Research Article

# Subclasses of Analytic Functions Associated with Generalised Multiplier Transformations 

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New subclasses of analytic functions in the open unit disc are introduced which are defined using generalised multiplier transformations. Inclusion theorems are investigated for functions to be in the classes. Furthermore, generalised Bernardi-Libera-Livington integral operator is shown to be preserved for these classes.

## 1. Introduction

Let $A$ denote the class of functions $f$ normalised by $f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}$ in the open unit disk $\mathbf{D}:=\{z \in \mathbf{C}:|z|<1\}$. Also let $S^{\star}, C$, and $K$ denote, respectively, the subclasses of A consisting of functions which are starlike, convex, and close to convex in $\mathbf{D}$. An analytic function $f$ is subordinate to an analytic function $g$, written $f(z)<g(z)(z \in \mathbf{D})$ if there exists an analytic function $w$ in $\mathbf{D}$ such that $w(0)=0$ and $|w(z)|<1$ for $|z|<1$ and $f(z)=g(w(z))$. In particular, if $g$ is univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbf{D}) \subset$ $g(\mathbf{D})$. The convolution of two analytic functions $\varphi(z)=\sum_{n=2}^{\infty} a_{n} z^{n}$ and $\psi(z)=\sum_{n=0}^{\infty} b_{n} z^{n}$ is defined by $\varphi(z) * \psi(z)=\sum_{n=0}^{\infty} a_{n} b_{n} z^{n}=\psi(z) * \varphi(z)$.

For any real numbers $k$ and $\lambda$ where $k \geq 0, \lambda \geq 0, c \geq 0$, Cǎtaş [1] defined the multiplier transformations $I(k, \lambda, c) f(z)$ by the following series:

$$
\begin{equation*}
I(k, \lambda, c) f(z)=z+\sum_{n=2}^{\infty}\left[\frac{1+\lambda(n-1)+c}{1+c}\right]^{k} a_{n} z^{n} \tag{1.1}
\end{equation*}
$$

Recently, some properties of functions using the multiplier transformations have been studied in [2-6]. Using the convolution, we extend the multiplier transformation in (1.1) to be a unified operator. The approach used is similar to Noor's [7], only we generalise and extend to include powers and uses the multiplier Cǎtaş as basis instead of the Ruscheweyh operator.

Set the function

$$
\begin{equation*}
f_{k, c}(z)=z+\sum_{n=2}^{\infty}\left[\frac{1+c}{1+\lambda(n-1)+c}\right]^{k} z^{n} \quad(k, \lambda \in \mathbf{R}, k \geq 0, \lambda \geq 0, c \geq 0) \tag{1.2}
\end{equation*}
$$

and note that, for $\lambda=1, f_{k, c}(z)$ is the generalised polylogarithm functions discussed in [8]. A new function $f_{k, c}^{\mu}(z)$ is defined in terms of the Hadamard product (or convolution) as follows:

$$
\begin{equation*}
f_{k, c}(z) * f_{k, c}^{\mu}(z)=\frac{z}{(1-z)^{\mu}} \quad(\mu>0) \tag{1.3}
\end{equation*}
$$

Motivated by [9-11] and analogous to (1.1), the following operator is introduced:

$$
\begin{align*}
I_{c}^{k}(\lambda, \mu) f(z) & =f_{k, c}^{\mu} * f(z) \\
& =z+\sum_{n=2}^{\infty} \frac{(\mu)_{n-1}}{(n-1)!}\left[\frac{1+\lambda(n-1)+c}{1+c}\right]^{k} a_{n} z^{n} \tag{1.4}
\end{align*}
$$

The operator $I_{c}^{k}(\lambda, \mu) f$ unifies other previously defined operators. For examples,
(i) $I_{c}^{k}(\lambda, 1) f$ is the $I_{1}(\delta, \lambda, l) f$ given in [1],
(ii) $I_{c}^{k}(1,1) f$ is the $I_{c}^{k} f$ given in [12],
also, for any integer $k$,
(iii) $I_{0}^{k}(\lambda, 1) f(z) \equiv D_{\lambda}^{k} f(z)$ given in [13],
(iv) $I_{0}^{k}(1,1) f(z) \equiv D^{k} f(z)$ given in [14],
(v) $I_{1}^{k}(1,1) f(z) \equiv I_{k} f(z)$ given in [15].

The following relations are easily derived using the following definition:

$$
\begin{gather*}
(1+c) I_{c}^{k+1}(\lambda, \mu) f(z)=(1-\lambda+c) I_{c}^{k}(\lambda, \mu) f(z)+\lambda z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}  \tag{1.5}\\
\mu I_{c}^{k}(\lambda, \mu+1) f(z)=z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}+(\mu-1) I_{c}^{k}(\lambda, \mu) f(z) \tag{1.6}
\end{gather*}
$$

Let $N$ be the class of all analytic and univalent functions $\phi$ in $\mathbf{D}$ and for which $\phi(\mathbf{D})$ is convex with $\phi(0)=1$ and $\operatorname{Re}\{\phi(z)\}>0$ for $z \in \mathbf{D}$. For $\phi, \psi \in N$, Ma and Minda [16] studied
the subclasses $S^{\star}(\phi), C(\phi)$, and $K(\phi, \psi)$ of the class $A$. These classes are defined using the principle of subordination as follows:

$$
\begin{gather*}
S^{\star}(\phi):=\left\{f: f \in A, \frac{z f^{\prime}(z)}{f(z)} \prec \phi(z) \text { in } \mathbf{D}\right\}, \\
C(\phi):=\left\{f: f \in A, 1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)} \prec \phi(z) \text { in } \mathbf{D}\right\},  \tag{1.7}\\
K(\phi, \psi):=\left\{f: f \in A, \exists g \in S^{\star}(\phi) \text { such that } \frac{z f^{\prime}(z)}{g(z)} \prec \psi(z) \text { in } \mathbf{D}\right\} .
\end{gather*}
$$

Obviously, we have the following relationships for special choices $\phi$ and $\psi$ :

$$
\begin{equation*}
S^{\star}\left(\frac{1+z}{1-z}\right)=S^{\star}, \quad C\left(\frac{1+z}{1-z}\right)=C, \quad K\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\right)=K . \tag{1.8}
\end{equation*}
$$

Using the generalised multiplier transformations $I_{c}^{k}(\lambda, \mu) f$, new classes $S_{c}^{k}(\lambda, \mu ; \phi), C_{c}^{k}(\lambda, \mu ; \phi)$ and $K_{c}^{k}(\lambda, \mu ; \phi, \psi)$ are introduced and defined below

$$
\begin{align*}
S_{c}^{k}(\lambda, \mu ; \phi) & :=\left\{f \in A: I_{c}^{k}(\lambda, \mu) f(z) \in S^{\star}(\phi)\right\}, \\
C_{c}^{k}(\lambda, \mu ; \phi) & :=\left\{f \in A: I_{c}^{k}(\lambda, \mu) f(z) \in C(\phi)\right\},  \tag{1.9}\\
K_{c}^{k}(\lambda, \mu ; \phi, \psi) & :=\left\{f \in A: I_{c}^{k}(\lambda, \mu) f(z) \in K(\phi, \psi)\right\} .
\end{align*}
$$

It can be shown easily that

$$
\begin{equation*}
f(z) \in C_{c}^{k}(\lambda, \mu ; \phi) \Longleftrightarrow z f^{\prime}(z) \in S_{c}^{k}(\lambda, \mu ; \phi) \tag{1.10}
\end{equation*}
$$

Janowski [17] introduced class $S^{\star}[A, B]=S^{\star}((1+A z) /(1+B z))$ and in particular for $\phi(z)=$ $(1+A z) /(1+B z)$, we set

$$
\begin{equation*}
S_{c}^{k}\left(\lambda, \mu ; \frac{1+A z}{1+B z}\right)=S_{k, c}^{\star}[\mu ; A, B] \quad(-1 \leq B<A \leq 1) \tag{1.11}
\end{equation*}
$$

In [18], the authors studied the inclusion properties for classes defined using DziokSrivastava operator. This paper investigates the similar properties for analytic functions in the classes defined by the generalised multiplier transformations $I_{c}^{k}(\lambda, \mu) f$. Furthermore, applications of other families of integral operators are considered involving these classes.

## 2. Inclusion Properties Involving $I_{c}^{K}(\lambda, \mu) f$

In proving our results, the following lemmas are needed.

Lemma 2.1 (see [19]). Let $\phi$ be convex univalent in $\mathbf{D}$, with $\phi(0)=1$ and $\operatorname{Re}[\kappa \phi(z)+\eta]>0(\kappa, \eta \in$ $\mathbf{C})$. If $p$ is analytic in $\mathbf{D}$ with $p(0)=1$, then

$$
\begin{equation*}
p(z)+\frac{z p^{\prime}(z)}{\kappa p(z)+\eta}<\phi(z) \Longrightarrow p(z)<\phi(z) \tag{2.1}
\end{equation*}
$$

Lemma 2.2 (see [20]). Let $\phi$ be convex univalent in $\mathbf{D}$ and $\omega$ be analytic in $\mathbf{D}$ with $\operatorname{Re}\{\omega(z)\} \geq 0$. If $p$ is analytic in $\mathbf{D}$ and $p(0)=\phi(0)$, then

$$
\begin{equation*}
p(z)+\omega(z) z p^{\prime}(z)<\phi(z) \Longrightarrow p(z)<\phi(z) \tag{2.2}
\end{equation*}
$$

Theorem 2.3. For any real numbers $k$ and $\lambda$ where $k \geq 0, \lambda \geq 0$ and $c \geq 0$.
Let $\phi \in N$ and $\operatorname{Re}\{\phi(z)+(1-\lambda+c) / \lambda\}>0$, then $S_{c}^{\bar{k}+1}(\lambda, \mu ; \phi) \subset S_{c}^{k}(\lambda, \mu ; \phi)(\mu>0)$.
Proof. Let $f \in S_{c}^{k+1}(\lambda, \mu ; \phi)$, and set $p(z)=\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) f(z)\right)$ where $p$ is analytic in $\mathbf{D}$ with $p(0)=1$. Rearranging (1.5), we have

$$
\begin{equation*}
\frac{(1+c) I_{c}^{k+1}(\lambda, \mu) f(z)}{I_{c}^{k}(\lambda, \mu) f(z)}=(1-\lambda+c)+\frac{\lambda z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) f(z)} \tag{2.3}
\end{equation*}
$$

Next, differentiating (2.3) and multiplying by $z$ gives

$$
\begin{align*}
\frac{z\left[I_{c}^{k+1}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k+1}(\lambda, \mu) f(z)} & =\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) f(z)}+\frac{z\left(\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) f(z)\right)\right)^{\prime}}{\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) f(z)\right)+(1-\lambda+c) / \lambda} \\
& =p(z)+\frac{z p^{\prime}(z)}{p(z)+(1-\lambda+c) / \lambda} . \tag{2.4}
\end{align*}
$$

Since $\left(z\left[I_{c}^{k+1}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k+1}(\lambda, \mu) f(z)\right) \prec \phi(z)$ and applying Lemma 2.1, it follows that $p \prec \phi$. Thus $f \in S_{c}^{k}(\lambda, \mu ; \phi)$.

Theorem 2.4. Let $k, \lambda \in \mathbf{R}, k \geq 0, \lambda \geq 0$, and $\mu \geq 1$. Then $S_{c}^{k}(\lambda, \mu+1 ; \phi) \subset S_{c}^{k}(\lambda, \mu ; \phi)(c \geq 0 ; \phi \in$ $N)$.

Proof. Let $f \in S_{c}^{k}(\lambda, \mu+1 ; \phi)$, and from (1.6), we obtain that

$$
\begin{equation*}
\frac{\mu I_{c}^{k}(\lambda, \mu+1) f(z)}{I_{c}^{k}(\lambda, \mu)}=\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu)}+(\mu-1) \tag{2.5}
\end{equation*}
$$

Making use of the differentiation on both sides in (2.5) and setting $p(z)=$ $\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) f(z)\right)$, we get the following:

$$
\begin{equation*}
\frac{z\left[I_{c}^{k}(\lambda, \mu+1) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu+1) f(z)}=p(z)+\frac{z p^{\prime}(z)}{p(z)+(\mu-1)} \prec \phi(z) . \tag{2.6}
\end{equation*}
$$

Since $\mu \geq 1$ and $\operatorname{Re}\{\phi(z)+(\mu-1)\}>0$, using Lemma 2.1, we conclude that $f \in S_{c}^{k}(\lambda, \mu ; \phi)$.
Corollary 2.5. Let $\lambda \geq 0, \mu \geq 1$, and $-1 \leq B<A \leq 1$. Then $S_{k+1, c}^{\star}[\mu ; A, B] \subset S_{k, c}^{\star}[\mu ; A, B]$ and $S_{k, c}^{\star}[\mu+1 ; A, B] \subset S_{k, c}^{\star}[\mu ; A, B]$.

Theorem 2.6. Let $\lambda \geq 0$ and $\mu \geq 1$. Then $C_{c}^{k+1}(\lambda, \mu ; \phi) \subset C_{c}^{k}(\lambda, \mu ; \phi)$ and $C_{c}^{k}(\lambda, \mu+1 ; \phi) \subset$ $C_{c}^{k}(\lambda, \mu ; \phi)$.

Proof. Using (1.10) and Theorem 2.3, we observe that

$$
\begin{align*}
f(z) \in C_{c}^{k+1}(\lambda, \mu ; \phi) & \Longleftrightarrow z f^{\prime}(z) \in S_{c}^{k+1}(\lambda, \mu ; \phi) \\
& \Longleftrightarrow z f^{\prime}(z) \in S_{c}^{k}(\lambda, \mu ; \phi) \\
& \Longleftrightarrow I_{c}^{k}(\lambda, \mu) z f^{\prime}(z) \in S^{\star}(\phi) \\
& \Longleftrightarrow z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime} \in S^{\star}(\phi)  \tag{2.7}\\
& \Longleftrightarrow I_{c}^{k}(\lambda, \mu) f(z) \in C(\phi) \\
& \Longleftrightarrow f \in C_{c}^{k}(\lambda, \mu ; \phi) .
\end{align*}
$$

To prove the second part of theorem, using the similar manner and applying Theorem 2.4, the result is obtained.

Theorem 2.7. Let $\lambda \geq 0, c \geq 0$ and $\operatorname{Re}\{(1-\lambda+c) / \lambda\}>0$.
Then $K_{c}^{k+1}(\lambda, \mu ; \phi, \psi) \subset K_{c}^{k}(\lambda, \mu ; \phi, \psi)$ and $K_{c}^{k}(\lambda, \mu+1 ; \phi, \psi) \subset K_{c}^{k}(\lambda, \mu ; \phi, \psi)(\phi, \psi \in N)$.
Proof. Let $f \in K_{c}^{k+1}(\lambda, \mu ; \phi, \psi)$. In view of the definition of the class $K_{c}^{k+1}(\lambda, \mu ; \phi, \psi)$, there is a function $g \in S_{c}^{k+1}(\lambda, \mu ; \phi)$ such that

$$
\begin{equation*}
\frac{z\left[I_{c}^{k+1}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k+1}(\lambda, \mu) g(z)}<\psi(z) \tag{2.8}
\end{equation*}
$$

Applying Theorem 2.3, then $g \in S_{c}^{k}(\lambda, \mu ; \phi)$ and let $q(z)=\left(z\left[I_{c}^{k}(\lambda, \mu) g(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) g(z)\right) \prec$ $\phi(z)$.

Let the analytic function $p$ with $p(0)=1$ as

$$
\begin{equation*}
p(z)=\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) g(z)} \tag{2.9}
\end{equation*}
$$

Thus, rearranging and differentiating (2.9), we have

$$
\begin{equation*}
\frac{\left[I_{c}^{k}(\lambda, \mu) z f^{\prime}(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) g(z)}=\frac{p(z)\left[I_{c}^{k}(\lambda, \mu) g(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) g(z)}+p^{\prime}(z) \tag{2.10}
\end{equation*}
$$

Making use (1.5), (2.9), (2.10), and $q(z)$, we obtain that

$$
\begin{align*}
& \frac{z\left[I_{c}^{k+1}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k+1}(\lambda, \mu) g(z)} \\
& \quad=\frac{\left[I_{c}^{k+1}(\lambda, \mu) z f^{\prime}(z)\right]}{I_{c}^{k+1}(\lambda, \mu) g(z)} \\
& \quad=\frac{(1-\lambda+c) I_{c}^{k}(\lambda, \mu) z f^{\prime}(z)+\lambda z\left[I_{c}^{k}(\lambda, \mu) z f^{\prime}(z)\right]^{\prime}}{(1-\lambda+c) I_{c}^{k}(\lambda, \mu) g(z)+\lambda z\left[I_{c}^{k}(\lambda, \mu) g(z)\right]^{\prime}} \\
& \quad=\frac{\left((1-\lambda+c) I_{c}^{k}(\lambda, \mu) z f^{\prime}(z)\right) /\left(I_{c}^{k}(\lambda, \mu) g(z)\right)+\left(\lambda z\left[I_{c}^{k}(\lambda, \mu) z f^{\prime}(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) g(z)\right)}{(1-\lambda+c)+\left(\lambda z\left[I_{c}^{k}(\lambda, \mu) g(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) g(z)\right)} \\
& \quad=\frac{(1-\lambda+c) p(z)+\lambda\left[p(z) q(z)+p^{\prime}(z)\right]}{(1-\lambda+c)+\lambda q(z)} \\
& \quad=p(z)+\frac{z p^{\prime}(z)}{q(z)+(1-\lambda+c) / \lambda} \prec \psi(z) . \tag{2.11}
\end{align*}
$$

Since $q(z)<\phi(z)$ and $\operatorname{Re}\{(1-\lambda+c) / \lambda\}>0$, then $\operatorname{Re}\{q(z)+(1-\lambda+c) / \lambda\}>0$. Using Lemma 2.2, we conclude that $p(z) \prec \psi(z)$ and thus $f \in K_{c}^{k}(\lambda, \mu ; \phi, \psi)$. By using similar manner and (1.6), we obtain the second result.

In summary, using subordination technique inclusion properties has been established for certain analytic functions defined via the generalised multiplier transformation.

## 3. Inclusion Properties Involving $F_{c} f$

In this section, we determine properties of generalised Bernardi-Libera-Livington integral operator defined by [21-24]

$$
\begin{align*}
F_{c}[f(z)] & =\frac{c+1}{z^{c}} \int_{0}^{z} t^{c-1} f(t) d t \quad(c>-1, \operatorname{Re} c \geq 0) \\
& =z+\sum_{n=2}^{\infty} \frac{c+1}{n+c} a_{n} z^{n} \tag{3.1}
\end{align*}
$$

and satisfies the following:

$$
\begin{equation*}
c I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]+z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}=(c+1) I_{c}^{k}(\lambda, \mu) f(z) \tag{3.2}
\end{equation*}
$$

Theorem 3.1. If $f \in S_{c}^{k}(\lambda, \mu ; \phi)$, then $F_{c} f \in S_{c}^{k}(\lambda, \mu ; \phi)$.
Proof. Let $f \in S_{c}^{k}(\lambda, \mu ; \phi)$, then $\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) f(z)\right) \prec \phi(z)$. Taking the differentiation on both sides of (3.2) and multiplying by $z$, we obtain

$$
\begin{equation*}
\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) f(z)}=\frac{z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]}+\frac{z\left(\left(z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right)\right)^{\prime}}{\left(z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right)+c} \tag{3.3}
\end{equation*}
$$

Setting $p(z)=\left(z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right)$, we have

$$
\begin{equation*}
\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) f(z)}=p(z)+\frac{z p^{\prime}(z)}{p(z)+c} \tag{3.4}
\end{equation*}
$$

Lemma 2.1 implies $\left(z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right) \prec \phi(z)$. Hence $F_{c} f \in S_{c}^{k}(\lambda, \mu ; \phi)$.

Theorem 3.2. Let $f \in C_{c}^{k}(\lambda, \mu ; \phi)$, then $F_{c} f \in C_{c}^{k}(\lambda, \mu ; \phi)$.
Proof. By using (1.10) and Theorem 3.1, it follows that

$$
\begin{align*}
f \in C_{c}^{k}(\lambda, \mu ; \phi) & \Longleftrightarrow z f^{\prime}(z) \in S_{c}^{k}(\lambda, \mu ; \phi) \Longrightarrow F_{c}\left[z f^{\prime}(z)\right] \in S_{c}^{k}(\lambda, \mu ; \phi)  \tag{3.5}\\
& \Longleftrightarrow z\left[F_{c}[f(z)]\right]^{\prime} \in S_{c}^{k}(\lambda, \mu ; \phi) \Longleftrightarrow F_{c}[f(z)] \in C_{c}^{k}(\lambda, \mu ; \phi)
\end{align*}
$$

Theorem 3.3. Let $\phi, \psi \in N$, and $f \in K_{c}^{k}(\lambda, \mu ; \phi, \psi)$, then $F_{c} f \in K_{c}^{k}(\lambda, \mu ; \phi, \psi)$.
Proof. Let $f \in K_{c}^{k}(\lambda, \mu ; \phi, \psi)$, then there exists function $g \in S_{c}^{k}(\lambda, \mu ; \phi)$ such that $\left(z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}\right) /\left(I_{c}^{k}(\lambda, \mu) g(z)\right) \prec \psi(z)$. Since $g \in S_{c}^{k}(\lambda, \mu ; \phi)$ therefore from Theorem 3.1, $F_{c}[f(z)] \in S_{c}^{k}(\lambda, \mu ; \phi)$. Then let

$$
\begin{equation*}
q(z)=\frac{z\left[I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]}<\phi(z) . \tag{3.6}
\end{equation*}
$$

Set

$$
\begin{equation*}
p(z)=\frac{z\left[I_{c}^{k}(\lambda, \mu) F_{c}[f(z)]\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]} \tag{3.7}
\end{equation*}
$$

By rearranging and differentiating (3.7), we obtain that

$$
\begin{equation*}
\frac{\left[I_{c}^{k}(\lambda, \mu) F_{c}\left[z f^{\prime}(z)\right]\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]}=\frac{p(z)\left[I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]}+\frac{\left[I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]\right] p^{\prime}(z)}{I_{c}^{k}(\lambda, \mu) F_{c}[g(z)]} . \tag{3.8}
\end{equation*}
$$

Making use (3.2), (3.7), and (3.6), it can be derived that

$$
\begin{equation*}
\frac{z\left[I_{c}^{k}(\lambda, \mu) f(z)\right]^{\prime}}{I_{c}^{k}(\lambda, \mu) g(z)}=p(z)+\frac{z p^{\prime}(z)}{c+q(z)} \tag{3.9}
\end{equation*}
$$

Hence, applying Lemma 2.2, we conclude that $p(z) \prec \psi(z)$, and it follows that $F_{c}[f(z)] \in$ $K_{c}^{k}(\lambda, \mu ; \phi, \psi)$.

For analytic functions in the classes defined by generalised multiplier transformations, the generalised Bernardi-Libera-Livington integral operator has been shown to be preserved in these classes.

## 4. Conclusion

Results involving functions defined using the generalised multiplier transformation, namely, inclusion properties and the Bernardi-Libera-Livington integral operator were obtained using subordination principles. In [18], similar results were discussed for functions defined using the Dziok-Srivastava operator.

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