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Research Article

On the Completely Positive and Positive Semidefinite-Preserving Cones—Part III

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This paper studies the extremals and other faces of the completely positive and positive semidefinite-preserving linear transformations.

1. Introduction

This paper is the third in a sequence giving the related theory of the cone $\pi(\mathrm{PSD}_n)$ of positive semidefinite-preserving linear transformations on the complex vector space of complex matrices of order n, denoted by M_n, and its self-dual subcone CP_n of the completely positive linear transformations. Following the papers of Barker et al. [1] and Yopp and Hill [2], this third paper studies the extremals and other faces of these cones. The cone of completely copositive linear transformations, coCP_n, fits nicely in this work as well.

For a list of all the cone-theoretic definitions used here, see [2]. For a list of characterization theorems for CP_n as well as HP_n, the Hermitian preservers, which are an ambient space for all the cones of this paper, see [3].

2. Classes of Extremals of $\pi(\mathrm{PSD}_n)$

Yopp and Hill [2, 4] have characterized the extremals of CP_n and coCP_n as follows.

Theorem 2.1. *Let $\mathcal{T} : M_n \to M_n$. Then,*

(i) $\mathcal{T} \in \operatorname{Ext}\mathrm{CP}_n$ *if and only if there exists $A \in M_n \setminus \{0\}$ such that $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A$;*

(ii) *if $\langle \mathcal{T} \rangle = \sum_{i=1}^{r} \overline{A}_i \otimes_K A_i$, where $A_1, \ldots, A_r \in M_n \setminus \{0\}$, then $\mathcal{T} \in \operatorname{Ext}\mathrm{CP}_n$ if and only if $A_i = \alpha_i A_1$ for all $i = 1, \ldots, r$;*

(iii) $\mathcal{T} \in \operatorname{Ext}\mathrm{CP}_n$ *if and only if there exist* $A_1, \ldots, A_r \in M_n \setminus \{0\}$ *and* $(d_{ij}) \in \mathrm{ExtPSD}_r$ *such that* $\langle \mathcal{T} \rangle = \sum_{i,j=1}^{r} d_{ij} \overline{A_i} \otimes_K A_j$.

Theorem 2.2. *Let* $\mathcal{T} : M_n \to M_n$ *be an element of* coCP_n. *Then,*

(i) $\mathcal{T} \in \operatorname{Ext}\mathrm{coCP}_n$ *if and only if* $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A \circ \mathbf{T}$ *for some* $A \in M_n \setminus \{0\}$;

(ii) *if* $\langle \mathcal{T} \rangle = \sum_{i=1}^{r} \overline{A_i} \otimes_K A_i \circ \mathbf{T}$, *where* $A_1, \ldots, A_r \in M_n \setminus \{0\}$, *then* $\mathcal{T} \in \operatorname{Ext}\mathrm{coCP}_n$ *if and only if* $A_i = \alpha_i A_1$ *for all* $i = 1, \ldots, r$;

(iii) $\mathcal{T} \in \operatorname{Ext}\mathrm{coCP}_n$ *if and only if there exist* $A_1, \ldots, A_r \in M_n \setminus \{0\}$ *and* $(d_{ij}) \in \mathrm{ExtPSD}_r$ *such that* $\langle \mathcal{T} \rangle = \sum_{i,j=1}^{r} d_{ij} \overline{A_i} \otimes_K A_j \circ \mathbf{T}$.

The remainder of this section focuses on the extremals of $\pi(\mathrm{PSD}_n)$. The major result of [5] follows.

Theorem 2.3. *Let* $A \in M_n$. *If* $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A$ *or* $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A \circ \mathbf{T}$, *then* $\mathcal{T}$ *is an extremal of* $\pi(\mathrm{PSD}_n)$, *and when the rank of* A *is 1 or* n, *the extremals are exposed.*

The following result, originally given in a different setting by Loewy and Schneider [6], yields a (different) large class of extremals, namely, the nonsingular maps. Further, we note that the analogous result does not hold for CP_n. We first give an alternative proof for the theorem.

Theorem 2.4. *Every nonsingular element of* $\pi(\mathrm{PSD}_n)$ *is an extremal of* $\pi(\mathrm{PSD}_n)$.

Proof. Let A be a nonsingular element of $\pi(\mathrm{PSD}_n)$. Since every nonsingular linear transformation is a vector space isomorphism, it follows from [2, Theorem 1.1(iv)] that $A(\mathrm{ExtPSD}_n) \subset \mathrm{ExtPSD}_n$. By [7, Theorem 2.B.2], it follows that A is an extremal of $\pi(\mathrm{PSD}_n)$. □

The following example shows that this result does not apply to CP_n. Let $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, and let $A_2 = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then, the linear transformation $\mathcal{T}$ with matrix representation

$$\langle \mathcal{T} \rangle = \sum_{i=1}^{2} \overline{A_i} \otimes_K A_i = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \tag{2.1}$$

is a nonsingular element of CP_2. However, we observe that there exists no $A \in M_2$ such that $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A$. It follows from [2, Theorem 3.4(i)] that $\mathcal{T}$ is not an extremal of CP_2.

From the subcones CP_n and coCP_n, we find two more classes of extremals of $\pi(\mathrm{PSD}_n)$.

Theorem 2.5. *Every extremal of* CP_n *and of* coCP_n *is also an extremal of* $\pi(\mathrm{PSD}_n)$.

Proof. Assume that $\mathcal{T} : M_n \to M_n$ is an extremal of CP_n. By [2, Theorem 3.4(i)], $\mathcal{T}$ is an extremal of CP_n if and only if there exists $A \in M_n$ such that $\langle \mathcal{T} \rangle = \overline{A} \otimes_K A$. Then, [5, Lemma 3.2] gives us that $\mathcal{T}$ is an extremal of $\pi(\mathrm{PSD}_n)$. An analogous argument implies that every extremal of coCP_n is also an extremal of $\pi(\mathrm{PSD}_n)$. □

By definition, all extreme rays of CP_n and coCP_n are also extreme rays of $\pi(\mathrm{PSD}_n)$. Also, Theorem 2.5 and [2, Theorem 3.4] give us the following extremals for $\pi(\mathrm{PSD}_n)$.

Theorem 2.6. *Let $\tau : M_n \to M_n$. Then,*

(i) *if $\langle\tau\rangle = \sum_{i=1}^{r} \overline{A_i}\otimes_K A_i$ or $\langle\tau\rangle = \sum_{i=1}^{r} \overline{A_i}\otimes_K A_i \circ \mathbf{T}$, where $A_1,\ldots,A_r \in M_n \setminus \{0\}$, and if $A_i = \alpha_i A_1$ for all $i = 1,\ldots,r$, then $\tau \in \operatorname{Ext}\pi(\mathrm{PSD}_n)$;*

(ii) *if there exist $A_1,\ldots,A_r \in M_n \setminus \{0\}$ and $(d_{ij}) \in \operatorname{Ext}\mathrm{PSD}_r$ such that $\langle\tau\rangle = \sum_{i=1}^{r} d_{ij}\overline{A_i}\otimes_K A_j$ or $\langle\tau\rangle = \sum_{i=1}^{r} d_{ij}\overline{A_i}\otimes_K A_j \circ \mathbf{T}$, then $\tau \in \operatorname{Ext}\pi(\mathrm{PSD}_n)$.*

3. Other Faces of CP_n and $\pi(\mathrm{PSD}_n)$

It is well known that the transpose map $\mathbf{T}$ is an element of $\pi(\mathrm{PSD}_n)$ but not CP_n. Since $\langle\mathbf{T}\rangle = I_n\otimes_K I_n\circ\mathbf{T}$, it follows that $\mathbf{T} \in \operatorname{Ext}(\pi(\mathrm{PSD}_n))$. Thus, $F = \Phi(\mathbf{T}) \trianglelefteq \pi(\mathrm{PSD}_n)$ but $F \ntrianglelefteq \mathrm{CP}_n$. While it is well known that CP_n is a subcone but not a face of $\pi(\mathrm{PSD}_n)$, it is an open question whether every proper face of CP_n (in the sense of a proper subset) is a face of $\pi(\mathrm{PSD}_n)$. Examples of such faces do exist. The Siler cone

$$K_1 = \{\tau \in \mathrm{HP}_n : A^*\tau(A) \in \mathrm{PSD}_n \,\forall A \in M_n(\mathbb{C})\} \tag{3.1}$$

is a one-dimensional face of both CP_n and $\pi(\mathrm{PSD}_n)$ [8, page 33].

In [2, Theorem 3.2], Yopp and Hill have characterized all the faces of CP_n as follows.

Theorem 3.1. *$F \trianglelefteq \mathrm{CP}_n$ if and only if there exist $B_1,\ldots,B_r \in M_n$ such that*

$$F = \left\{\tau \in \mathrm{CP}_n : \langle\tau\rangle = \sum_{i=1}^{s}\overline{A_i}\otimes_K A_i \text{ where } \operatorname{vec}\left(A_i^*\right)^* x = 0 \text{ whenever } x \in \bigcap_{i=1}^{r}\operatorname{vec}\left(B_i^*\right)^{\perp}\right\}. \tag{3.2}$$

One way in which we could show that a face F of CP_n is not a face of $\pi(\mathrm{PSD}_n)$ would be to exhibit (at least) one element of F that lies in the interior (equivalently, not in the boundary) of $\pi(\mathrm{PSD}_n)$, as we could then apply the following result due to Barker and Schneider [9, Corollary 2.18].

Theorem 3.2. *If $F \trianglelefteq K$ and $F \neq K$, then $F \subset \operatorname{bd} K$.*

In [13], Kye gives us a criterion by which we can determine whether a given linear transformation is an element of $\operatorname{int}\pi(\mathrm{PSD}_n)$ as follows. Let $v \in \mathbb{C}^n$ be nonzero, and let P_v denote the one-dimensional projection matrix corresponding to the projection onto the subspace spanned by v. Note that one-dimensional projections are precisely the extremals of PSD_n, as characterized by Barker and Carlson [11] and further by Hill and Waters [12, Theorem 3.8]. This leads to Kye's result [13, Proposition 4.1].

Theorem 3.3. *The linear map $\tau \in \pi(\mathrm{PSD}_n)$ is an interior point of $\pi(\mathrm{PSD}_n)$ if and only if $\tau(P_v)$ is nonsingular for all extremals P_v.*

A more common characterization of the interior of a positive cone of linear maps when specialized to $\pi(\mathrm{PSD}_n)$ gives the following:

$$\operatorname{int}\pi(\mathrm{PSD}_n) = \{\tau \in \pi(\mathrm{PSD}_n) : \tau(\mathrm{PSD}_n \setminus \{0\}) \subseteq \mathrm{PD}_n\}. \tag{3.3}$$

Thus, instead of requiring that the image of any nonzero positive semidefinite matrix be positive definite, as in the standard characterization, Kye's characterization instead requires that the image of any extremal be nonsingular.

We continue this section with two examples due to Kye [13].

The first example is the trace map, which is easily seen to be an element of $\pi(\mathrm{PSD}_n)$. For nonzero $v \in \mathbb{C}^n$, we have that

$$\operatorname{tr}(P_v) = (\overline{v}_1 v_1 + \overline{v}_2 v_2 + \cdots + \overline{v}_n v_n) I_n, \tag{3.4}$$

which is nonsingular, giving us that the trace map is an element of $\operatorname{int}\pi(\mathrm{PSD}_n)$.

Kye's second example is the linear map

$$\operatorname{tran}_K : A \longmapsto K^{1/2} A^T K^{1/2}, \tag{3.5}$$

where $K^{1/2}$ is the standard square root of a positive definite matrix (cf. [10]). We know that $A \in \mathrm{PSD}_n \Rightarrow \mathrm{A}^{\mathrm{T}} \in \mathrm{PSD}_n$. Since K is positive definite, $K^{1/2}$ is also positive definite, and Sylvester's law of inertia gives us that A^T and $K^{1/2}A^TK^{1/2}$ have the same inertias, so that $K^{1/2}A^TK^{1/2} \in \mathrm{PSD}_n$. Thus, $\operatorname{tran}_K \in \pi(\mathrm{PSD}_n)$ for all positive definite matrices K. Kye claims, for any $K \in \mathrm{PD}_n$, $\operatorname{tran}_K \in \operatorname{int}\pi(\mathrm{PSD}_n)$. Actually, it is not (making it an element of bd $\pi(\mathrm{PSD}_n)$). To see this, let $K = I_n$, which is positive definite. Then, $K^{1/2} = I_n$. In this case,

$$\operatorname{tran}_{I_n} : A \longmapsto I_n^{1/2} A^T I_n^{1/2} = A^T \tag{3.6}$$

for $A \in M_n$, that is, $\operatorname{tran}_{I_n}$ is simply the transpose map, an extremal of $\pi(\mathrm{PSD}_n)$, as we observed at the beginning of this section, and thus not an interior point of $\pi(\mathrm{PSD}_n)$.

4. Toward a Result Concerning Faces in Shared Boundaries

In Section 3, we have encountered a number of linear maps that lie in the intersection of the boundaries of CP_n and $\pi(\mathrm{PSD}_n)$. We further study this area, which leads us to a still-open question: is a face of CP_n which lies in the boundary of $\pi(\mathrm{PSD}_n)$ necessarily also a face of $\pi(\mathrm{PSD}_n)$?

We do have the following.

Theorem 4.1. *Let $F \leq L \leq K$ be a chain of (sub)cones. If $F \trianglelefteq K$, then $F \trianglelefteq L$.*

Proof. Assume that $F \trianglelefteq K$ and that $x, y - x \in L$, and $y \in F$. Since $L \leq K$, it follows that $x, y - x \in K$. Since $F \trianglelefteq K$, we have that $x \in F$. Thus, $F \trianglelefteq L$. □

In [2], Yopp and Hill give several results concerning the boundary of CP_n, one of which, when combined with its analogue for $\pi(\mathrm{PSD}_n)$ (also found in [2]), yields a sufficient condition for a linear map to be in the intersection of their boundaries.

Theorem 4.2. *Let $\tau \in \mathrm{CP}_n$, and suppose that there exist $A_1, \ldots, A_r \in M_n$ such that $\langle \tau \rangle = \sum_{i=1}^{r} \overline{A_i} \otimes_K A_i$. If $r < n^2$, then $\tau \in \operatorname{bd} \mathrm{CP}_n$.*

Theorem 4.3. *Let $\tau \in \pi(\mathrm{PSD}_n)$, and suppose that there exist $A_1, \ldots, A_r \in M_n$ such that $\langle \tau \rangle = \sum_{i=1}^{r} \epsilon_i \overline{A_i} \otimes_K A_i$ where $\epsilon = \pm 1$. If $r < n$, then $\tau \in \operatorname{bd} \pi(\mathrm{PSD}_n)$.*

Combining these results gives the following.

Theorem 4.4. *Let $\tau \in \mathrm{CP}_n$, and suppose that there exist $A_1, \ldots, A_r \in M_n$ such that $\langle \tau \rangle = \sum_{i=1}^{r} \overline{A_i} \otimes_K A_i$. If $r < n$, then $\tau \in \operatorname{bd} \mathrm{CP}_n \cap \operatorname{bd} \pi(\mathrm{PSD}_n)$.*

Proof. Assume $r < n$. Since $r < n \leq n^2$ for $n = 1, 2, \ldots, \tau \in \operatorname{bd} \mathrm{CP}_n$. If we let $\epsilon_i = 1$ for $i = 1, 2, \ldots, r$, then $\tau \in \operatorname{bd} \pi(\mathrm{PSD}_n)$ and the conclusion is immediate. □

Finally, combining Theorems 3.1 and 4.4, we obtain the following sufficient condition for a face of CP_n to also be a face of $\pi(\mathrm{PSD}_n)$:

Theorem 4.5. *F is a face of both CP_n and $\pi(\mathrm{PSD}_n)$ if and only if there exist $B_1, \ldots, B_r \in M_n$ such that*

$$F = \left\{ \tau \in \mathrm{CP}_n : \langle \tau \rangle = \sum_{i=1}^{s} \overline{A_i} \otimes_K A_i \text{ where } s < n, \text{ and } \operatorname{vec}\left(A_i^*\right)^* x = 0 \text{ whenever } x \in \bigcap_{i=1}^{r} \operatorname{vec}\left(B_i^*\right)^{\perp} \right\}. \tag{4.1}$$

Results by Yopp and Hill [2] finish the known material of this section.

Theorem 4.6. *Let $\tau \in \mathrm{CP}_n$. Then, $\tau \in \operatorname{int} \mathrm{CP}_n$ if and only if there exist linearly independent matrices $A_1, \ldots, A_{n^2}$ such that $\langle \tau \rangle = \sum_{i=1}^{n^2} \overline{A_i} \otimes_K A_i$.*

Theorem 4.7. *Let $\tau, \mathcal{S} \in \pi(\mathrm{PSD}_n)$. If $\tau \in \Phi_{\pi(\mathrm{PSD}_n)}(\mathcal{S})$, then $\tau(A) \in \Phi_{\mathrm{PSD}_n}(\mathcal{S}(A))$ for every $A \in \mathrm{PSD}_n$.*

Proof. By Corollary 2.10 of [9, page 222], $\tau \in \Phi_{\pi(\mathrm{PSD}_n)}(\mathcal{S})$ if and only if there exists $\lambda > 0$ such that $\mathcal{S} - \lambda\tau \in \pi(\mathrm{PSD}_n)$. Therefore, for every $A \in \mathrm{PSD}_n$, $\mathcal{S}(A) - \lambda\tau(A) \in \pi(\mathrm{PSD}_n)$. It follows that $\tau(A) \in \Phi_{\mathrm{PSD}_n}(\mathcal{S}(A))$ for every $A \in \mathrm{PSD}_n$. □

The material of this paper leaves an open problem that we first state in generality and then in the setting of the paper.

Conjecture 4.8. *Let K, L, and F be cones such that $F \subsetneq L \subsetneq K$. If $F \trianglelefteq L$ and $F \subset \operatorname{bd} K$, then $F \trianglelefteq K$.*

Conjecture 4.9. *Let F be a proper subcone of CP_n. If $F \trianglelefteq \mathrm{CP}_n$ and $F \subset \operatorname{bd} \pi(\mathrm{PSD}_n)$, then $F \trianglelefteq \pi(\mathrm{PSD}_n)$.*

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