=(q-1)^{n-k} \sigma[n, k]_{q}^{1, \log _{q} 2}
$$

By taking $\beta=1$ and $r=\log _{q} 2,(2.13)$ and (2.15) yield

$$
\begin{gather*}
{\left[\begin{array}{l}
n+1 \\
k+1
\end{array}\right]_{q}=\sum_{j=k}^{n}\left(q^{k+1}\right)^{n-j}\left[\begin{array}{l}
j \\
k
\end{array}\right]_{q}} \\
{\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}=\sum_{j=0}^{n-k}(-1)^{j} q^{j k+\binom{j+1}{2}}\left[\begin{array}{c}
n+1 \\
k+j+1
\end{array}\right]_{q}} \tag{2.17}
\end{gather*}
$$

which are exactly the recurrence relations obtained in [13]. When $q \rightarrow 1$, these further give the Hockey Stick identities.

## 3. Explicit Formulas and Generating Functions

The next theorem is analogous to that relation in (1.1). This is necessary in obtaining one of the explicit formulas for $\sigma[n, k]_{q}^{\beta, r}$ and the orthogonality and inverse relations of $\phi_{\alpha, \gamma}[n, k]_{q}$ and $\sigma[n, k]_{q}^{\beta, r}$.

Theorem 3.1. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ satisfies the following relation:

$$
\begin{equation*}
\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r}\left\langle t \mid[\beta]_{q}\right\rangle_{k}^{q}=\left(t+[r]_{q}\right)^{n} \tag{3.1}
\end{equation*}
$$

Proof. We proceed by induction on $n$. Clearly, (3.1) is true for $n=0$. Assume that it is true for $n>0$. Then using Definition 2.1,

$$
\begin{align*}
& \sum_{k=0}^{n+1} \sigma[n+1, k]_{q}^{\beta, r}\left\langle t \mid[\beta]_{q}\right\rangle_{k}^{q} \\
& \quad=\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r}\langle t|\left\langle[\beta]_{q}\right\rangle_{k+1}^{q}+\sum_{k=0}^{n}\left([k \beta]_{q}+[r]_{q}\right) \sigma[n, k]_{q}^{\beta, r}\left\langle t \mid[\beta]_{q}\right\rangle_{k}^{q}  \tag{3.2}\\
& \quad=\left(t+[r]_{q}\right) \sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r}\left\langle t \mid[\beta]_{q}\right\rangle_{k}^{q} \\
& \quad=\left(t+[r]_{q}\right)\left(t+[r]_{q}\right)^{n}=\left(t+[r]_{q}\right)^{n+1}
\end{align*}
$$

The new $q$-analogue of Newton's Interpolation Formula in [14] states that, for

$$
\begin{equation*}
f_{q}(x)=a_{0}+a_{1}\left[x-x_{0}\right]_{q}+\cdots+a_{m}\left[x-x_{0}\right]_{q}\left[x-x_{1}\right]_{q}\left[x-x_{m-1}\right]_{q} \tag{3.3}
\end{equation*}
$$

we have

$$
\begin{align*}
f_{q}(x)= & f_{q}\left(x_{0}\right)+\frac{\Delta_{q^{h}, h} f_{q}\left(x_{0}\right)\left[x-x_{0}\right]_{q}}{[1]_{q^{h}}![h]_{q}}+\frac{\Delta_{q^{h}, h}^{2} f_{q}\left(x_{0}\right)\left[x-x_{0}\right]_{q}\left[x-x_{1}\right]_{q}}{[2]_{q^{h}}![h]_{q}^{2}}  \tag{3.4}\\
& +\cdots+\frac{\Delta_{q^{h}, h}^{m} f_{q}\left(x_{0}\right)\left[x-x_{0}\right]_{q}\left[x-x_{1}\right]_{q} \ldots\left[x-x_{m-1}\right]_{q}}{[m]_{q^{h}}![h]_{q}^{m}}
\end{align*}
$$

where $x_{k}=x_{0}+k h, k=1,2, \ldots$ such that when $x_{0}=0$, this can be simplified as

$$
\begin{align*}
f_{q}(x)= & f_{q}(0)+\frac{\Delta_{q^{h}, h} f_{q}(0)[x]_{q}}{[1]_{q^{h}}![h]_{q}}+\frac{\Delta_{q^{h}, h}^{2} f_{q}(0)[x]_{q}[x-h]_{q}}{[2]_{q^{n}}![h]_{q}^{2}}  \tag{3.5}\\
& +\cdots+\frac{\Delta_{q^{h}, h}^{m} f_{q}(0)[x]_{q}[x-h]_{q} \ldots[x-(m-1) h]_{q}}{[m]_{q^{h}}![h]_{q}^{m}} .
\end{align*}
$$

Using (3.1) with $t=[x]_{q}$, we get

$$
\begin{equation*}
\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r}\left\langle[x]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}=\left([x]_{q}+[r]_{q}\right)^{n} \tag{3.6}
\end{equation*}
$$

which can be expressed further as

$$
\begin{equation*}
\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r} q^{\beta\binom{k}{2}}[x]_{q}[x-\beta]_{q} \cdots[x-(k-1) \beta]_{q}=\left([x]_{q}+[r]_{q}\right)^{n} \tag{3.7}
\end{equation*}
$$

Applying the above Newton's Interpolation Formula and the identity in [14]

$$
\Delta_{q, h}^{n} f(x)=\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n  \tag{3.8}\\
k
\end{array}\right]_{q} f(x+(n-k) h)
$$

we get

$$
\begin{align*}
\sigma[n, k]_{q}^{\beta, r} q^{\beta}\binom{k}{2} & =\frac{\Delta_{q^{\beta}, \beta}^{k} f_{q}(0)}{[k]_{q^{\beta}}![\beta]_{q}^{k}}  \tag{3.9}\\
& =\frac{1}{\left.[k]_{q^{\beta}!}!\beta\right]_{q}^{k}} \sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left([j \beta]_{q}+[r]_{q}\right)^{n} .
\end{align*}
$$

With $\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}=q^{\beta\binom{k}{2}}[k \beta]_{q}[(k-1) \beta]_{q} \cdots[\beta]_{q}=q^{\beta\binom{k}{2}}[k]_{q^{\beta}}![\beta]_{q}^{k}$, we obtain the following explicit formula.

Theorem 3.2. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ is equal to

$$
\sigma[n, k]_{q}^{\beta, r}=\frac{1}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k  \tag{3.10}\\
j
\end{array}\right]_{q^{\beta}}\left([j \beta]_{q}+[r]_{q}\right)^{n} .
$$

Remark 3.3. We can also prove Theorem 3.2 using the $q$-binomial inversion formula in (2.6). That is, by taking $t=[k \beta]_{q^{\prime}}$ (3.1) gives

$$
\begin{align*}
\left([k \beta]_{q}+[r]_{q}\right)^{n} & =\sum_{j=0}^{n} \sigma[n, j]_{q}^{\beta, r}\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{j}^{q} \\
& =\sum_{j=0}^{k}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left\{\frac{\sigma[n, j]_{q}^{\beta, r}\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{j}^{q}}{\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}}\right\} \tag{3.11}
\end{align*}
$$

Applying (2.6), we obtain

$$
\frac{\sigma[n, j]_{q}^{\beta, r}\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}}{\left[\begin{array}{l}
k  \tag{3.12}\\
k
\end{array}\right]_{q^{\beta}}}=\sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left([j \beta]_{q}+[r]_{q}\right)^{n} .
$$

This is precisely the explicit formula in Theorem 3.2.
Remark 3.4. Note that $\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q} \rightarrow k!\beta^{k},\left[\begin{array}{c}k \\ j\end{array}\right]_{q^{\beta}} \rightarrow\binom{k}{j}$, and $\left([j \beta]_{q}+[r]_{q}\right)^{n} \rightarrow(j \beta+r)^{n}$ as $q \rightarrow 1$. Thus, using property $(P 4), \sigma[n, k]_{q}^{\beta, r} \rightarrow S_{k}^{n}(\mathbf{c})$ as $q \rightarrow 1$. This implies that $\sigma[n, k]_{q}^{\beta, r}$ is a proper $q$-analogue of $S_{k}^{n}(\mathbf{c})$.

Now, using the explicit formula in Theorem 3.2, we obtain

$$
\begin{align*}
\sum_{n \geq 0} \sigma[n, j]_{q}^{\beta, r} \frac{t^{n}}{n!} & =\frac{1}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \sum_{n \geq 0}\left\{\sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left([j \beta]_{q}+[r]_{q}\right)^{n}\right\} \frac{t^{n}}{n!} \\
& =\frac{\sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left\{\sum_{n \geq 0}\left([j \beta]_{q}+[r]_{q}\right)^{n} t^{n} / n!\right\}}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \\
& =\frac{\sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k \\
j
\end{array}\right]_{q^{\beta}}\left\{\sum_{n \geq 0}\left(\sum_{i=0}^{n}\left(\left([r]_{q} t\right)^{i} / i!\right)\left(\left([j \beta]_{q} t\right)^{n-i} /(n-i)!\right)\right)\right\}}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} . \tag{3.13}
\end{align*}
$$

Applying Cauchy's formula for the product of two power series [3], we get

$$
\sum_{n \geq 0} \sigma[n, k]_{q}^{\beta, r} \frac{t^{n}}{n!}=\frac{1}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k  \tag{3.14}\\
j
\end{array}\right]_{q^{\beta}}\left\{\sum_{\lambda \geq 0} \frac{\left([r]_{q} t\right)^{\lambda}}{\lambda!} \sum_{\mu \geq 0} \frac{\left([j \beta]_{q} t\right)^{\mu}}{\mu!}\right\}
$$

Thus,

$$
\sum_{n \geq 0} \sigma[n, k]_{q}^{\beta, r} \frac{t^{n}}{n!}=\frac{1}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \sum_{j=0}^{k}(-1)^{k-j} q^{\beta}\binom{k-j}{2}\left[\begin{array}{c}
k  \tag{3.15}\\
j
\end{array}\right]_{q^{\beta}} e^{\left([j \beta]_{q}+[r]_{q}\right) t}
$$

Applying the above identity for $\Delta_{q, h}^{n} f$ to the function $f$ defined by

$$
\begin{equation*}
f(x)=\frac{e^{\left([x \beta]_{q}+[r]_{q}\right) t}}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}} \tag{3.16}
\end{equation*}
$$

we can further express the above generating function in terms of a $q$-difference operator. More precisely,

$$
\begin{equation*}
\sum_{n \geq 0} \sigma[n, k]_{q}^{\beta, r} \frac{t^{n}}{n!}=\left\{\Delta_{q}^{k}\left(\frac{e^{\left([x \beta]_{q}+[r]_{q}\right) t}}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}}\right)\right\}_{x=0} \tag{3.17}
\end{equation*}
$$

This is a kind of exponential generating function for $\sigma[n, k]_{q}^{\beta, r}$ which is included in the next theorem. Together with this, a rational generating function for $\sigma[n, k]_{q}^{\beta, r}$ is also stated in the
theorem that will be used to derive another explicit formula for $\sigma[n, k]_{q}^{\beta, r}$ in homogeneous symmetric function form.

Theorem 3.5. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ satisfies the exponential generating function

$$
\begin{equation*}
\Phi_{k}(t)=\sum_{n \geq 0} \sigma[n, k]_{q}^{\beta, r} \frac{t^{n}}{n!}=\left[\Delta_{q^{\beta}}^{k}\left(\frac{e^{\left([x \beta]_{q}+[r]_{q}\right) t}}{\left\langle[k \beta]_{q} \mid[\beta]_{q}\right\rangle_{k}^{q}}\right)\right]_{x=0} \tag{3.18}
\end{equation*}
$$

and the rational generating function

$$
\begin{equation*}
\psi_{k}(t)=\sum_{n \geq k} \sigma[n, k]_{q}^{\beta, r} t^{n}=\frac{t^{k}}{\prod_{j=0}^{k}\left(1-\left([j \beta]_{q}+[r]_{q}\right) t\right)} \tag{3.19}
\end{equation*}
$$

Proof. We are done with the proof of the first generating function. We are left to prove the second one and we are going to prove this by induction on $k$. For $k=0$, we have

$$
\begin{equation*}
\psi_{0}(t)=\sum_{n \geq 0} \sigma[n, 0]_{q}^{\beta, r} t^{n}=\sum_{n \geq 0}[r]_{q}^{n} t^{n}=\frac{1}{\left(1-[r]_{q} t\right)} \tag{3.20}
\end{equation*}
$$

With $k>0$ and using Definition 2.1, we obtain

$$
\begin{align*}
\psi_{k}(t) & =\sum_{n \geq k} \sigma[n, k]_{q}^{\beta, r} t^{n} \\
& =t \sum_{n \geq k} \sigma[n-1, k-1]_{q}^{\beta, r} t^{n-1}+\left([k \beta]_{q}+[r]_{q}\right) t \sum_{n \geq k} \sigma[n-1, k]_{q}^{\beta, r} t^{n-1}  \tag{3.21}\\
& =t \psi k-1(t)+\left([k \beta]_{q}+[r]_{q}\right) t \psi(t)
\end{align*}
$$

Hence,

$$
\begin{equation*}
\psi_{k}(t)=\frac{t}{1-\left([k \beta]_{q}+[r]_{q}\right) t} \psi_{k-1}(t) \tag{3.22}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\psi_{k}(t)=\frac{t^{k}}{\prod_{j=0}^{k}\left(1-\left([j \beta]_{q}+[r]_{q}\right) t\right)} \tag{3.23}
\end{equation*}
$$

The rational generating function in Theorem 3.5 can then be expressed as

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{s_{1}+s_{2}+\cdots+s_{k}=n-k} \prod_{j=0}^{k}\left([j \beta]_{q}+[r]_{q}\right)^{s_{j}} \tag{3.24}
\end{equation*}
$$

This sum may be written further as follows.
Theorem 3.6. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the explicit formula for $\sigma[n, k]_{q}^{\beta, r}$ in homogeneous symmetric function form is given by

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{0 \leq j_{1} \leq j_{2} \leq \cdots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k}\left(\left[j_{i} \beta\right]_{q}+[r]_{q}\right) . \tag{3.25}
\end{equation*}
$$

This explicit formula is necessary in giving combinatorial interpretation of $\sigma[n, k]_{q}^{\beta, r}$ in the context of 0-1 tableau. Note that when $\beta=1$ and $r=0$, Theorem 3.6 yields

$$
\begin{equation*}
\sigma[n, k]_{q}^{1,0}=\sum_{0 \leq j_{1} \leq j_{2} \leq \cdots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k}\left[j_{i}\right]_{q}=S_{q}[n, k] \tag{3.26}
\end{equation*}
$$

the $q$-Stirling numbers of the second kind [12]. Moreover, taking $\beta=1$ and $r=\log _{q} 2$, Theorem 3.6 reduces to

$$
\begin{align*}
\sigma[n, k]_{q}^{1, \log _{q} 2} & =\sum_{0 \leq j_{1} \leq j_{2} \leq \cdots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k}\left(\left[j_{i}\right]_{q}+\left[\log _{q} 2\right]_{q}\right)  \tag{3.27}\\
& =(q-1)^{k-n} \sum_{0 \leq j_{1} \leq j_{2} \leq \cdots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k} q^{j_{i}} .
\end{align*}
$$

Using the representation given in [15] for the $q$-binomial coefficients, we have

$$
\left[\begin{array}{l}
n  \tag{3.28}\\
k
\end{array}\right]_{q}=(q-1)^{n-k} \sigma[n, k]_{1 q}^{1, \log _{q} 2}
$$

This is the identity that we used in Section 2.

## 4. Combinatorial Interpretation of $\sigma[n, k]_{q}^{\beta, r}$

Definition 4.1 (see [15]). A 0-1 tableau is a pair $\varphi=(\lambda, f)$, where

$$
\begin{equation*}
\lambda=\left(\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{k}\right) \tag{4.1}
\end{equation*}
$$

| 0 | 0 | 0 |  | 1 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 |  |  |
| 1 | 0 | 0 |  |  |
| 0 | 1 |  |  |  |
| 0 |  |  |  |  |

Figure 1: The 0-1 tableau $\varphi=(\lambda, f)$.
is a partition of an integer $m$ and $f=\left(f_{i j}\right)_{1 \leq j \leq \lambda_{i}}$ is a "filling" of the cells of corresponding Ferrers diagram of the shape $\lambda$ with 0 's and 1 's, such that there is exactly one " 1 " in each column.

Using the partition $\lambda=(5,3,3,2,1)$, we can construct 60 distinct $0-1$ tableaux. Figure 1 below shows one of these tableaux with $f_{14}=f_{15}=f_{23}=f_{31}=f_{42}=1, f_{i j}=0$ elsewhere such that $1 \leq j \leq \lambda_{i}$.

Definition 4.2 (see [15]). An $A$-tableau is a list $\phi$ of column $c$ of a Ferrer's diagram of a partition $\lambda$ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A=\left(a_{i}\right)_{i \geq 0}$, a strictly increasing sequence of nonnegative integers.

Let $\omega$ be a function from the set of nonnegative integers $N$ to a ring K. Suppose $\Phi$ is an $A$-tableau with $r$ columns of lengths $|c| \leq h$. Then, we set

$$
\begin{equation*}
\omega_{A}(\Phi)=\prod_{c \in \Phi} \omega(|c|) . \tag{4.2}
\end{equation*}
$$

Note that $\Phi$ might contain a finite number of columns whose lengths are zero since $0 \in A=$ $\{0,1,2, \ldots, k\}$ and if $\omega(0) \neq 0$.

From this point onward, whenever an $A$-tableau is mentioned, it is always associated with the sequence $A=\{0,1,2, \ldots, k\}$.

We are now ready to mention the following theorem.
Theorem 4.3. Let $\omega: N \rightarrow K$ denote a function from $N$ to a ring $K$ (column weights according to length) which is defined by $\omega(|c|)=[|c| \beta]_{q}+[r]_{q}$ where $\beta$ and $\gamma$ are complex numbers, and $|c|$ is the length of column $c$ of an $A$-tableau in $T^{A}(k, n-k)$. Then

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{\phi \in T^{A}(k, n-k)} \prod_{c \in \phi} \omega(|c|) . \tag{4.3}
\end{equation*}
$$

Proof. This can easily be proved using Definition 4.2 and Theorem 3.6.
Now, we demonstrate simple combinatorics of 0-1 tableaux to obtain certain relation for $\sigma[n, k]_{q}^{\beta, r}$. To start with, we have, from Theorem 4.3,

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{\phi \in T^{A}(k, n-k)} \omega_{A}(\Phi), \tag{4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{A}(\Phi)=\prod_{c \in \phi}\left([|c| \beta]_{q}+[r]_{q}\right), \quad|c| \in\{0,1,2, \ldots, k\} . \tag{4.5}
\end{equation*}
$$

Substituting $j_{i}=|c|$, we obtain

$$
\begin{equation*}
\omega_{A}(\Phi)=\prod_{i=1}^{n-k}\left(\left[j_{i} \beta\right]_{q}+[r]_{q}\right), \quad j_{i} \in\{0,1,2, \ldots, k\} \tag{4.6}
\end{equation*}
$$

Let $[r]_{q}=c_{1}+c_{2}$ where $c_{1}=[r]_{q}-\left[r_{2}\right]_{q}$ and $c_{2}=\left[r_{2}\right]_{q}$ for some numbers $r_{1}$ and $r_{2}$. Then, with $\omega^{*}(j)=[j \beta]_{q}+c_{2}$, we have

$$
\begin{equation*}
\omega_{A}(\Phi)=\sum_{l=0}^{n-k} c_{1}^{n-k-l} \sum_{q_{1} \leq q_{2} \leq \cdots \leq q_{1}, q_{i} \in\left\{j_{1}, j_{2}, \ldots, j_{n-k}\right\}} \prod_{i=1}^{l} \omega^{*}\left(q_{i}\right) . \tag{4.7}
\end{equation*}
$$

Now, we are going to count the number of tableaux with $n-k$ columns such that $n-k-$ $r$ columns are of weight $c_{1}$ and $r$ columns are of weight $\omega^{*}\left(q_{i}\right), q_{i} \in\{0,1,2, \ldots, k\}$. Note that there are $\binom{n-k}{r}$ tableaux with $r$ columns whose lengths are taken from the lengths of the columns of $\Phi$. Since there is a one-to-one correspondence between weights $\omega\left(j_{i}\right)$ and $A$-tableaux, the number of $A$-tableaux $\Phi$ in $T^{A}(k, n-k)$ is equal to the number of possible multisets $\left\{j_{1}, j_{2}, \ldots, j_{n-k}\right\}$ with $j_{i}$ in $\{0,1,2, \ldots, k\}$. That is,

$$
\begin{equation*}
\left|T^{A}(k, n-k)\right|=\binom{n}{k} . \tag{4.8}
\end{equation*}
$$

Thus, for all $\Phi \in T^{A}(k, n-k)$, we can generate $\binom{n}{k}\binom{n-k}{r}$ tableaux with $r$ columns whose weights are $\omega^{*}\left(j_{i}\right), j_{i} \in\{0,1,2, \ldots, k\}$. However, there are only

$$
\begin{equation*}
\left|T^{A}(k, r)\right|=\binom{r+k}{r} \tag{4.9}
\end{equation*}
$$

distinct tableaux with $r$ columns whose lengths are in $\{0,1,2, \ldots, k\}$. Hence, every distinct tableau with $n-k$ columns, $r$ of which are of weight other than $c_{1}$, appears

$$
\begin{equation*}
\frac{\binom{n}{k}\binom{n-k}{r}}{\binom{r+k}{r}}=\binom{n}{r+k} \tag{4.10}
\end{equation*}
$$

times in the collection. Thus,

$$
\begin{equation*}
\sum_{\Phi \in T^{A}(k, n-k)} \omega_{A}(\Phi)=\sum_{r=0}^{n-k}\binom{n}{r+k} c_{1}^{n-k-r} \sum_{\phi \in \bar{B}_{r}} \prod_{c \in \phi} \omega^{*}(|c|) \tag{4.11}
\end{equation*}
$$

where $\bar{B}_{r}$ denotes the set of all tableaux $\psi$ having $r$ columns of weights $\omega^{*}\left(j_{i}\right)=\left[j_{i} \beta\right]_{q}+c_{2}$. Reindexing the double sum, we get

$$
\begin{equation*}
\sum_{\Phi \in T^{A}(k, n-k)} \omega_{A}(\Phi)=\sum_{j=k}^{n}\binom{n}{j} c_{1}^{n-j} \sum_{\phi \in \bar{B}_{j-k}} \prod_{\bar{c} \in \phi} \omega^{*}(|c|), \tag{4.12}
\end{equation*}
$$

where $\bar{B}_{j-k}$ is the set of all tableaux with $j-k$ columns of weights $\omega^{*}\left(j_{i}\right)=\left[j_{i} \beta\right]_{q}+c_{2}$ for each $i=1,2, \ldots, j-k$. Clearly, $\bar{B}_{j-k}=T^{A}(k, j-k)$. Therefore,

$$
\begin{equation*}
\sum_{\Phi \in T^{A}(k, n-k)} \omega_{A}(\Phi)=\sum_{j=k}^{n}\binom{n}{j} c_{1}^{n-j} \sum_{\phi \in T^{A}(k, j-k)} \omega_{A}(\phi) \tag{4.13}
\end{equation*}
$$

Applying Theorem 4.3 completes the proof of the following theorem.
Theorem 4.4. For nonnegative integers $n$ and $k$ and complex numbers $\beta$ and $r$, the $q$-analogue $\sigma[n, k]_{q}^{\beta, r}$ satisfies the following identity:

$$
\begin{equation*}
\sigma[n, k]_{q}^{\beta, r}=\sum_{j=k}^{n}\binom{n}{j} q^{(n-j) r_{2}}\left[r_{1}\right]_{q}^{n-j} \sigma[j, k]_{q}^{\beta, r_{2}} \tag{4.14}
\end{equation*}
$$

where $r=r_{1}+r_{2}$.
Taking $\beta=1, r_{2}=0$, and $r=r_{1}=\log _{q} 2$, Theorem 4.4 gives

$$
\begin{equation*}
(q-1)^{n-k} \sigma[n, k]_{q}^{1, \log _{q} 2}=\sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k} \sigma[j, k]_{q}^{1,0} \tag{4.15}
\end{equation*}
$$

Using (2.16) and (3.26), we obtain

$$
\left[\begin{array}{l}
n  \tag{4.16}\\
k
\end{array}\right]_{q}=\sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k} S_{q}[j, k]
$$

the Carlitz identity in [12]. Hence, we can consider the identity in Theorem 4.4 as a generalization of the above Carlitz identity.

## 5. Orthogonality and Inverse Relations

We notice that (1.4) can be written as

$$
\begin{equation*}
\sum_{k=0}^{m} F_{r,-a}(m, k) t^{k}=p_{m}^{\mathrm{a}}(t) \tag{5.1}
\end{equation*}
$$

Using (1.1), it can easily be shown that

$$
\begin{equation*}
\sum_{k=n}^{m} F_{r,-a}(m, k) S_{n}^{k}(\mathbf{a})=\sum_{k=n}^{m} S_{k}^{m}(\mathbf{a}) F_{r,-a}(k, n)=\delta_{m n} \tag{5.2}
\end{equation*}
$$

where $\delta_{m n}$ is the Kronecker delta defined by $\delta_{m n}=1$ if $m=n$, and $\delta_{m n}=0$ if $m \neq n$. Moreover, the following inverse relations hold:

$$
\begin{align*}
& f_{n}=\sum_{k=0}^{n} S_{k}^{n}(\mathbf{a}) g_{k} \Longleftrightarrow g_{n}=\sum_{k=0}^{n} F_{r,-a}(n, k) f_{k}  \tag{5.3}\\
& f_{k}=\sum_{n \geq k} S_{k}^{n}(\mathbf{a}) g_{n} \Longleftrightarrow g_{k}=\sum_{n \geq k} F_{r,-a}(n, k) f_{n} \tag{5.4}
\end{align*}
$$

Relation (5.2) is exactly the orthogonality relation for $r$-Whitney numbers that appeared in [6]. Consequently, the generating functions in $(P 2)$ and $(P 3)$ can be transformed, respectively, using (5.4) into the following identities:

$$
\begin{align*}
& \sum_{n \geq k} F_{r,-a}(n, k) \frac{k!}{x^{k}} \frac{1}{r^{n} n!} e^{a x}\left(e^{r x}-1\right)^{n}=1 \\
& \sum_{n \geq k} F_{r,-a}(n, k) \frac{x^{n-k}}{\prod_{j=0}^{n}(1-(r j+a) x)}=1 \tag{5.5}
\end{align*}
$$

which will reduce to the following interesting identities for $F_{\alpha, r}(n, k)$ when $x=1$ :

$$
\begin{gather*}
\sum_{n \geq k} \frac{F_{\alpha, \gamma}(n, k) k!\left(e^{\alpha}-1\right)^{n}}{\alpha^{n} n!e^{r}}=1,  \tag{5.6}\\
\sum_{n \geq k} \frac{F_{\alpha, \gamma}(n, k)}{(1+\gamma \mid \alpha)_{n+1}}=1
\end{gather*}
$$

Note that the number $F_{\alpha, \gamma}(n, k)$ can be expressed in terms of the unified generalization of Stirling numbers by Hsu and Shiue [16] as $F_{\alpha, \gamma}(n, k)=S(n, k ; \alpha, 0, \gamma)$. Hence, the identity in (5.6) coincides with the identity in [17, Theorem 9] by taking $x=1+\gamma$.

Parallel to (5.2), (5.3), and (5.4), we will establish in this section the orthogonality and inverse relations of $\phi_{\alpha, \gamma}[n, k]_{q}$ and $\sigma[n, k]_{q}^{\beta, r}$.

To derive the orthogonality relation for $\phi_{\alpha, \gamma}[n, k]_{q}$ and $\sigma[n, k]_{q}^{\beta, r}$, we need to rewrite first (1.5) and (3.1). By taking $\gamma=\log _{q}\left(2-q^{r}\right)$, (1.5) gives

$$
\begin{equation*}
\sum_{k=0}^{n} \phi_{\alpha, \log _{q}\left(2-q^{r}\right)}[n, k]_{q} t^{k}=\left\langle t-[r]_{q} \mid[\alpha]\right\rangle_{n^{\prime}}^{q} \tag{5.7}
\end{equation*}
$$

and, by replacing $t$ with $t-[r]_{q^{\prime}}$ (3.1) yields

$$
\begin{equation*}
\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r}\left\langle t-[r]_{q} \mid[\beta]\right\rangle_{k}^{q}=t^{n} \tag{5.8}
\end{equation*}
$$

Using (5.8), (5.7) can be expressed as

$$
\begin{align*}
\left\langle t-[r]_{q} \mid[\beta]\right\rangle_{m}^{q} & =\sum_{k=0}^{m} \phi_{\beta, \log _{q}\left(2-q^{r}\right)}[m, k]_{q}\left\{\sum_{n=0}^{k} \sigma[k, n]_{q}^{\beta, r}\left\langle t-[r]_{q} \mid[\beta]_{q}\right\rangle_{n}^{q}\right\}  \tag{5.9}\\
& =\sum_{n=0}^{m}\left\{\sum_{k=n}^{m} \phi_{\beta, \log _{q}\left(2-q^{r}\right)}[m, k]_{q} \sigma[k, n]_{q}^{\beta, r}\right\}\left\langle t-[r]_{q} \mid[\beta]_{q}\right\rangle_{n}^{q} .
\end{align*}
$$

Thus

$$
\begin{equation*}
\sum_{k=n}^{m} \phi_{\beta, l_{0}\left(2-q^{r}\right)}[m, k]_{q} \sigma[k, n]_{q}^{\beta, r}=\delta_{m n} \quad(m \geq n) . \tag{5.10}
\end{equation*}
$$

Theorem 5.1. For nonnegative integers $m, n$, and $k$ and complex numbers $\beta$ and $r$, the following orthogonality relation holds:

$$
\begin{equation*}
\sum_{k=n}^{m} \phi_{\beta, \bar{r}}[m, k]_{q} \sigma[k, n]_{q}^{\beta, r}=\sum_{k=n}^{m} \sigma[m, k]_{q}^{\beta, r} \phi_{\beta, \bar{r}}[k, n]_{q}=\delta_{m n} \quad(m \geq n), \tag{5.11}
\end{equation*}
$$

where $\bar{r}=\log _{q}\left(2-q^{r}\right)$.
Remark 5.2. It can easily be shown that $\bar{r}=\log _{q}\left(2-q^{r}\right) \rightarrow-r$ as $q \rightarrow 1$. This implies that $\phi_{\beta, \bar{r}}[m, k]_{q} \rightarrow F_{\beta,-r}(m, k)$ as $q \rightarrow 1$. Since $\sigma[k, n]_{q}^{\beta, r} \rightarrow S_{n}^{k}(\mathbf{c})$ as $q \rightarrow 1$, (5.11) yields (5.2) easily.

Remark 5.3. Let $M_{1}$ and $M_{2}$ be two $n \times n$ matrices whose entries are $\phi_{\beta, \bar{r}}[i, j]_{q}$ and $\sigma[i, j]_{q}^{\beta, r}$, respectively. That is, $M_{1}=\left(\phi_{\beta, \bar{r}}[i, j]_{q}\right)_{0 \leq i, j \leq n}$ and $M_{2}=\left(\sigma[i, j]_{q}^{\beta, r}\right)_{0 \leq i, j \leq n}$. Then using Theorem 5.1, $M_{1} M_{2}=M_{2} M_{1}=I_{n}$, the identity matrix of order $n$. This implies that $M_{1}$ and $M_{2}$ are orthogonal matrices.

Using the orthogonality relation in Theorem 5.1, we can easily prove the following inverse relation.

Theorem 5.4. For nonnegative integers $m, n$, and $k$, and complex numbers $\beta$ and $r$, the following inverse relation holds:

$$
\begin{equation*}
f_{n}=\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r} g_{k} \Longleftrightarrow g_{n}=\sum_{k=0}^{n} \phi_{\beta, \bar{r}}[n, k]_{q} f_{k}, \tag{5.12}
\end{equation*}
$$

where $\bar{r}=\log _{q}\left(2-q^{r}\right)$.

Proof. Given $f_{n}=\sum_{k=0}^{n} \sigma[n, k]_{q}^{\beta, r} g_{k}$, we have

$$
\begin{align*}
\sum_{k=0}^{n} \phi_{\beta, \bar{r}}[n, k]_{q} f_{k} & =\sum_{k=0}^{n} \phi_{\beta, \bar{r}}[n, k]_{q}\left\{\sum_{j=0}^{k} \sigma[k, j]_{q}^{\beta, r} g_{j}\right\} \\
& =\sum_{j=0}^{k}\left\{\sum_{k=j}^{n} \phi_{\beta, \bar{r}}[n, k]_{q} \sigma[k, j]_{q}^{\beta, r}\right\} g_{j}  \tag{5.13}\\
& =\sum_{j=0}^{k} \delta_{n j} g_{j}=g_{n} .
\end{align*}
$$

The converse can be shown similarly.
One can easily prove the following inverse relation.
Theorem 5.5. For nonnegative integers $m, n$, and $k$ and complex numbers $\beta$ and $r$, the following inverse relation holds:

$$
\begin{equation*}
f_{k}=\sum_{n=0}^{\infty} \sigma[n, k]_{q}^{\beta, r} g_{n} \Longleftrightarrow g_{k}=\sum_{n=0}^{\infty} \phi_{\beta, \bar{r}}[n, k]_{q} f_{n} \tag{5.14}
\end{equation*}
$$

where $\bar{r}=\log _{q}\left(2-q^{r}\right)$.
Remark 5.6. The exponential and rational generating functions in Theorem 3.5 can be transformed into the following identities for the $q$-analogue of $F_{\alpha, \gamma}(n, k)$ :

$$
\begin{align*}
& \sum_{n \geq 0} \phi_{\beta, \bar{r}}[n, k]_{q} \frac{k!}{t^{k}} \Delta_{q}^{n}\left(\frac{e^{\left([x \beta]_{q}+[r]_{q}\right) t}}{\left\langle[n \beta]_{q} \mid[\beta]\right\rangle_{n}^{q}}\right)_{x=0}=1,  \tag{5.15}\\
& \sum_{n \geq 0} \phi_{\beta, \bar{r}}[n, k]_{q} \frac{t^{n-k}}{\prod_{j=0}^{n}\left(1-\left([j \beta]_{q}+[r]_{q}\right) t\right)}=1,
\end{align*}
$$

when $q \rightarrow 1$, (5.15) will exactly give (5.5), respectively.

## Acknowledgments

The authors wish to thank the referees for reading and evaluating the paper. This research was partially funded by the Commission on Higher Education-Philippines and Mindanao State University-Main Campus, Marawi City, Philippines.

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