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Research Article

A Generalization for *n*-Cocycles

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We will give generalized definitions called type II n-cocycles and weak quasi-bialgebra and also show properties of type II n-cocycles and some results about weak quasi-bialgebras, for instance, construct a new structure of tensor product algebra over a module algebra on weak quasi-bialgebras.

1. Introduction

We will introduce a new generalized definition called type II n-cocycle; the namely, relax the invertible condition of associator in n-cocycle up to adding an associator satisfies all forms in definition of n-cocycle together with the original associator, and both need not to be invertible for each other; then we give examples to illustrate it clearly. Majid have shown many results about n-cocycle in [1], and we obtain some results including cohomologous concept through this new definition, main properties of type II n-cocycles, and its simple classification.

It is well known that quasi-bialgebras and quasi-Hopf algebras play important roles in quantum group theory, and these concepts were introduced by Drinfel'd in [2] who relaxed the coassociative law of Δ up to conjugation. In this paper, we will show a new definition called weak quasi-bialgebra, a generalization of quasi-bialgebras, and there are simple examples to illustrate. Authors show results for weak quasi-bialgebras in place of quasi-bialgebras (cf. [1, 3]), including that there exists an algebra structure on $A \otimes H$, a generalization of the algebra product in [3], where H is a weak quasi-bialgebra and A is a H-module algebra.

We follow all the notation and conventions in [1], throughout the paper. In the following, k will be a fixed field throughout, and all algebras, coalgebras, vector spaces, and so forth are over k automatically unless specified. We recall definitions as follows.

Definition 1.1. For any bialgebra, if there is an invertible element $\theta \in H^{\otimes n}$ such that

$$\partial\theta = (\Delta_0\theta\Delta_2\theta\cdots\Delta_s\theta)(\Delta_1\theta\Delta_3\theta\cdots\Delta_t\theta) = 1, \tag{1.1}$$

where integers s and t are max even number and max odd number, respectively, in $\{0, 1, \ldots, n\}$, we call θ an n-cocycle. If $\epsilon_i(\theta) = 1$ $(0 \leqslant i \leqslant n)$, then the cocycle is counital. We define $\Delta_i = id \otimes \cdots \otimes \Delta \otimes \cdots \otimes id$, $\epsilon_i = id \otimes \cdots \otimes \epsilon \otimes \cdots \otimes id$ $(1 \leqslant i \leqslant n)$,$\Delta_0 = 1 \otimes \theta$, and $\Delta_{n+1} = \theta \otimes 1$.

Definition 1.2. Let H be a k-algebra with unit and homomorphisms $\Delta : H \to H \otimes H, \epsilon : H \to k$. If there exists a counital 3-cocycle $\Phi \in H^{\otimes 3}$ rendering that, for all $h \in H$,

$$\begin{gathered}(id \otimes \Delta)\Delta h = \Phi(\Delta \otimes id)(\Delta h)\Phi^{-1}, \\ (\epsilon \otimes id)\Delta h = 1_k \otimes h, \qquad (id \otimes \epsilon)\Delta h = h \otimes 1_k.\end{gathered} \tag{1.2}$$

Then H is called a quasi-bialgebra together with coproduct Δ and counit ϵ, and call Φ an associator.

We will denote the tensor components of Φ, ϕ with big and small letters, respectively, for instance,

$$\begin{gathered}\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum Y^1 \otimes Y^2 \otimes Y^3 = \sum Z^1 \otimes Z^2 \otimes Z^3 \text{ etc.}, \\ \phi = \sum x^1 \otimes x^2 \otimes x^3 = \sum y^1 \otimes y^2 \otimes y^3 = \sum z^1 \otimes z^2 \otimes z^3 \text{ etc.},\end{gathered} \tag{1.3}$$

where x^i is the ith factor.

Definition 1.3. Let H be a quasi-bialgebra and A a vector space. If A has a multiplication and the unit 1_A obeying that

$$\begin{gathered}a(bc) = \sum\left(X^1 \cdot a\right)\left[\left(X^2 \cdot b\right)\left(X^3 \cdot c\right)\right], \\ h \cdot (ab) = \sum(h_1 \cdot a)(h_2 \cdot b), \qquad h \cdot 1_A = \epsilon(h)1_A,\end{gathered} \tag{1.4}$$

for any $a, b, c \in A$ and $h \in H$, where $\cdot :< H \otimes A \to A$ is the H-module structure map of A, then say A is a left H-module algebra.

In bialgebras, the composition of any two algebra homomorphisms Δ_i, Δ_j satisfies the equality $\Delta_i\Delta_j = \Delta_{j+1}\Delta_i$ for all $i \leq j$. We will use this equation frequently.

2. Type II Cocycles and Weak Quasi-Bialgebras

Definition 2.1. An associative algebra H with unit over a commutative ring R called a fake bialgebra, if there are two algebra homomorphisms $\Delta : H \to H \otimes H$ and $\epsilon : H \to R$.

Definition 2.2. Let H be a fake bialgebra, and elements $\Phi, \phi \in H^{\otimes n}$. Denote

$$\begin{aligned} \partial(\Phi,\phi) &= (\Delta_0\Phi\Delta_2\Phi\cdots\Delta_s\Phi)\big(\Delta_1\phi\Delta_3\phi\cdots\Delta_t\phi\big), \\ \chi(\Phi,\phi) &= (\Delta_t\Phi\Delta_{t-2}\Phi\cdots\Delta_1\Phi)\big(\Delta_s\phi\Delta_{s-2}\phi\cdots\Delta_0\phi\big), \end{aligned} \tag{2.1}$$

where integers s and t in $\{0,1,\ldots,n\}$ are max even number and max odd number, respectively. If there exists an element $\psi(\Phi,\phi) \in H^{\otimes n+1}$ with $\psi(\Phi,\phi)\partial(\Phi,\phi) = \chi(\Phi,\phi)$, which makes $\partial(\Phi,\phi)$ and Φ, ϕ satisfy all equalities that one side of equalities is not a single item at least, similar to (1.1). Then call the pair (Φ,ϕ) a type II n-cocycle for H and denote it by $(\Phi,\phi)_n^{\mathrm{II}}$. The cocycle $(\Phi,\phi)_n^{\mathrm{II}}$ is called counital, if both Φ and ϕ are counital.

There is a nature way to define type I n-cocycle $(\Phi,\phi)_n^{\mathrm{I}}$ similarly. If we require the type II n-cocycle $(\Phi,\phi)_n^{\mathrm{II}}$ for H to satisfy all transformations of (1.1), but each side of formulas must have one item at least, then the type II n-cocycle $(\Phi,\phi)_n^{\mathrm{II}}$ is called type I n-cocycle and denoted by $(\Phi,\phi)_n^{\mathrm{I}}$.

We write ∂, ψ, and χ briefly for $\partial(\Phi,\phi)$, $\psi(\Phi,\phi)$, and $\chi(\Phi,\phi)$ without confusions, respectively. To clarify a new definition above, we give simple examples on a fake bialgebra H. In the following, we discuss type II n-cocycles $(\Phi,\phi)_n^{\mathrm{II}}$ only.

Example 2.3. Cocycle $(\Phi,\phi)_1^{\mathrm{II}}$ means both Φ and ϕ are in H, obeying that

$$\partial\Delta_1\Phi = \Delta_0\Phi\Delta_2\Phi, \qquad \Delta_0\phi\partial = \Delta_2\Phi\Delta_1\phi. \tag{2.2}$$

And there is $\psi \in H^{\otimes 2}$ such that

$$\Delta_2\phi\Delta_0\phi = \Delta_1\phi\psi, \qquad \Delta_1\Phi\Delta_2\phi = \psi\Delta_0\Phi, \tag{2.3}$$

where $\partial = \Delta_0\Phi\Delta_2\Phi\Delta_1\phi$.

Example 2.4. Cocycle $(\Phi,\phi)_3^{\mathrm{II}}$ and $\partial = \Delta_0\Phi\Delta_2\Phi\Delta_4\Phi\Delta_1\phi\Delta_3\phi$, $\psi \in H^{\otimes 4}$, where $\Phi, \phi \in H^{\otimes 3}$, satisfy that

$$\begin{gathered} \partial\Delta_3\Phi\Delta_1\Phi = \Delta_0\Phi\Delta_2\Phi\Delta_4\Phi, \qquad \partial\Delta_3\Phi\Delta_1\Phi\Delta_4\phi = \Delta_0\Phi\Delta_2\Phi, \\ \partial\Delta_3\Phi = \Delta_0\Phi\Delta_2\Phi\Delta_4\Phi\Delta_1\phi, \qquad \big(\Delta_0\phi\big)\partial\Delta_3\Phi\Delta_1\Phi = \Delta_2\Phi\Delta_4\Phi, \text{ etc.}, \\ \Delta_3\Phi\Delta_1\Phi = \psi\Delta_0\Phi\Delta_2\Phi\Delta_4\Phi, \qquad \Delta_3\Phi\Delta_1\Phi\Delta_4\phi = \psi\Delta_0\Phi\Delta_2\Phi, \\ \Delta_3\Phi\Delta_1\Phi\Delta_4\phi\Delta_2\phi = \psi\Delta_0\Phi, \qquad \Delta_1\Phi\Delta_4\phi\Delta_2\phi = \big(\Delta_3\phi\big)\psi\Delta_0\Phi, \text{ etc.} \end{gathered} \tag{2.4}$$

Observing examples, we can see that ∂ and Φ are replaced by ψ and ϕ, respectively, after moving to a corresponding place in the other side of equations, and vice versa. Obviously, n-cocycle θ must be a type II 2-cocycle $(\theta,\theta^{-1})_n^{\mathrm{II}}$ and $\partial(\theta,\theta^{-1}) = \psi(\theta,\theta^{-1}) = 1$.

Example 2.5. Let A be an associative algebra with an idempotent $q \in A$ over a field k. Define $\Delta : A \to A \otimes A$ by $\Delta(a) = a \otimes a$ and $\epsilon : A \to k$ by $\epsilon(a) = 0_k$, for all $a \in A$. It is clear that A is a fake bialgebra. We set $\Phi = \phi = q$ and $\psi = q \otimes q$, then $\partial = \chi = q \otimes q$. It is easy to check that (Φ,ϕ) is a type II 1-cocycle.

Proposition 2.6. *Let $(\Phi, \phi)_n^{\mathrm{II}}$ be a cocycle for a fake bialgebra H, and denote*

$$\partial = (\Delta_0\Phi\Delta_2\Phi \cdots \Delta_s\Phi)\big(\Delta_1\phi\Delta_3\phi \cdots \Delta_t\phi\big),$$
$$\chi = (\Delta_t\Phi\Delta_{t-2}\Phi \cdots \Delta_1\Phi)\big(\Delta_s\phi\Delta_{s-2}\phi \cdots \Delta_0\phi\big), \tag{2.5}$$

where integers s and t in $\{0, 1, \ldots, n\}$ are max even number and max odd number, respectively. Then one has the following.

(1) $\chi = \psi\Delta_0(\Phi\phi) = \Delta_t(\Phi\phi)\psi = (\Delta_t\Phi\Delta_{t-2}\Phi \cdots \Delta_1\Phi)(\Delta_1\phi\Delta_3\phi \cdots \Delta_t\phi) = \chi\partial = \chi^2$, *and* $\partial = \Delta_0(\Phi\phi)\partial = \partial\Delta_t(\Phi\phi) = (\Delta_0\Phi\Delta_2\Phi \cdots \Delta_s\Phi)(\Delta_s\phi\Delta_{s-2}\phi \cdots \Delta_0\phi) = \partial\chi = \partial^2$.

(2) *If ∂ is commutative with χ, then $\chi = \partial$. Especially, if either ∂ or χ is zero, then the other one is zero too. On the other hand, if either of elements ∂ and χ is not zero, then the rest elements in set $\{\partial,\ \chi,\ \psi\}$ are not zero.*

(3) *If $\Delta_0(\Phi\phi) - 1$ is a left unit and Φ is not a right zero divisor, then $\partial = \chi = \psi = 0$.*

(4) *If Φ has a right inverse Φ_R^{-1}, so do ∂, ψ, and χ. Similarly, if ϕ has a left inverse ϕ_L^{-1}, so do ∂, ψ, and χ.*

Proof. (1) We obtain that $\chi = \Delta_t(\Phi\phi)\psi$ by

$$\big(\Delta_t\phi\big)\psi = \Delta_{t-2}\Phi \cdots \Delta_3\Phi\Delta_1\Phi\Delta_s\phi \cdots \Delta_2\phi\Delta_0\phi \tag{2.6}$$

and $\chi = \psi\Delta_0(\Phi\phi)$ by

$$\psi\Delta_0\Phi = \Delta_t\Phi \cdots \Delta_3\Phi\Delta_1\Phi\Delta_s\phi \cdots \Delta_4\phi\Delta_2\phi, \tag{2.7}$$

since Δ_i is a homomorphism. And

$$\chi = \psi\partial = \big(\psi\Delta_0\Phi \cdots \Delta_s\Phi\big)\Delta_1\phi \cdots \Delta_t\phi = (\Delta_t\Phi \cdots \Delta_1\Phi)\big(\Delta_1\phi \cdots \Delta_t\phi\big). \tag{2.8}$$

Analogously, we have that

$$\Delta_0\big(\Phi\phi\big)\partial = \Delta_0\Phi\big(\big(\Delta_0\phi\big)\partial\big) = \Delta_0\Phi\big(\Delta_2\Phi \cdots \Delta_s\Phi\Delta_1\phi \cdots \Delta_t\phi\big) = \partial,$$
$$\partial\Delta_t\big(\Phi\phi\big) = (\partial\Delta_t\Phi)\Delta_t\phi = \big(\Delta_0\Phi \cdots \Delta_s\Phi\Delta_1\phi \cdots \Delta_{t-2}\phi\big)\Delta_t\phi = \partial. \tag{2.9}$$

Then, we get easily that $\chi = \psi\partial = \psi\Delta_0(\Phi\phi)\partial = \chi\partial$ and

$$\begin{aligned}\partial\chi &= (\partial\Delta_t\Phi\Delta_{t-2}\Phi \cdots \Delta_1\Phi)\Delta_1\phi \cdots \Delta_{t-2}\phi\Delta_t\phi \\ &= \Delta_0\Phi\Delta_2\Phi \cdots \Delta_s\Phi\Delta_1\phi \cdots \Delta_{t-2}\phi\Delta_t\phi = \partial.\end{aligned} \tag{2.10}$$

Finally, there is the equality that

$$\begin{aligned}\partial = \partial\chi &= (\partial\Delta_t\Phi\Delta_{t-2}\Phi \cdots \Delta_1\Phi)\big(\Delta_s\phi\Delta_{s-2}\phi \cdots \Delta_0\phi\big) \\ &= (\Delta_0\Phi\Delta_2\Phi \cdots \Delta_s\Phi)\big(\Delta_s\phi\Delta_{s-2}\phi \cdots \Delta_0\phi\big).\end{aligned} \tag{2.11}$$

We, last, compute that

$$\partial^2 = (\partial\chi)\partial = \partial(\chi\partial) = \partial\chi = \partial, \qquad \chi^2 = (\chi\partial)\chi = \chi(\partial\chi) = \chi\partial = \chi. \tag{2.12}$$

Therefore ∂ and χ are idempotent.

(2) Obviously, we get this by statement (1). Let $\partial \neq 0$ and assume $\chi = 0$, from the previous part, that yields to $\partial = 0$ contradicting $\partial \neq 0$. Therefore χ must be zero, and $\psi = 0$ for the same reason.

(3) The equality $(\Delta_0(\Phi\phi) - 1)\partial = 0$ suggests that $\partial = 0$, and then $\chi = 0$. It is clear that $\Delta_0\Phi \cdots \Delta_s\Phi = 0$ because $\partial\Delta_t\Phi \cdots \Delta_1\Phi = \Delta_0\Phi \cdots \Delta_s\Phi$. In addition, $\Delta_t\Phi \cdots \Delta_3\Phi\Delta_1\Phi = \psi\Delta_0\Phi\Delta_2\Phi \cdots \Delta_s\Phi = 0$ implies that

$$\psi\Delta_0\Phi = \Delta_t\Phi \cdots \Delta_3\Phi\Delta_1\Phi\Delta_s\phi \cdots \Delta_4\phi\Delta_2\phi = 0. \tag{2.13}$$

As a result, we have $\psi = 0$ if Φ is not a right zero divisor.

(4) It is easy to obtain that $\partial\Delta_t\Phi \cdots \Delta_3\Phi\Delta_1\Phi\Delta_s\Phi_R^{-1} \cdots \Delta_2\Phi_R^{-1}\Delta_0\Phi_R^{-1} = 1$ and $\psi\Delta_0\Phi \cdots \Delta_s\Phi\Delta_1\Phi_R^{-1} \cdots \Delta_t\Phi_R^{-1} = 1$ as $\partial\Delta_t\Phi \cdots \Delta_1\Phi = \Delta_0\Phi \cdots \Delta_s\Phi$ and $\psi\Delta_0\Phi \cdots \Delta_s\Phi = \Delta_t\Phi \cdots \Delta_1\Phi$, respectively. But then $\chi = \psi\partial$ and χ has a right inverse. Likewise, we can prove the rest part. □

Furthermore, if ∂ or χ has a one-side inverse, it makes sense that $\partial = \chi = \psi = 1$ since both ∂ and χ are idempotent. We also have that $\Phi\phi = 1$ which indicates Φ is a left unit and ϕ a right unit, by $\Delta_0(\Phi\phi)\partial = \partial$ if $\partial = 1$. Hence ∂ and χ cannot be anything but the identity element if one of them is a one-side unit.

Corollary 2.7. *The following statements are equivalent.*

(1) Φ *has a right inverse.*

(2) $\partial = \chi = \psi = 1$.

(3) ϕ *has a left inverse.*

(4) ∂ *is a one-side unit.*

(5) χ *is a one-side unit.*

(6) ψ *has a left inverse.*

Proof. (Sketch of Proof).
Check by (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1). □

Equality $(\Delta_0(\Phi\phi) - 1)\partial = 0$ suggests that classification of ∂ is divided into three types. The first type $\partial = 0$, and the second $\partial = 1$ if $\Delta_0(\Phi\phi) = 1$, that is, $\Phi\phi = 1$. The last one is that ∂ is a right zero divisor.

Example 2.8. In algebra $\mathbb{Z}_6$ over the integer ring $\mathbb{Z}$, we define $\Delta : \mathbb{Z}_6 \to \mathbb{Z}_6 \otimes \mathbb{Z}_6$ given by $\Delta\overline{3} = \overline{3} \otimes \overline{3}$ and $\Delta\overline{x} = \overline{x} \otimes \overline{3}$ for any $\overline{x} \in \mathbb{Z}_6 - \{\overline{3}\}$, and $\epsilon : \mathbb{Z}_6 \to \mathbb{Z}$ by $\epsilon(\overline{y}) = y$ for all $\overline{y} \in \mathbb{Z}_6$ such that $(\mathbb{Z}_6, \Delta, \epsilon)$ is a fake bialgebra. Set $\Phi = \phi = \overline{3} \in \mathbb{Z}_6$ such that $\partial = \overline{3} \otimes \overline{3}$. The product of any two elements in the set $\{\Delta_i\Phi, \partial\}$ $(i = 0, 1, 2)$ equals $\overline{3} \otimes \overline{3}$, obviously. We also set $\psi = \partial$; then, it is easy to prove that $(\Phi, \phi)_1^{\mathrm{II}}$ is a cocycle and a right zero-divisor ∂.

Proposition 2.9. *Let H be a fake bialgebra with counital law of ϵ. If $\Phi, \phi \in H$ and $\epsilon(\Phi) = 1$ ($\epsilon(\phi) = 1$, resp.), then $\partial(\Phi, \phi)$ ($\chi(\Phi, \phi)$, resp.) is counital if and only if $\Phi\phi = 1$ ($\phi\Phi = 1$, resp.).*

Proof. Since that $\partial(\Phi, \phi) = \Delta_0\Phi\Delta_2\Phi\Delta_1\phi = (\Phi \otimes \Phi)\Delta\phi$, we have $\epsilon_i\partial(\Phi, \phi) = \epsilon_i(\Phi \otimes \Phi)\epsilon_i\Delta\phi = \epsilon(\Phi)\Phi\phi = \Phi\phi$ rendering that $\epsilon_i\partial(\Phi, \phi) = 1$ if and only if $\Phi\phi = 1$, where $i = 1, 2$. □

Proposition 2.10. *Let H be a fake bialgebra with coassociative law of Δ and there are elements Φ, ϕ in H. If $\partial(\Phi, \phi)$ obeys $\partial\Delta_1\Phi = \Delta_0\Phi\Delta_2\Phi$, then $\Delta_3\partial\Delta_1\partial = \Delta_0\partial\Delta_2\partial$. Especially, ∂ is a 2-cocycle if ∂ is invertible if H is a bialgebra.*

Proof. To obtain the result, we observe that

$$\begin{aligned}
\Delta_3\partial\Delta_1\partial &= (\partial \otimes 1)(\Delta \otimes id)\partial = (\partial \otimes 1)(\Delta \otimes id)\big((\Phi \otimes \Phi)\Delta_1\phi\big) \\
&= (\partial \otimes 1)(\Delta\Phi \otimes \Phi)\big(\Delta_1\Delta_1\phi\big) = (\partial \otimes 1)(\Delta\Phi \otimes \Phi)\big(\Delta_2\Delta_1\phi\big) \\
&= (\partial\Delta\Phi \otimes \Phi)\big(\Delta_2\Delta_1\phi\big) = (\Phi \otimes \Phi \otimes \Phi)\Delta_2\Delta_1\phi \\
&= (1 \otimes \partial)(\Phi \otimes \Delta\Phi)\Delta_2\Delta_1\phi = (1 \otimes \partial)\Delta_2(\Phi \otimes \Phi)\ \Delta_2\Delta_1\phi \\
&= \Delta_0\partial\Delta_2\partial.
\end{aligned} \tag{2.14}$$

□

We have known that $\theta^{\gamma} = (\gamma \otimes \gamma)\theta\Delta\gamma^{-1}$ is cohomologous to θ for a bialgebra H if θ is a counital 2-cocycle, which was mentioned by Majid in [1]. Let H be a bialgebra, $\Phi, \phi \in H$ and cocycle $(\sigma, \delta)_2^{\mathrm{II}}$ for H. Denote that $\sigma^{(\Phi,\phi)} = \Delta_0\Phi\Delta_2\Phi\sigma\delta\sigma\Delta_1\phi$ and $\delta^{(\Phi,\phi)} = \Delta_1\Phi\delta\sigma\delta\Delta_2\phi\Delta_0\phi$. Then we have the following.

Proposition 2.11. *If equality $\partial(\sigma, \delta)\Delta_2\delta\Delta_0\delta\Delta_3\sigma = \Delta_1\delta$ holds and $\partial(\sigma, \delta)$ is a commutative element in set $\{\Delta_1\sigma, \Delta_3\delta, \Delta_3\sigma\}$, and $1 \otimes \Delta(\phi\Phi)(= \Delta(\phi\Phi) \otimes 1)$ commutes with any element in set $\{\Delta_1\sigma, \Delta_1\delta, \Delta_2\sigma, \Delta_2\delta\}$ as well, then*

$$\Delta_0\sigma^{(\Phi,\phi)}\Delta_2\sigma^{(\Phi,\phi)} = \Delta_3\sigma^{(\Phi,\phi)}\Delta_1\sigma^{(\Phi,\phi)}. \tag{2.15}$$

Proof. A long equality showed that

$$\begin{aligned}
\Delta_0\sigma^{(\Phi,\phi)}\Delta_2\sigma^{(\Phi,\phi)} &= \big(1 \otimes (\Phi \otimes \Phi)\sigma\delta\sigma\Delta\phi\big)(id \otimes \Delta)\big(\sigma\delta\sigma\Delta\phi\big) \\
&= (\Phi \otimes \Phi \otimes \Phi)\Delta_0(\sigma\delta\sigma)\Delta_0\Delta_1\phi\Delta_2\Delta_0\Phi\Delta_2(\sigma\delta\sigma)\Delta_2\Delta_1\phi \\
&= (\Phi \otimes \Phi \otimes \Phi)\Delta_0(\sigma\delta\sigma)\Delta_2\Delta_0\big(\phi\Phi\big)\Delta_2(\sigma\delta\sigma)\Delta_1\Delta_1\phi \\
&= (\Phi \otimes \Phi \otimes \Phi)\Delta_0(\sigma\delta\sigma)\Delta_2(\sigma\delta\sigma)\Delta_1\Delta_2\big(\phi\Phi\big)\Delta_1\Delta_1\phi \\
&= (\Phi \otimes \Phi \otimes \Phi)\Delta_0(\sigma\delta)\Delta_0\sigma\Delta_2\sigma\Delta_2\delta\Delta_2\sigma\Delta_1\Delta_2\big(\phi\Phi\big)\Delta_1\Delta_1\phi \\
&= (\Phi \otimes \Phi \otimes \Phi)\Delta_0(\sigma\delta)\partial(\sigma, \delta)\Delta_3\sigma\Delta_1\sigma\Delta_2\delta\Delta_2\sigma\Delta_1\Delta_2\big(\phi\Phi\big)\Delta_1\Delta_1\phi \\
&= (\Phi \otimes \Phi \otimes \Phi)\partial(\sigma, \delta)\Delta_3\sigma\Delta_3\delta\psi(\sigma, \delta)\Delta_0\sigma\Delta_2\sigma\Delta_1\Delta_2\big(\phi\Phi\big)\Delta_1\Delta_1\phi
\end{aligned}$$

$$
\begin{aligned}
&= (\Phi\otimes\Phi\otimes\Phi)\Delta_3(\sigma\delta)\partial(\sigma,\delta)\psi(\sigma,\delta)\partial(\sigma,\delta)\Delta_3\sigma\Delta_1\sigma\Delta_1\Delta_2(\phi\Phi)\Delta_1\Delta_1\phi\\
&= (\Phi\otimes\Phi\otimes\Phi)\Delta_3(\sigma\delta)\partial(\sigma,\delta)\Delta_3\sigma\Delta_1\sigma\Delta_2\delta\Delta_0\delta\Delta_3\sigma\Delta_1\sigma\Delta_1\Delta_2(\phi\Phi)\Delta_1\Delta_1\phi\\
&= (\Phi\otimes\Phi\otimes\Phi)\Delta_3(\sigma\delta\sigma)\Delta_1\sigma\partial(\sigma,\delta)\Delta_2\delta\Delta_0\delta\Delta_3\sigma\Delta_1\sigma\Delta_1\Delta_2(\phi\Phi)\Delta_1\Delta_1\phi\\
&= (\Phi\otimes\Phi\otimes\Phi)\Delta_3(\sigma\delta\sigma)\Delta_1(\sigma\delta\sigma)\Delta_1\Delta_2(\phi\Phi)\Delta_1\Delta_1\phi\\
&= (\Phi\otimes\Phi\otimes\Phi)(\sigma\delta\sigma\otimes 1)(\Delta(\phi\Phi)\otimes 1)(\Delta\otimes id)(\sigma\delta\sigma\Delta\phi)\\
&= ((\Phi\otimes\Phi)\sigma\delta\sigma\Delta\phi\otimes 1)(\Delta\otimes id)((\Phi\otimes\Phi)\sigma\delta\sigma\Delta\phi)\\
&= \Delta_3\sigma^{(\Phi,\phi)}\Delta_1\sigma^{(\Phi,\phi)}.
\end{aligned}
\tag{2.16}
$$

□

There exists a similar version for $\delta(\Phi,\phi)$, namely, the following preposition.

Proposition 2.12. *If there is the equation $\Delta_1\delta\Delta_3\delta\psi(\sigma,\delta)\Delta_0\sigma = \Delta_2\delta$ and $\partial(\sigma,\delta)$ commutes with any element in set $\{\Delta_0\sigma,\Delta_0\delta\}$, and $1\otimes\Delta(\phi\Phi)(=\Delta(\phi\Phi)\otimes 1)$ is a commutative element in set $\{\Delta_0\sigma,\Delta_0\delta,\Delta_3\sigma,\Delta_3\delta\}$ as well, then*

$$
\Delta_1\delta^{(\Phi,\phi)}\Delta_3\delta^{(\Phi,\phi)} = \Delta_0\delta^{(\Phi,\phi)}\Delta_2\delta^{(\Phi,\phi)}. \tag{2.17}
$$

Proposition 2.13. *Let H be a bialgebra and $(\Phi,\phi)_2^{II}$ a counital cocycle for H, and define $\Delta_{(\Phi,\phi)}(h) = \Phi\phi\Phi\Delta(h)\phi\Phi\phi$ for all $h\in H$, then the algebra H with original ϵ and $\Delta_{(\Phi,\phi)}$ consists a new coalgebra if $\chi\Delta_3\Phi\Delta_1\Phi = \Delta_0\Phi\Delta_2\Phi$. Moreover, $\Delta_{(\Phi,\phi)}$ is an algebra map if $(\phi\Phi)^3 = 1$, then algebra H is a bialgebra with comultiplication $\Delta_{(\Phi,\phi)}$.*

Proof. It is clear that $(\epsilon\otimes id)\Delta_{(\Phi,\phi)}(h) = (id\otimes\epsilon)\Delta_{(\Phi,\phi)}(h) = h$. So we only need to show the coassociative law of $\Delta_{(\Phi,\phi)}$. For all $h\in H$, we obtain

$$
\begin{aligned}
(\Delta_{(\Phi,\phi)}\otimes id)\Delta_{(\Phi,\phi)}(h) &= (\Delta_{(\Phi,\phi)}\otimes id)(\Phi\phi\Phi\Delta(h)\phi\Phi\phi)\\
&= \Delta_3(\Phi\phi\Phi)\Delta_1(\Phi\phi\Phi)\Delta_1\Delta_1(h)\Delta_1(\phi\Phi\phi)\Delta_3(\phi\Phi\phi)\\
&= \Delta_3(\Phi\phi)\psi\Delta_0\Phi\Delta_2\Phi\Delta_1\phi\Delta_1\Phi\Delta_2\Delta_1(h)\Delta_1\phi\Delta_1\Phi\Delta_2\phi\Delta_0\phi\partial\Delta_3(\Phi\phi)\\
&= \chi\partial\Delta_3\Phi\Delta_1\Phi\Delta_2\Delta_1(h)\Delta_1\phi\Delta_3\phi\psi\partial\\
&= \chi\Delta_3\Phi\Delta_1\Phi\Delta_2\Delta_1(h)\Delta_1\phi\Delta_3\phi\chi\\
&= \partial\Delta_3\Phi\Delta_1\Phi\Delta_2\Delta_1(h)\Delta_2\phi\Delta_0\phi\partial\chi\\
&= \partial\psi\Delta_0\Phi\Delta_2\Phi\Delta_2\Delta_1(h)\Delta_2\phi\Delta_2\Phi\Delta_1\phi\Delta_3\phi\chi\\
&= \Delta_0(\Phi\phi)\partial\Delta_3\Phi\Delta_1\Phi\Delta_2\Phi\Delta_2\Delta_1(h)\Delta_2\phi\Delta_2\Phi\Delta_1\phi\Delta_3\phi\psi\Delta_0(\Phi\phi)
\end{aligned}
$$

$$\begin{aligned}
&= \Delta_0(\Phi\phi)\Delta_0\Phi\Delta_2\Phi\Delta_2(\phi\Phi)\Delta_2\Delta_1(h)\Delta_2(\phi\Phi)\Delta_2\phi\Delta_0\phi\Delta_0(\Phi\phi) \\
&= \Delta_0(\Phi\phi\Phi)\Delta_2(\Phi\phi\Phi)\Delta_2\Delta_1(h)\Delta_2(\phi\Phi\phi)\Delta_0(\phi\Phi\phi) \\
&= \Delta_0(\Phi\phi\Phi)(id\otimes\Delta)(\Phi\phi\Phi\Delta(h)\phi\Phi\phi)\Delta_0(\phi\Phi\phi) \\
&= \left(id\otimes\Delta_{(\Phi,\phi)}\right)\Delta_{(\Phi,\phi)}(h).
\end{aligned} \tag{2.18}$$

Finally, for any $g \in H$,

$$\begin{aligned}
\Delta_{(\Phi,\phi)}(hg) &= \Phi\phi\Phi\Delta(hg)\phi\Phi\phi \\
&= \Phi\phi\Phi\Delta(h)\phi\Phi\phi\Phi\phi\Phi\Delta(g)\phi\Phi\phi \\
&= \Delta_{(\Phi,\phi)}(h)\Delta_{(\Phi,\phi)}(g).
\end{aligned} \tag{2.19}$$

□

Definition 2.14. Let (H, Δ, ϵ) be a fake bialgebra. If there exists a cocycle $(\Phi, \phi)_3^{\mathrm{II}}$ for H obeying that

$$\begin{gathered}
(id\otimes\Delta)\Delta(h)\Phi = \Phi(\Delta\otimes id)\Delta(h), \qquad \phi(id\otimes\Delta)\Delta(h) = (\Delta\otimes id)\Delta(h)\phi, \\
(id\otimes\epsilon)\Delta(h) = (\epsilon\otimes id)\Delta(h) = h,
\end{gathered} \tag{2.20}$$

for all $h \in H$, then H is called a weak quasi-bialgebra.

Example 2.15. Let H be an associate algebra over field k, where the characteristic of k is not 2. And H is a 4-dimensional vector space with basis $\{1, i, j, ij\}$ obeying that $i^2 = i$, $j^2 = j$, and $ij = ji$. We define homomorphisms $\Delta : H \to H \otimes H$ given by $\Delta(i) = i \otimes j$, $\Delta(j) = j \otimes 1$ and $\epsilon : H \to k$ given by $\epsilon(i) = \epsilon(j) = 0$. Obviously, H is a fake bialgebra. Set $\Phi = j \otimes j \otimes j$ and $\varphi = j \otimes 1 \otimes 1 \otimes j$, then $(\Phi, \Phi)_3^{\mathrm{II}}$ is a cocycle with holding $\partial = \chi = j \otimes j \otimes j \otimes j$. It is routine to check $(H, \Delta, \epsilon, (\Phi, \Phi)_3^{\mathrm{II}})$ is a weak quasi-bialgebra.

We relax Definition 1.3 by setting that H is a weak quasi-bialgebra so that we can define an algebra structure on $A \otimes H$, if A is a left H-module algebra and H a weak quasi-bialgebra, given by

$$(a\#h)(b\#g) = \sum\left(y^1X^1x^1\cdot a\right)\left(y^2X^2x^2h_1\cdot b\right)\#y^3X^3x^3h_2g \tag{2.21}$$

for all $a, b \in A$, $h, g \in H$, while $a\#h$ is equal to $a \otimes h$ here.

Theorem 2.16. *Let H be a weak quasi-bialgebra and A a left H-module algebra. Then $A\#H$ is an associative algebra under the multiplication mentioned above and $1_A\#1_H$ is the unit.*

Proof. For all $a, b,$ and $c \in A$ and $h, g,$ and $l \in H$, we easily get that, by properties of ϵ,

$$\begin{aligned}
(1\#1)(a\#h) &= \left(y^1X^1x^1 \cdot 1\right)\left(y^2X^2x^2 \cdot a\right)\#y^3X^3x^3h \\
&= \epsilon\left(y^1X^1x^1\right)\left(y^2X^2x^2 \cdot a\right)\#y^3X^3x^3h = a\#h, \\
(a\#h)(1\#1) &= \left(y^1X^1x^1 \cdot a\right)\left(y^2X^2x^2h_1 \cdot 1\right)\#y^3X^3x^3h_2 \\
&= \left(y^1X^1x^1 \cdot a\right)\epsilon\left(y^2X^2x^2h_1\right)\#y^3X^3x^3h_2 = a\#h.
\end{aligned} \tag{2.22}$$

Now we show the associative law:

$$\begin{aligned}
\left[(a\#h)(b\#g)\right](c\#l) &= \left[\left(y^1X^1x^1 \cdot a\right)\left(y^2X^2x^2h_1 \cdot b\right)\#y^3X^3x^3h_2g\right](c\#l) \\
&= \left(w^1Y^1z^1 \cdot \left(y^1X^1x^1 \cdot a\right)\left(y^2X^2x^2h_1 \cdot b\right)\right) \\
&\quad \times \left(w^2Y^2z^2\left(y^3X^3x^3h_2g\right)_1 \cdot c\right)\#w^3Y^3z^3\left(y^3X^3x^3h_2g\right)_2 l
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&= \left(\left(w^1{}_1Y^1{}_1z^1{}_1y^1X^1x^1 \cdot a\right)\left(w^1{}_2Y^1{}_2z^1{}_2y^2X^2x^2h_1 \cdot b\right)\right) \\
&\quad \times \left(w^2Y^2z^2y^3{}_1X^3{}_1x^3{}_1h_{21}g_1 \cdot c\right)\#w^3Y^3z^3y^3{}_2X^3{}_2x^3{}_2h_{22}g_2l.
\end{aligned} \tag{2.24}$$

But

$$\begin{aligned}
\Delta_1\phi\Delta_1\Phi\Delta_1\phi\Delta_3\phi\Delta_3\Phi\Delta_3\phi &= \Delta_1\phi\Delta_1\Phi\Delta_4\phi\Delta_2\phi(\Delta_0\phi)\partial\Delta_3\Phi\Delta_3\phi \\
&= \Delta_1\phi\Delta_1\Phi\Delta_4\phi\Delta_2\phi(\Delta_0\phi)\partial \\
&= \Delta_1\phi(\Delta_3\phi)\varphi\partial,
\end{aligned} \tag{2.25}$$

and then we obtain that (2.23) is

$$\begin{aligned}
&\left(\left(z^1{}_1x^1\varphi^1\partial^1 \cdot a\right)\left(z^1{}_2x^2\varphi^2\partial^2h_1 \cdot b\right)\right)\left(z^2x^3{}_1\varphi^3\partial^3h_{21}g_1 \cdot c\right)\#z^3x^3{}_2\varphi^4\partial^4h_{22}g_2l \\
&\quad = \left(Y^1z^1{}_1x^1\varphi^1\partial^1 \cdot a\right)\left(\left(Y^2z^1{}_2x^2\varphi^2\partial^2h_1 \cdot b\right)\left(Y^3z^2x^3{}_1\varphi^3\partial^3h_{21}g_1 \cdot c\right)\right)\#z^3x^3{}_2\varphi^4\partial^4h_{22}g_2l.
\end{aligned} \tag{2.26}$$

On the other hand, the equation

$$\begin{aligned}
\Delta_4\Phi\Delta_1\phi(\Delta_3\phi)\varphi\partial &= \Delta_2\phi(\Delta_0\phi)\partial\varphi\partial = \Delta_2\phi(\Delta_0\phi)\partial\chi \\
&= \Delta_2\phi(\Delta_0\phi)\partial\varphi\Delta_0\Phi\Delta_0\phi \\
&= \Delta_2\phi\Delta_2\Phi\Delta_4\Phi\Delta_1\phi(\Delta_3\phi)\varphi\Delta_0\Phi\Delta_0\phi \\
&= \Delta_2\phi\Delta_2\Phi\Delta_2\phi\Delta_0\phi\Delta_0\Phi\Delta_0\phi
\end{aligned} \tag{2.27}$$

makes (2.26) equal

$$\begin{aligned}
&\left(w^1Y^1z^1\cdot a\right)\left(\left(w^2{}_1Y^2{}_1z^2{}_1y^1X^1x^1h_1\cdot b\right)\left(w^2{}_2Y^2{}_2z^2{}_2y^2X^2x^2h_{21}g_1\cdot c\right)\right)\#w^3Y^3z^3y^3X^3x^3h_{22}g_2l\\
&\quad=\left(w^1Y^1z^1\cdot a\right)\left(\left(w^2{}_1Y^2{}_1z^2{}_1h_{11}y^1X^1x^1\cdot b\right)\left(w^2{}_2Y^2{}_2z^2{}_2h_{12}y^2X^2x^2g_1\cdot c\right)\right)\#w^3Y^3z^3h_2y^3X^3x^3g_2l\\
&\quad=\left(w^1Y^1z^1\cdot a\right)\left(w^2Y^2z^2h_1\cdot\left(\left(y^1X^1x^1\cdot b\right)\left(y^2X^2x^2g_1\cdot c\right)\right)\right)\#w^3Y^3z^3h_2y^3X^3x^3g_2l\\
&\quad=(a\#h)\left[(b\#g)(c\#l)\right].
\end{aligned}\tag{2.28}$$

Hence, $[(a\#h)(b\#g)](c\#l)=(a\#h)[(b\#g)(c\#l)]$. □

If ϕ is an inverse of Φ, then the multiplication becomes that

$$(a\#h)(b\#g)=\sum\left(x^1\cdot a\right)\left(x^2h_1\cdot b\right)\#x^3h_2g,\tag{2.29}$$

which is as exact as the one in [3].

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