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Research Article

A Comparative Study on the Stability of Laplace-Adomian Algorithm and Numerical Methods in Generalized Pantograph Equations

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The main objective of this paper is to examine the stability and convergence of the Laplace-Adomian algorithm to approximate solutions of the pantograph-type differential equations with multiple delays. This is done by comparatively investigating it with other methods.

1. Introduction

Delay differential equations (DDEs) are a large and important class of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. They often arise in wide and diverse range of applications in population studies [1, 2], economics [3], medical biology [4, 5], controls of mechanical systems [6], and so forth. The pantograph equation is one of the most important kinds of DDEs. The name pantograph was used by Iserles and Liu [7] to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name.

In recent years, some promising approximate analytical solutions have been proposed, such as the Taylor collocation method [8], Bernstein polynomials [9], spline method [10, 11], and other methods are reviewed in [2, 12, 13].

In this work we consider the following problems.

Problem 1.

$$u'(t) = \beta u(t) + f(t, u(t), u(q_1 t), u(q_2 t), \ldots, u(q_\ell t)),$$
$$u(0) = u_0. \tag{1}$$

Problem 2.

$$u^{(m)}(t) = f(t, u(t), u(q_1 t), u(q_2 t), \ldots, u(q_\ell t)),$$
$$\sum_{k=0}^{m-1} C_{ik} u^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1, \tag{2}$$

where f is an analytical function, C_{ik}, λ_i, and $\beta \in \ldots C$; $0 < q_i < 1$, $i = 1, 2, \ldots, \ell$.

The aim of this paper is to employ and examine the stability of the Laplace-Adomian algorithm (LAA) for solving Problems 1 and 2. This method was first proposed by Khuri [14], who applied the scheme to a class of nonlinear differential equations. In this method the solution is given as an infinite series usually converging very rapidly to the exact solution of the problem.

A major advantage of this method is that it is free from round-off errors and without any discretization or restrictive assumptions. Therefore, results obtained by LAA are more accurate and efficient. LAA has been shown to accurately and easily approximate solutions of large class of linear and nonlinear ODEs and PDEs [14–16]; for example, Ongun [17] employed LAA to give an approximate solution of

nonlinear ordinary differential equation systems, such as a model for HIV infection of $CD4^+$ T cells, Wazwaz [18] also used this method for handling the nonlinear Volterra integro-differential equations, Khan and Faraz [19] modified LAA to obtain series solutions of the boundary layer equation, and Yusufoglu [20] adapted LAA to solve the Duffing equation.

The numerical technique of LAA basically illustrates how the Laplace transforms are used to approximate the solution of the nonlinear differential equations by manipulating the decomposition method that was first introduced by Adomian [21, 22].

2. Laplace-Adomian Algorithm

To illustrate the basic idea of the Laplace-Adomian algorithm, we consider the following nonlinear operator:

$$u'(t) = Ru + Nu + f(t), \tag{3}$$

with the initial condition

$$u(0) = u_0, \tag{4}$$

where R is a linear operator, N is a nonlinear operator, and $f(t)$ is a given analytical function.

The technique consists first of applying the Laplace transform (denoted throughout this paper by $\mathcal{L}$) to both sides of (3), to get

$$\mathcal{L}[u'(t)] = \mathcal{L}[Ru] + \mathcal{L}[Nu] + \mathcal{L}[f(t)]. \tag{5}$$

Applying the formulas of the Laplace transform, we obtain

$$\mathcal{L}[u(t)] = \mathcal{H}(s) + \frac{1}{s}\mathcal{L}[Ru] + \frac{1}{s}\mathcal{L}[Nu], \tag{6}$$

where

$$\mathcal{H}(s) = \frac{1}{s}(u(0) + \mathcal{L}[f(t)]). \tag{7}$$

Suppose the answer to (3) is as follows:

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{8}$$

where the terms $u_n(t)$ are to be recursively computed, the nonlinear operator Nu is decomposed as follows:

$$Nu(t) = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where $A_n$is an infinite series of the Adomian polynomials $u_0, u_1, \ldots, u_n$, calculated by the formula [21]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{10}$$

Substituting (8) and (9) into (6) leads to

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(t)\right] = \mathcal{H}(s) + \frac{1}{s}\mathcal{L}\left[R\sum_{n=0}^{\infty} u_n\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{11}$$

Using the linearity of the Laplace transform gives (11) as

$$\sum_{n=0}^{\infty}\mathcal{L}[u_n(t)] = \mathcal{H}(s) + \frac{1}{s}\sum_{n=0}^{\infty}\mathcal{L}[Ru_n] + \frac{1}{s}\sum_{n=0}^{\infty}\mathcal{L}[A_n]. \tag{12}$$

Matching both sides of (11) yields

$$\mathcal{L}[u_0(t)] = \frac{1}{s}(u(0) + \mathcal{L}[f(t)]) = \mathcal{H}(s), \tag{13}$$

$$\mathcal{L}[u_1(t)] = \frac{1}{s}\mathcal{L}[Ru_0] + \frac{1}{s}\mathcal{L}[A_0], \tag{14}$$

$$\mathcal{L}[u_2(t)] = \frac{1}{s}\mathcal{L}[Ru_1] + \frac{1}{s}\mathcal{L}[A_1]. \tag{15}$$

Generally

$$\mathcal{L}[u_{n+1}(t)] = \frac{1}{s}\mathcal{L}[Ru_n] + \frac{1}{s}\mathcal{L}[A_n], \quad n \geq 0. \tag{16}$$

Applying the inverse Laplace transform to (13) gives the initial approximation

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s}(u(0) + \mathcal{L}[f(t)])\right] = H(t). \tag{17}$$

Substituting this value of u_0 into the inverse Laplace transform of (14) gives u_1. The other terms $u_2, u_3, \ldots$ can be obtained recursively in similar fashion from

$$u_{n+1}(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Ru_n] + \frac{1}{s}\mathcal{L}[A_n]\right], \quad n \geq 0. \tag{18}$$

Using (18), we can rewrite them as follows to obtain u:

$$\begin{aligned}
u_0 &= \mathcal{L}^{-1}\left[\frac{1}{s}\left(u(0)+\mathcal{L}[f(t)]\right)\right] = H(t),\\
u_1 &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Ru_0]+\frac{1}{s}\mathcal{L}[A_0]\right]\\
&= (\mathcal{L}^{-1}s^{-1}\mathcal{L}R)u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
u_2 &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Ru_1]+\frac{1}{s}\mathcal{L}[A_1]\right]\\
&= \mathcal{L}^{-1}[s^{-1}\mathcal{L}[R((\mathcal{L}^{-1}s^{-1}\mathcal{L}R)u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0)]\\
&\qquad +s^{-1}\mathcal{L}[A_1]]\\
&= (\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^2u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
&\quad +(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_1\\
u_3 &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Ru_2]+\frac{1}{s}\mathcal{L}[A_2]\right]\\
&= (\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^3u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^2(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
&\quad +(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_1+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_2,\\
&\vdots\\
u_{n+1} &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Ru_n]+\frac{1}{s}\mathcal{L}[A_n]\right]\\
&= (\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^{n+1}u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^n(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
&\quad +(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^{n-1}(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_1\\
&\quad +\cdots+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_n,\\
&\vdots
\end{aligned} \tag{19}$$

Subsiutiting the values of $u_0, u_1, u_2, \ldots$ into (8) gives

$$\begin{aligned}
u(t) &= H(t)+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)u_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
&\quad +(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^2u_0\\
&\quad +(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0\\
&\quad +\cdots+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^nu_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^{n-1}\\
&\quad \times(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_0+(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^{n-2}(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_1\\
&\quad +\cdots+(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_n+\cdots\\
&= \sum_{m=0}^{\infty}(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^mH(t)\\
&\quad +\sum_{m=1}^{\infty}\sum_{k=0}^{m-1}(\mathcal{L}^{-1}s^{-1}\mathcal{L}R)^{m-k-1}(\mathcal{L}^{-1}s^{-1}\mathcal{L})A_k.
\end{aligned} \tag{20}$$

The sum of m terms $u_0, u_1, u_2, \ldots, u_{m-1}$ in (8), that is, $Q_m = \sum_{n=0}^{m-1} u_n$, where $m \to \infty$, Q_m tends to u. This means that Q_m is an appropriate approximation of u. The terms in the above series soon tend to zero where $1/(mn)!$ has been the coefficients of calculations derived from the operation $(\mathcal{L}^{-1}s^{-1}\mathcal{L})$, m is the number of terms, and n is the order of the operation derivation. Therefore, it has a rapid convergence.

3. Test Problems

In this section, we will apply Laplace-Adomian algorithm to solve the pantograph delay equation. The absolute errors in Tables 1–4 are the values of $|u(t) - \sum_{i=0}^{n} u_i(t)|$ at selected points. All iterates are calculated by using Matlab 7.

3.1. Stability of the Laplace-Adomian Algorithm. A method is said to be stable when the obtained solution undergoes small variations as there are slight variations in inputs and parameters and when probable perturbations in parameters that are effective in equations and conditions prevailing them do not introduce, in comparison to the physical reality of the problem, any perturbations in what is returned. We propose here to compare the Laplace-Adomian algorithm with other numerical methods by offering examples and examining the stability of the Laplace-Adomian algorithm.

Example 1 (Example (Evans and Raslan [13])). Consider the following pantograph equation:

$$u'(t) = \frac{1}{2}u(t)+\frac{1}{2}e^{(1/2)t}u\left(\frac{t}{2}\right), \quad 0 \le t \le 1,\ u(0) = 1. \tag{21}$$

The exact solution is $u(t) = e^t$. Applying the result of (17) gives us

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1. \tag{22}$$

The iteration formula (18) for this example is

$$u_{n+1}(t) = \mathcal{L}^{-1}\left[\frac{1}{2s}\mathcal{L}\left[u_n+e^{(t/2)t}u_n\left(\frac{t}{2}\right)\right]\right], \quad n \ge 0. \tag{23}$$

In Table 1 we make a comparison between the absolute errors obtained by the Bernstien series [9] in column 2, together with the spline method [10, 11] in columns 3 and 4, and finally, the Laplace-Adomian algorithm with $n = 5, 6$, and 7 in the last three columns. Given the following absolute errors that were obtained from these methods, we can conclude that the rate of convergence of LAA is higher than the other methods. However, taking a closer look at the errors of LAA shows that the error increases significantly more than the other methods in the bottom of the columns. This means that the stability of LAA is decreasing more than the other methods.

Table 1: Comparison of the absolute errors for Example 1.

t	Bernstein series [9]	Spline method, $h = 0.001$		Present method		
	$N = 7$	$m = 2$ [10]	$m = 3$ [11]	$n = 5$	$n = 6$	$n = 7$
0.2	$0.3004E-8$	$0.198E-7$	$1.37E-11$	$6.737E-9$	$9.726E-11$	$1.220E-12$
0.4	$0.3610E-8$	$0.473E-7$	$3.27E-11$	$4.530E-7$	$1.301E-8$	$3.250E-10$
0.6	$0.4024E-8$	$0.847E-7$	$5.86E-11$	$5.428E-6$	$2.326E-7$	$8.675E-9$
0.8	$0.4161E-8$	$0.135E-6$	$9.54E-11$	$3.212E-5$	$1.825E-6$	$9.033E-8$
1.0	$0.8934E-7$	$0.201E-6$	$1.43E-10$	$1.292E-4$	$9.125E-6$	$5.617E-7$

Table 2: Comparison of the absolute errors for $q = 0.2$ for Example 2.

t	Taylor collocation method [8]		Present method		
	$N = 10$	$N = 12$	$n = 2$	$n = 3$	$n = 4$
2^{-1}	$0.200E-9$	$0.124E-9$	$1.600E-2$	$2.052E-3$	$2.088E-4$
2^{-2}	$0.100E-9$	$0.974E-10$	$2.137E-3$	$1.353E-4$	$6.823E-6$
2^{-3}	$0.100E-9$	$0.700E-10$	$2.763E-4$	$8.686E-6$	$2.181E-7$
2^{-4}	$0.100E-9$	$0.914E-10$	$3.513E-5$	$5.504E-7$	$6.915E-9$
2^{-5}	$0.100E-9$	$0.528E-10$	$4.430E-6$	$3.464E-8$	$2.684E-10$
2^{-6}	0.000	$0.195E-10$	$5.561E-7$	$2.172E-9$	$7.417E-11$

Example 2 (Example (Muroya et al. [23])). Consider the following pantograph equation:

$$u'(t) = -u(t) + \frac{q}{2}u(qt) - \frac{q}{2}e^{-qt}, \quad 0 \le t \le 1,\ u(0) = 1. \tag{24}$$

The exact solution is $u(t) = e^{-t}$. Applying the result of (17), gives us

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(1 - \mathcal{L}\left[\frac{q}{2}e^{-qt}\right]\right)\right] = \frac{1}{2} + \frac{1}{2}e^{-qt}. \tag{25}$$

The other terms of the sequences u_n can be obtained directly from (18) as in general

$$u_{n+1}(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[-u_n + \frac{q}{2}u_n(qt)\right]\right], \quad n \ge 0. \tag{26}$$

Table 2 compares the results of the Laplace-Adomian algorithm and the Taylor collocation method [8]. It can be seen that the Laplace-Adomian algorithm is weaker in the stability than Taylor collocation method and yet is stronger in the convergence.

Example 3 (Example (Liu and Li [12])). Consider the multipantograph delay equation with variable coefficients

$$u'(t) = -u(t) + \mu_1(t)u(0.5t) + \mu_2(t)u(0.25t), \qquad 0 \le t \le 1,\ u(0) = 1. \tag{27}$$

Here $\mu_1(t) = -e^{-0.5t}\sin(0.5t)$, and $\mu_2(t) = -2e^{-0.75t}\cos(0.5t)\sin(0.25t)$. The initial approximation for this example is

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1. \tag{28}$$

And from (18), we obtain

$$\begin{aligned} u_{n+1}(t) &= \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left[-u_n - e^{-(1/2)t}\sin\left(\frac{t}{2}\right)u_n\left(\frac{t}{2}\right)\right.\right. \\ &\qquad \left.\left. -2e^{-(3/4)t}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{4}\right)u_n\left(\frac{t}{4}\right)\right]\right), \quad n \ge 0. \end{aligned} \tag{29}$$

Comparisons of approximate solutions for few terms with exact solution $u(t) = e^{-t}\cos(t)$ are illustrated in Table 3.

Example 4. Consider the pantograph equation of third order

$$u'(t) = 1 - \sin(t) - 2u^2\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{2}\right)u\left(\frac{t}{2}\right), \qquad 0 \le t \le 1,\ u(0) = 0. \tag{30}$$

Let us start with an initial approximation

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[1 - \sin(t)]\right] = t - 1 + \cos(t). \tag{31}$$

The other terms of the sequences u_n can be obtained directly from (18)

$$u_{n+1}(t) = \mathcal{L}^{-1}\left[\frac{2}{s}\mathcal{L}\left[-A_n\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)u_n\left(\frac{t}{2}\right)\right]\right], \quad n \ge 0, \tag{32}$$

Table 3: Absolute errors for Example 3.

t	Exact value	The present method		
		$n = 1$	$n = 2$	$n = 3$
0.2	$8.0241E-1$	$2.080E-2$	$1.404E-3$	$7.026E-5$
0.4	$6.1740E-1$	$8.483E-2$	$1.164E-2$	$1.169E-3$
0.6	$4.5295E-1$	$1.918E-1$	$4.018E-2$	$6.092E-3$
0.8	$3.1305E-1$	$3.387E-1$	$9.639E-2$	$1.964E-2$
1.0	$1.9876E-1$	$5.206E-1$	$1.889E-1$	$4.851E-2$

Table 4: Absolute errors for Example 4.

t	Exact value	The present method		
		$n = 1$	$n = 2$	$n = 3$
0.2	$1.9866E-1$	$5.524E-4$	$4.655E-6$	$2.134E-8$
0.4	$3.8941E-1$	$3.548E-3$	$2.313E-5$	$2.240E-6$
0.6	$5.6464E-1$	$9.196E-3$	$1.063E-4$	$2.572E-5$
0.8	$7.1735E-1$	$1.562E-2$	$9.314E-4$	$1.334E-4$
1.0	$8.4147E-1$	$1.931E-2$	$3.470E-3$	$4.523E-4$

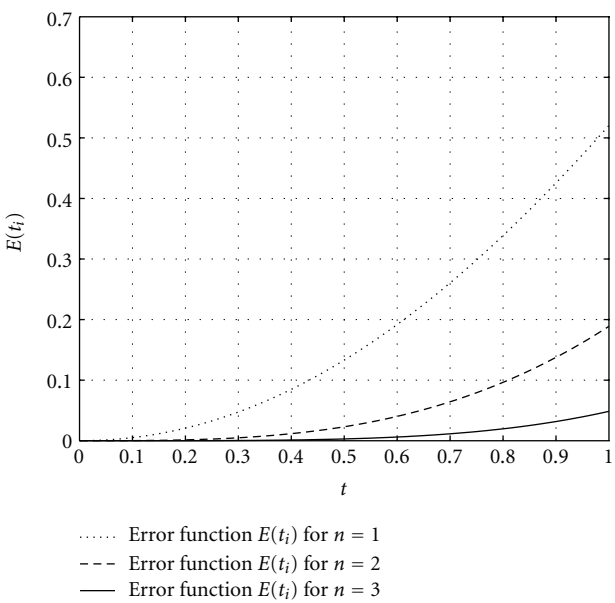

Figure 1: Error $E(t_i)$ for $n = 1, 2$ and 3 for Example 3.

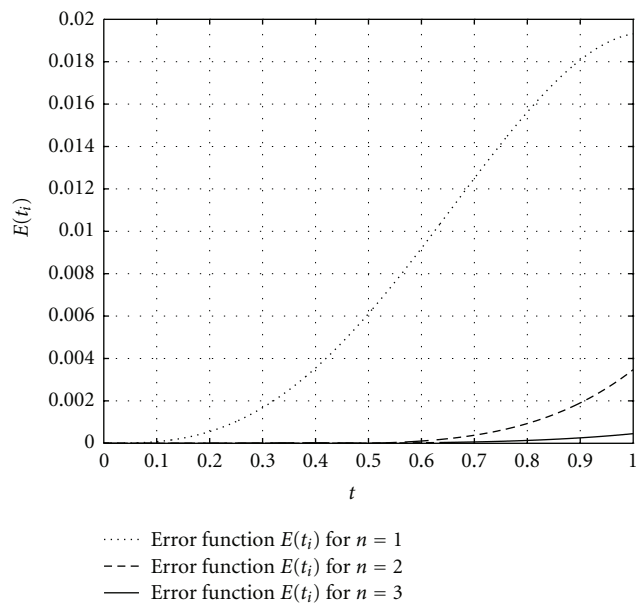

Figure 2: Error $E(t_i)$ for $n = 1, 2$ and 3 for Example 4.

where A_n is the Adomian polynomials that represent the nonlinear term $u^2(x/2)$. The Adomian polynomials for $A(u(t)) = u^2(t/2)$ are given by

$$A_0 = u_0^2\left(\frac{t}{2}\right), \tag{33}$$

$$A_1 = 2u_0\left(\frac{t}{2}\right)u_1\left(\frac{t}{2}\right), \tag{34}$$

$$A_2 = u_1^2\left(\frac{t}{2}\right) + 2u_0\left(\frac{t}{2}\right)u_2\left(\frac{t}{2}\right), \tag{35}$$

$$\vdots \tag{36}$$

Table 4 shows the absolute error of the numerical approximation by using a few iterations.

The results for Examples 3–4 in Figures 1 and 2, respectively, show that the approximations converge rapidly and only a few iterations are sufficient to obtain accurate solutions.

Example 5. Consider the multipantograph equation

$$u'(t) = -\frac{5}{6}u(t) + 4u\left(\frac{t}{2}\right) + 9u\left(\frac{t}{3}\right) + t^2 - 1, \quad 0 \le t \le 1,\ u(0) = 1. \tag{37}$$

Following the procedures in the previous examples gives us

$$u_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{2}{s^4} - \frac{1}{s^2}\right] = 1 - t + \frac{t^3}{3}. \tag{38}$$

According to (18), we obtain

$$\begin{aligned} u_1(t) &= \frac{73}{6}t - \frac{25}{12}t^2, \\ u_2(t) &= \frac{1825}{72}t^2 - \frac{175}{216}t^3, \\ u_3(t) &= \frac{12775}{1296}t^3, \\ u_n(t) &= 0, \quad \forall n \geq 4. \end{aligned} \tag{39}$$

Thus

$$u(t) = \sum_{n=0}^{\infty} u_n(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3 \tag{40}$$

which is the exact solution.

4. Conclusion

In this paper, the Laplace-Adomian algorithm has been successfully applied to the generalized pantograph equations with multiple delays to obtain high approximate solutions or exact solutions with little iterations used. Moreover, the Laplace-Adomian algorithm is simple and easy to use and stronger in convergence compared with other methods. Despite these advantages, it is seen that its stability is lower than other numerical methods.

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