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Research Article

θ-$\mathcal{I}_g$-*Closed* Sets

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We define θ-$\mathcal{I}_g$-*Closed* sets and discuss their properties. Using these sets, we characterize $\mathcal{T}_{1/2}$-*spaces* and $\mathcal{T}_{\mathcal{I}}$-*Spaces*.

1. Introduction and Preliminaries

An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(\cdot)^\star : \wp(X) \to \wp(X)$ called a *local function* [1] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^\star(X, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I},$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. A *Kuratowski closure operator* $\mathrm{cl}^\star(\cdot)$ for a topology $\tau^\star(\mathcal{I}, \tau)$ called the $\star$-*topology*, finer than τ, is defined by $\mathrm{cl}^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [2]. When there is no confusion we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an *ideal space*. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\star$-*closed* [3] if $A^\star \subset A$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_g$-*closed* [4] if $A^\star \subset U$ whenever $A \subset U$ and U is open. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$-*open* if $X - A$ is $\mathcal{I}_g$-*closed*. An ideal space $(X, \tau, \mathcal{I})$ is said to be a $\mathcal{T}_{\mathcal{I}}$-*space* [4] if every $\mathcal{I}_g$-*closed* set is $\star$-*closed*. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-*locally* $\star$-*closed* [5] if there exist an open set U and a $\star$-*closed* set F such that $A = U \cap F$. If $\mathcal{I} = \{\emptyset\}$, then $\mathcal{I}$-*locally* $\star$-*closed* sets coincide with locally *closed* sets.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $\mathrm{int}^\star(A)$ will denote the interior of A in $(X, \tau^\star)$. A subset A of a topological space (X, τ) is said to be a g-*closed* set [6] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open. A subset A

of a topological space (X,τ) is said to be a *g-open* set if $X - A$ is a *g-closed* set. A space (X,τ) is said to be a $\mathcal{T}_{1/2}$-*space* [6] if every *g-closed* set is a closed set.

For a subset A of a space (X,τ), the θ-*interior* [7] of A is the union of all open sets of X whose closures contained in A and is denoted by $\mathrm{int}_\theta(A)$. The subset A is called θ-*open* if $A = \mathrm{int}_\theta(A)$. The complement of a θ-*open* set is called a θ-*closed* set. Equivalently, $A \subset X$ is called θ-*closed* [7] if $A = \mathrm{cl}_\theta(A)$, where $\mathrm{cl}_\theta(A) = \{x \in X \mid \mathrm{cl}(U) \cap A \neq \emptyset$ for all $U \in \tau(x)\}$. The family of all θ-*open* sets of X forms a *topology* [7] on X, which is coarser than τ and is denoted by τ_θ. A subset A of a topological space (X,τ) is said to be a θ-*g-closed set* [8] if $\mathrm{cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is open. A subset A of a space (X,τ) is said to be a θ-*g-open set* [8] if $X - A$ is a θ-*g-closed* set. A subset A of a space (X,τ) is said to be a Λ-*set* [9, 10] if $A = A^\Lambda$, where $A^\Lambda = \cap \{U \in \tau \mid A \subset U\}$.

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be θ-$\mathcal{I}$-*closed* [11] if $\mathrm{cl}^\star_\theta(A) = A$, where $\mathrm{cl}^\star_\theta(A) = \{x \in X \mid A \cap \mathrm{cl}^\star(U) \neq \phi$ for all $U \in \tau(x)\}$. A is said to be θ-$\mathcal{I}$-*open* if $X - A$ is θ-$\mathcal{I}$-*closed*. If $\mathcal{I} = \{\emptyset\}$, $\mathrm{cl}^\star_\theta(A) = \mathrm{cl}_\theta(A)$. If $\mathcal{I} = \wp(X)$, $\mathrm{cl}^\star_\theta(A) = \mathrm{cl}(A)$. For a subset A of X, $\mathrm{int}_\theta I(A) = \cup \{U \in \tau \mid \mathrm{cl}^\star(U) \subset A\}$ [11]. We denote this $\mathrm{int}_\theta I(A)$ by $\mathrm{int}^\star_\theta(A)$. The family of all θ-$\mathcal{I}$-*open* sets of $(X,\tau,\mathcal{I})$ is a topology and it is denoted by $\tau_{\theta\text{-}\mathcal{I}}$ (see [11, Theorem 1]).

Lemma 1.1 (see [11, Corollary 4 if Theorem 2]). $\tau_\theta \subset \tau_{\theta\text{-}\mathcal{I}} \subset \tau$.

Lemma 1.2 (see [11, Proposition 3]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then, we have*

(1) *if $\mathcal{I} = \{\phi\}$ or $\mathcal{I} = \mathcal{N}$, where $\mathcal{N}$ is the ideal of nowhere dense sets of (X,τ), then $\tau_{\theta\text{-}\mathcal{I}} = \tau_\theta$,*

(2) *if $\mathcal{I} = \{\phi\}$, then $\tau_{\theta\text{-}\mathcal{I}} = \tau$.*

Lemma 1.3 (see [5, Theorem 2.13]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_g$-closed if and only if every open set is $\star$-closed.*

Lemma 1.4 (see [11, Proposition 1]). *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a subset of X. Then A is θ-$\mathcal{I}$-open if and only if $\mathrm{int}^\star_\theta(A) = A$.*

Lemma 1.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a subset of X. Then $\mathrm{cl}^\star_\theta(A) = \{x \in X \mid U \cap \mathrm{cl}^\star(A) \neq \phi$ for all $U \in \tau(x)\}$ is closed.*

Proof. If $x \in \mathrm{cl}(\mathrm{cl}^\star_\theta(A))$ and $U \in \tau(x)$, then $U \cap \mathrm{cl}^\star_\theta(A) \neq \phi$. Then, $y \in U \cap \mathrm{cl}^\star_\theta(A)$ for some $y \in X$. Since $U \in \tau(y)$ and $y \in \mathrm{cl}^\star_\theta(A)$, from the definition of $\mathrm{cl}^\star_\theta(A)$ we have $A \cap \mathrm{cl}^\star(U) \neq \phi$. Therefore, $x \in \mathrm{cl}^\star_\theta(A)$. So $\mathrm{cl}(\mathrm{cl}^\star_\theta(A)) \subset \mathrm{cl}^\star_\theta(A)$ and hence $\mathrm{cl}^\star_\theta(A)$ is closed. □

Lemma 1.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a subset of X. Then, $X - \mathrm{cl}^\star_\theta(X - A) = \mathrm{int}^\star_\theta(A)$.*

Proof. $x \in X - \mathrm{cl}^\star_\theta(X - A)$ if and only if $x \notin \mathrm{cl}^\star_\theta(X - A)$ if and only if there exist $U \in \tau(x)$ such that $(X - A) \cap \mathrm{cl}^\star(U) = \phi$ if and only if $x \in U$ and, $\mathrm{cl}^\star(U) \subset (A)$ if and only if $x \in U \subset \mathrm{int}^\star_\theta(A)$. □

2. θ-$\mathcal{I}_g$- *Closed* Sets

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be a θ-$\mathcal{I}_g$-*closed* set if $\mathrm{cl}^\star_\theta(A) \subset U$ whenever $A \subset U$ and U is open. Every θ-$\mathcal{I}$-*closed* set is a θ-$\mathcal{I}_g$-*closed* set. If $\mathcal{I} = \{\emptyset\}$, then $\mathrm{cl}^\star_\theta(A) = \mathrm{cl}_\theta(A)$ and hence θ-$\mathcal{I}_g$-*closed* sets coincide with θ-*g-closed* sets. If $\mathcal{I} = \wp(X)$, then

$\mathrm{cl}_\theta^\star(A) = \mathrm{cl}(A)$ and hence θ-$\mathcal{I}_g$-*closed* sets coincide with *g-closed* sets. Since $\mathrm{cl}^\star(A) \subset \mathrm{cl}(A) \subset \mathrm{cl}_\theta^\star(A) \subset \mathrm{cl}_\theta(A)$, we have the following inclusion diagram:

$$\theta\text{-}g\text{-}closed \longrightarrow \theta\text{-}\mathcal{I}_g\text{-}closed \longrightarrow g\text{-}closed \longrightarrow \mathcal{I}_g\text{-}closed. \tag{2.1}$$

Example 2.1. shows that a *g-closed* set needs not to be θ-$\mathcal{I}_g$-*closed*, and Example 2.2 shows that θ-$\mathcal{I}_g$-*closed* set needs not to be a θ-*g-closed* set.

Example 2.1. Let $X = \{a,b,c,d\}$, $\tau = \{\phi,\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\},X\}$, and $\mathcal{I} = \{\phi,\{a\},\{c\},\{a,c\}\}$. Let $A = \{c\}$. Then A is closed and hence *g-closed*. But A is not θ-$\mathcal{I}_g$-*closed* because $A \subset \{b,c\}$ and $\mathrm{cl}_\theta^\star(A) = X \not\subset \{b,c\}$.

Example 2.2. Let X and τ be the same as in Example 2.1. Let $\mathcal{I} = \{\phi,\{a\},\{b\},\{a,b\}\}$ and $A = \{c\}$. Then A is a θ-$\mathcal{I}$-*closed* and hence θ-$\mathcal{I}_g$-*closed*. Since $A \subset \{b,c\}$ and $\mathrm{cl}_\theta(A) = X \not\subset \{b,c\}$, A is not θ-*g-closed*.

Theorem 2.3. *If A is a subset of an ideal space $(X,\tau,\mathcal{I})$, then the following are equivalent.*

(a) *A is θ-$\mathcal{I}_g$-closed.*

(b) *For all $x \in \mathrm{cl}_\theta^\star(A)$, $\mathrm{cl}(\{x\}) \cap A \neq \phi$.*

(c) *$\mathrm{cl}_\theta^\star(A) - A$ contains no nonempty closed set.*

Proof. $(a) \Rightarrow (b)$. Suppose $x \in \mathrm{cl}_\theta^\star(A)$. If $\mathrm{cl}(\{x\}) \cap A = \phi$, then $A \subset X - \mathrm{cl}(\{x\})$. Since A is θ-$\mathcal{I}_g$-*closed*, $\mathrm{cl}_\theta^\star(A) \subset X - \mathrm{cl}(\{x\})$. It is a contradiction to the fact that $x \in \mathrm{cl}_\theta^\star(A)$. This proves (b).

$(b) \Rightarrow (c)$. Suppose $F \subset \mathrm{cl}_\theta^\star(A) - A$, F is closed and $x \in F$. Since $F \subset X - A$ and F closed, $\mathrm{cl}(\{x\}) \cap A \subset \mathrm{cl}(F) \cap A = F \cap A = \phi$. Since $x \in \mathrm{cl}_\theta^\star(A)$, by (b), $\mathrm{cl}(\{x\}) \cap A \neq \phi$, a contradiction which proves (c).

$(c) \Rightarrow (a)$. Let U be an open set containing A. Since $\mathrm{cl}_\theta^\star(A)$ is closed, $\mathrm{cl}_\theta^\star(A) \cap (X - U)$ is closed and $\mathrm{cl}_\theta^\star(A) \cap (X - U) \subset \mathrm{cl}_\theta^\star(A) - A$. By hypothesis, $\mathrm{cl}_\theta^\star(A) \cap (X - U) = \phi$ and hence $\mathrm{cl}_\theta^\star(A) \subset U$. Thus, A is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.3, we get Corollary 2.4 which gives characterizations for θ-*g-closed* sets. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.3, we get Corollary 2.5 which gives characterizations for *g-closed* sets.

Corollary 2.4. *If A is a subset of a topological space (X,τ), then the following are equivalent.*

(a) *A is θ-g-closed.*

(b) *For all $x \in \mathrm{cl}_\theta(A)$, $\mathrm{cl}(\{x\}) \cap A \neq \phi$.*

(c) *$\mathrm{cl}_\theta(A) - A$ contains no nonempty closed set.*

Corollary 2.5 (see [12, Theorem 2.2]). *If A is a subset of a topological space (X,τ), then the following are equivalent.*

(a) *A is g-closed.*

(b) *For all $x \in \mathrm{cl}(A)$, $\mathrm{cl}(\{x\}) \cap A \neq \phi$.*

(c) *$\mathrm{cl}(A) - A$ contains no nonempty closed set.*

The following Corollary 2.6 shows that in $\mathcal{T}_1$-*space*, θ-$\mathcal{I}_g$-*closed* sets are θ-$\mathcal{I}$-*closed*, the proof of which follows from Theorem 2.3(c). Corollary 2.7 gives the relation between θ-$\mathcal{I}_g$-*closed* and θ-$\mathcal{I}$-*closed* sets.

Corollary 2.6. *If $(X, \tau, \mathcal{I})$ is a $\mathcal{T}_1$-space and A is θ-$\mathcal{I}_g$-closed then A is a θ-$\mathcal{I}$-closed set.*

Corollary 2.7. *If $(X, \tau, \mathcal{I})$ is an ideal space and A is a θ-$\mathcal{I}_g$-closed set, then the following are equivalent.*

(a) *A is a θ-$\mathcal{I}$-closed set.*

(b) $\mathrm{cl}^\star_\theta(A) - A$ *is a closed set.*

Proof. $(a) \Rightarrow (b)$. If A is θ-$\mathcal{I}$-*closed*, then $\mathrm{cl}^\star_\theta(A) - A = \phi$ and so $\mathrm{cl}^\star_\theta(A) - (A)$ is closed.

$(b) \Rightarrow (a)$. If $\mathrm{cl}^\star_\theta(A) - (A)$ is closed, since A is θ-$\mathcal{I}_g$-*closed*, by Theorem 2.3(c), $\mathrm{cl}^\star_\theta(A) - (A) = \phi$ and so A is θ-$\mathcal{I}$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Corollary 2.7, we get Corollary 2.8. If we put $\mathcal{I} = \wp(X)$ in Corollary 2.7, we get Corollary 2.9.

Corollary 2.8. *If $(X, \tau,)$ is a topological space and A is a θ-g-closed set, then the following are equivalent.*

(a) *A is a θ-closed set.*

(b) $\mathrm{cl}_\theta(A) - A$ *is a closed set.*

Corollary 2.9 (see [6, Corollary 2.3]). *If (X, τ) is an topological space and A is a g-closed set, then the following are equivalent.*

(a) *A is a closed set.*

(b) $\mathrm{cl}(A) - A$ *is a closed set.*

Theorem 2.10. *If every open set of an ideal space $(X, \tau, \mathcal{I})$ is $\star$-closed, then every g-closed set is θ-$\mathcal{I}_g$-closed.*

Proof. Since every open set is $\star$-*closed*, $\mathrm{cl}^\star(U) = U$ for every $U \in \tau$. Therefore, for every subset A of X, $\mathrm{int}^\star_\theta(A) = \cup\{U \in \tau \mid \mathrm{cl}^\star(U) \subset A\} = \cup\{U \in \tau \mid U \subset A\} = \mathrm{int}(A)$. So $\mathrm{cl}^\star_\theta(A) = \mathrm{cl}(A)$ for every subset A of X. This implies that every g-*closed* set is θ-$\mathcal{I}_g$-*closed*. □

Corollary 2.11. *If every subset of an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}_g$-closed, then every g-closed set is θ-$\mathcal{I}_g$-closed.*

The proof follows from Lemma 1.3 and Theorem 2.10.

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then every subset of X is θ-$\mathcal{I}_g$-closed if and only if every open set is θ-$\mathcal{I}$-closed.*

Proof. Suppose every subset of X is θ-$\mathcal{I}_g$-*closed*. If U is open, then U is θ-$\mathcal{I}_g$-*closed* and so $\mathrm{cl}^\star_\theta(U) \subset U$. Hence U is θ-$\mathcal{I}$-*closed*. Conversely, suppose $A \subset U$ and U is open. Since every open set is θ-$\mathcal{I}$-*closed*, $\mathrm{cl}^\star_\theta(A) \subset U$ and so A is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.12, we get Corollary 2.13. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.12, we get Corollary 2.14.

Corollary 2.13. *Let (X, τ) be a topological space. Then every subset of X is θ-g-closed if and only if every open set is θ-closed.*

Corollary 2.14 (see [6, Theorem 2.10]). *Let (X, τ) be a topological space. Then every subset of X is g-closed if and only if every open set is closed.*

Theorem 2.15. *If every θ-$\mathcal{I}_g$-closed set of an ideal space $(X, \tau, \mathcal{I})$ is θ-closed, then (X, τ) is a $\mathcal{T}_1$ space.*

Proof. Suppose $\{x\}$ is not closed for some $x \in X$. Then, $B = X - \{x\}$ is not open. So B is θ-$\mathcal{I}_g$-*closed*. By hypothesis, B is θ-*closed*. Therefore, $\{x\}$ is θ-*open*. So $\{x\}$ is both open and closed, a contradiction. Hence, (X, τ) is a $\mathcal{T}_1$-*space*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.15, we get Corollary 2.16.

Corollary 2.16. *If every θ-g-closed set of a space (X, τ) is θ-closed, then (X, τ) is a $\mathcal{T}_1$ space.*

Theorem 2.17. *Intersection of a θ-$\mathcal{I}_g$-closed set and a θ-$\mathcal{I}$-closed set is always θ-$\mathcal{I}_g$-closed.*

Proof. Let A be a θ-$\mathcal{I}_g$-*closed* set and F a θ-$\mathcal{I}$-*closed* set of an ideal space $(X, \tau, \mathcal{I})$. Suppose $A \cap F \subset U$ and U is open in X. Then, $A \subset U \cup (X - F)$. Now $X - F$ is θ-$\mathcal{I}$-*open* and hence open. So $U \cup (X - F)$ is an open set containing A. Since A is θ-$\mathcal{I}_g$-*closed*, $\mathrm{cl}^\star_\theta(A) \subset U \cup (X - F)$. Therefore, $\mathrm{cl}^\star_\theta(A) \cap F \subset U$ which implies that $\mathrm{cl}^\star_\theta(A \cap F) \subset U$. So $A \cap F$ is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.17, we get Corollary 2.18. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.17, we get Corollary 2.19.

Corollary 2.18 (see [8, Proposition 3.11]). *Intersection of a θ-g-closed set and a θ-closed set is always θ-g-closed.*

Corollary 2.19 (see [6, Corollary 2.7]). *Intersection of a g-closed set and a closed set is always a g-closed set.*

Theorem 2.20. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}_g$-closed if and only if $\mathrm{cl}^\star_\theta(A) \subset A^\Lambda$.*

Proof. Suppose A is θ-$\mathcal{I}_g$-*closed* and $x \in \mathrm{cl}^\star_\theta(A)$. If $x \notin A^\Lambda$, then there exists an open set U such that $A \subset U$, but $x \notin U$. Since A is θ-$\mathcal{I}_g$-*closed*, $\mathrm{cl}^\star_\theta(A) \subset U$ and so $x \notin \mathrm{cl}^\star_\theta(A)$, a contradiction. Therefore, $\mathrm{cl}^\star_\theta(A) \subset A^\Lambda$. Conversely, suppose that $\mathrm{cl}^\star_\theta(A) \subset A^\Lambda$. If $A \subset U$ and U is open, then $A^\Lambda \subset U$ and so $\mathrm{cl}^\star_\theta(A) \subset U$. Therefore, A is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.20, we get Corollary 2.21. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.20, we get Corollary 2.22.

Corollary 2.21. *A subset A of a space (X, τ) is θ-g-closed if and only if $\mathrm{cl}_\theta(A) \subset A^\Lambda$.*

Corollary 2.22. *A subset A of a space (X, τ) is g-closed if and only if $\mathrm{cl}(A) \subset A^\Lambda$.*

Theorem 2.23. *Let A be a Λ-set of an ideal space $(X, \tau, \mathcal{I})$. Then A is θ-$\mathcal{I}_g$-closed if and only if A is θ-$\mathcal{I}$-closed.*

Proof. Suppose A is θ-$\mathcal{I}_g$-*closed*. By Theorem 2.20, $\mathrm{cl}^\star_\theta(A) \subset A^\Lambda = A$, since A is a Λ-*set*. Therefore, A is θ-$\mathcal{I}$-*closed*. Converse follows from the fact that every θ-$\mathcal{I}$-*closed* is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.23, we get Corollary 2.24. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.23, we get Corollary 2.25.

Corollary 2.24. *Let A be a Λ-set of a space (X, τ). Then A is θ-g-closed if and only if A is θ-closed.*

Corollary 2.25. *Let A be a Λ-set of a space (X, τ). Then A is g-closed if and only if A is closed.*

Theorem 2.26. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. If A^{Λ} is θ-$\mathcal{I}_g$-closed, then A is also θ-$\mathcal{I}_g$-closed.*

Proof. Suppose that A^{Λ} is a θ-$\mathcal{I}_g$-*closed* set. If $A \subset U$ and U is open, then $A^{\Lambda} \subset U$. Since A^{Λ} is θ-$\mathcal{I}_g$-*closed*, $\mathrm{cl}^{\star}_{\theta}(A^{\Lambda}) \subset U$. But, $\mathrm{cl}^{\star}_{\theta}(A) \subset \mathrm{cl}^{\star}_{\theta}(A^{\Lambda})$. Therefore, A is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.26, we get Corollary 2.27. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.26, we get Corollary 2.28.

Corollary 2.27. *Let (X, τ) be a topological space and $A \subset X$. If A^{Λ} is θ-g-closed, then A is also θ-g-closed.*

Corollary 2.28. *Let (X, τ) be a space and $A \subset X$. If A^{Λ} is g-closed set, then A is also g-closed.*

Theorem 2.29. *For an ideal space $(X, \tau, \mathcal{I})$, the following are equivalent.*

(a) *Every θ-$\mathcal{I}_g$-closed set is θ-$\mathcal{I}$-closed.*

(b) *Every singleton of X is closed or θ-$\mathcal{I}$-open.*

Proof. $(a) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then $A = X - \{x\} \notin \tau$ and then A is trivially θ-$\mathcal{I}_g$-*closed*. By (a), A is θ-$\mathcal{I}$-*closed*. Hence $\{x\}$ is θ-$\mathcal{I}$-*open*.

$(b) \Rightarrow (a)$. Let A be a θ-$\mathcal{I}_g$-*closed* set and let $x \in \mathrm{cl}^{\star}_{\theta}(A)$. We have the following cases.

Case 1. $\{x\}$ is closed. By Theorem 2.3, $\mathrm{cl}^{\star}_{\theta}(A) - A$ does not contain a nonempty closed subset. This shows $\{x\} \in A$.

Case 2. $\{x\}$ is θ-$\mathcal{I}$-*open*. Then, $\{x\} \cap A \neq \phi$. Hence, $x \in A$.

Thus in both cases $x \in A$ and so $A = \mathrm{cl}^{\star}_{\theta}(A)$, that is, A is θ-$\mathcal{I}$-*closed*, which proves (a). □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.29, we get Corollary 2.30. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.29, we get Corollary 2.31.

Corollary 2.30. *For an ideal space (X, τ), the following are equivalent.*

(a) *Every θ-g-closed set is θ-closed.*

(b) *Every singleton of X is closed or θ-open.*

Corollary 2.31 (see [13, Theorem 2.5]). *For an ideal space (X, τ), the following are equivalent.*

(a) *Every g-closed set is closed.*

(b) *Every singleton of X is closed or open.*

Theorem 2.32. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then A is θ-$\mathcal{I}_g$-closed if and only if $A = F - N$, where F is θ-$\mathcal{I}$-closed and N contains no nonempty closed set.*

Proof. If A is θ-$\mathcal{I}_g$-*closed*, then by Theorem 2.3, $N = \mathrm{cl}_\theta^\star(A) - A$ contains no nonempty closed set. If $F = \mathrm{cl}_\theta^\star(A)$, then F is θ-$\mathcal{I}$-*closed* such that $F - N = \mathrm{cl}_\theta^\star(A) - (\mathrm{cl}_\theta^\star(A) - A) = \mathrm{cl}_\theta^\star(A) \cap ((X - \mathrm{cl}_\theta^\star(A)) \cup A) = A$. Conversely, suppose $A = F - N$, where F is θ-$\mathcal{I}$-*closed* and N contains no nonempty closed set. Let U be an open set such that $A \subset U$. Then, $F - N \subset U$ which implies that $F \cap (X - U) \subset N$. Now, $A \subset F$ and F is θ-$\mathcal{I}$-*closed* implies that $\mathrm{cl}_\theta^\star(A) \cap (X - U) \subset \mathrm{cl}_\theta^\star(F) \cap (X - U) \subset F \cap (X - U) \subset N$. Since θ-$\mathcal{I}$-*closed* sets are closed, $\mathrm{cl}_\theta^\star(A) \cap (X - U)$ is closed. By hypothesis, $\mathrm{cl}_\theta^\star(A) \cap (X - U) = \phi$ and so $\mathrm{cl}_\theta^\star(A) \subset U$, which implies that A is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.32, we get Corollary 2.33. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.32, we get Corollary 2.34.

Corollary 2.33. *Let (X, τ) be a space and $A \subset X$. Then A is θ-g-closed subset of X if and only if $A = F - N$, where F is θ-closed and N contains no nonempty closed set.*

Corollary 2.34 (see [12, Corollary 2.3]). *Let (X, τ) be a space and $A \subset X$. Then A is g-closed if and only if $A = F - N$, where F is closed and N contains no nonempty closed set.*

Theorem 2.35. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is a θ-$\mathcal{I}_g$-closed subset of X and $A \subset B \subset \mathrm{cl}_\theta^\star(A)$, then B is also θ-$\mathcal{I}_g$-closed.*

Proof. $\mathrm{cl}_\theta^\star(B) - B \subset \mathrm{cl}_\theta^\star(A) - A$, and since $\mathrm{cl}_\theta^\star(A) - A$ has no nonempty closed subset, neither does $\mathrm{cl}_\theta^\star(B) - B$. By Theorem 2.3, B is θ-$\mathcal{I}_g$-*closed*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.35, we get Corollary 2.36. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.35, we get Corollary 2.37.

Corollary 2.36. *Let (X, τ) be a space. If A is a θ-g-closed subset of X and $A \subset B \subset \mathrm{cl}_\theta(A)$, then B is also θ-g-closed.*

Corollary 2.37 (see [6, Theorem 2.8]). *Let (X, τ) be a space. If A is a g-closed subset of X and $A \subset B \subset \mathrm{cl}(A)$, then B is also g-closed.*

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be θ-$\mathcal{I}_g$-*open* if $X - A$ is θ-$\mathcal{I}_g$-*closed*.

Theorem 2.38. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}_g$-open if and only if $F \subset \mathrm{int}_\theta^\star(A)$ whenever F is closed and $F \subset A$.*

Proof. Suppose A is a θ-$\mathcal{I}_g$-*open* set and F is a closed set contained in A, then $X - A \subset X - F$ and $X - F$ is open. Since $X - A$ is θ-$\mathcal{I}_g$-*closed*, $\mathrm{cl}_\theta^\star(X - A) \subset (X - F)$ and so $F \subset X - \mathrm{cl}_\theta^\star(X - A) = \mathrm{int}_\theta^\star(A)$. Conversely, suppose $X - A \subset U$ and $X - U$ is closed. By hypothesis, $X - U \subset \mathrm{int}_\theta^\star(A)$, which implies that $\mathrm{cl}_\theta^\star(X - A) = X - \mathrm{int}_\theta^\star(A) \subset U$. Therefore, $X - A$ is θ-$\mathcal{I}_g$-*closed* and hence A is θ-$\mathcal{I}_g$-*open*. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.38, we get Corollary 2.39. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.38, we get Corollary 2.40.

Corollary 2.39. *A subset A of a space (X, τ) is θ-g-open if and only if $F \subset \mathrm{int}_\theta(A)$ whenever F is closed and $F \subset A$.*

Corollary 2.40 (see [6, Theorem 4.2]). *A subset A of a space (X, τ) is g-open if and only if $F \subset \mathrm{int}(A)$ whenever F is closed and $F \subset A$.*

Theorem 2.41. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset U$. Then the following are equivalent.*

(a) *A is θ-$\mathcal{I}_g$-closed.*

(b) *$A \cup (X - \mathrm{cl}_\theta^\star(A))$ is θ-$\mathcal{I}_g$-closed.*

(c) *$\mathrm{cl}_\theta^\star(A) - A$ is θ-$\mathcal{I}_g$-open.*

Proof. $(a) \Rightarrow (b)$. Suppose A is θ-$\mathcal{I}_g$-*closed*. If U is any open set containing $A \cup (X - \mathrm{cl}_\theta^\star(A))$, then $X - U \subset X - (A \cup (X - \mathrm{cl}_\theta^\star(A)) = \mathrm{cl}_\theta^\star(A) - A$. Since A is θ-$\mathcal{I}_g$-*closed*, by Theorem 2.3(c), it follows that $X - U = \phi$ and so $X = U$. Since X is the only open set containing $A \cup (X - \mathrm{cl}_\theta^\star(A))$, $A \cup (X - \mathrm{cl}_\theta^\star(A))$ is θ-$\mathcal{I}_g$-*closed*.

$(b) \Rightarrow (a)$. Suppose $A \cup (X - \mathrm{cl}_\theta^\star(A))$ is θ-$\mathcal{I}_g$-*closed*. If F is any closed set contained in $\mathrm{cl}_\theta^\star(A) - A$, then $A \cup (X - \mathrm{cl}_\theta^\star(A)) \subset X - F$ and $X - F$ is open. Therefore, $\mathrm{cl}_\theta^\star(A \cup (X - \mathrm{cl}_\theta^\star(A)) \subset X - F$, which implies that $\mathrm{cl}_\theta^\star(A) \cup \mathrm{cl}_\theta^\star(X - \mathrm{cl}_\theta^\star(A)) \subset X - F$ and so $X \subset X - F$; it follows that $F = \phi$. Hence A is θ-$\mathcal{I}_g$-*closed*.

The equivalence of (b) and (c) follows from the fact that $X - (\mathrm{cl}_\theta^\star(A) - A) = A \cup (X - \mathrm{cl}_\theta^\star(A))$. □

If we put $\mathcal{I} = \{\phi\}$ in Theorem 2.41, we get Corollary 2.42. If we put $\mathcal{I} = \wp(X)$ in Theorem 2.41, we get Corollary 2.43.

Corollary 2.42. *Let (X, τ) be a space and $A \subset U$. Then the following are equivalent.*

(a) *A is θ-g-closed.*

(b) *$A \cup (X - \mathrm{cl}_\theta(A))$ is θ-g-closed.*

(c) *$\mathrm{cl}_\theta(A) - A$ is θ-g-open.*

Corollary 2.43. *Let (X, τ) be an ideal space and $A \subset U$. Then the following are equivalent.*

(a) *A is g-closed.*

(b) *$A \cup (X - \mathrm{cl}(A))$ is g-closed.*

(c) *$\mathrm{cl}(A) - A$ is g-open.*

3. Characterization of $\mathcal{T}_{1/2}$ and $\mathcal{T}_{\mathcal{I}}$-*Space*

Theorem 3.1. *In an ideal space $(X, \tau, \mathcal{I})$, the following are equivalent.*

(a) *Every θ-g-closed set is closed.*

(b) *(X, τ) is a $\mathcal{T}_{1/2}$-space.*

(c) *Every θ-$\mathcal{I}_g$-closed set is closed.*

Proof. $(a) \Leftrightarrow (b)$. Equivalence of (a) and (b) follows from Theorem 4.1 of [8].

$(b) \Rightarrow (c)$. Let A be a θ-$\mathcal{I}_g$-*closed* set. Since every θ-$\mathcal{I}_g$-*closed* set is g-*closed*, A is g-*closed*. By hypothesis, A is closed.

$(c) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then $B = X - \{x\}$ is not open. So B is θ-$\mathcal{I}_g$-*closed*. By hypothesis, B is closed and so $\{x\}$ is open. By Corollary 2.31, (X, τ) is a $\mathcal{T}_{1/2}$-*space*. □

Theorem 3.2. *In an ideal space $(X, \tau, \mathcal{I})$ the following, are equivalent.*

(a) *Every θ-g-closed set is $\star$-closed.*

(b) $(X, \tau, \mathcal{I})$ *is a* $\mathcal{T}_{\mathcal{I}}$*-Space.*

(c) *Every* θ-$\mathcal{I}_g$*-closed set is* $\star$*-closed.*

Proof. $(a) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then X is the only open set containing $X - \{x\}$ and so $X - \{x\}$ is θ-g-*closed*. By hypothesis, $X - \{x\}$ is $\star$-*closed*. Equivalently $\{x\}$ is $\star$-*open*. Thus, every singleton set in X is either closed or $\star$-*open*. By Theorem 3.3 of [4], $(X, \tau, \mathcal{I})$ is a $\mathcal{T}_{\mathcal{I}}$-*Space*.

$(b) \Rightarrow (a)$. The proof follows from the fact that every θ-g-*closed* set is $\mathcal{I}_g$-*closed*.

$(b) \Rightarrow (c)$. The proof follows from the fact that every set is θ-$\mathcal{I}_g$-*closed* $\mathcal{I}_g$-*closed*.

$(c) \Rightarrow (b)$. Let $x \in X$. If $\{x\}$ is not closed, then X is the only open set containing $x - \{x\}$ and so $x - \{x\}$ is θ-$\mathcal{I}_g$-*closed*. By hypothesis, $X - \{x\}$ is $\star$-*closed*. Thus, $\{x\}$ is $\star$-*open*. Therefore, every singleton set in X is either $\star$-*open* or closed. By Theorem of 3.3 [4], $(X, \tau, \mathcal{I})$ is a $\mathcal{T}_{\mathcal{I}}$-*Space*. □

The proof of the Corollary 3.3 follows from Theorem 3.2 and Theorem 3.10 of [5].

If we put $\mathcal{I} = \{\phi\}$ in Corollary 3.3, we get Corollary 3.4.

Corollary 3.3. *In an ideal space* $(X, \tau, \mathcal{I})$*, the following are equivalent.*

(a) *Every* θ-g*-closed set is* $\star$*-closed.*

(b) *Every* θ-$\mathcal{I}_g$*-closed set is* $\star$*-closed.*

(c) *Every* $\mathcal{I}_g$*-closed set is an* $\mathcal{I}$*-locally* $\star$*-closed set.*

Corollary 3.4. *In a topological space* (X, τ)*, the following are equivalent.*

(a) *Every* θ-g*-closed set is closed.*

(b) *Every* g*-closed set is a locally closed set.*

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