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Research Article

Optimal Control of Diffusions with Hard Terminal State Restrictions

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A maximum principle is proved for certain problems of optimal control of diffusions where hard end constraints occur. The results apply to several dimensional problems, where some of the state equations involve Brownian motions, but not the equations corresponding to states being hard restricted at the terminal time.

1. Introduction

Various types of maximum principles have been proved for problems of control of diffusions in case of no or soft terminal state restrictions; see for example, Kushner [1], Haussmann [2], Peng [3], and Yong and Zhou [4]. Maximum principles for problem with hard terminal restrictions are proved for certain types of continuous time piecewise deterministic problems in Seierstad [5, 6]. Singular controls are sometimes introduced in various problems with certain types of hard restrictions, but below we merely consider problems where the controls appearing may be said to be absolutely continuous with respect to Lebesgue measure. The restriction to such controls makes it harder to operate with hard terminal state restrictions; in fact we can only work with such restrictions on states governed by differential equations not containing a Brownian motion. Brownian motion will only appear in differential equations of states unconstrained at the terminal time. So the problem we consider is a problem of control of diffusions where hard terminal restrictions are placed on states not governed by differential equations containing a Brownian motion; these states, however, can be influenced by other states directly influenced by Brownian motions. Below, a maximum principle is stated and proved for such problems. To the authors knowledge, maximum principles have not been stated for such problems before. Because the states are stochastic, the state space is infinite

dimensional; so to obtain a maximum principle, one must impose a condition amounting to demand sufficient variability of the first order variations in the problem.

2. The Control Problem and the Statement of the Necessary Condition

Let $T > 0$ and let x_0 be a given point in the Euclidean space $X := \mathbb{R}^{n^*}$, let π be a projection from X into $Y := \mathbb{R}^{m^*}$, $m^* < n^*$, such that $\pi x = (x_1, \ldots, x_{m^*})$ $(x = (x_1, \ldots, x_{n^*}))$ and let U be a Borel subset of a Euclidean space. Furnish the interval $J := [0, T]$ with the Lebesgue measure. Let $(\Omega, \Phi, \Phi_t, P)$ be a filtered probability space, (i.e., for $t \in [0, T]$, the Φ_t's are sub-σ-algebras of the given σ-algebra Φ of subsets of Ω, $\Phi_s \subset \Phi_t$ if $s < t$, and P is a probability measure on Φ), and assume that (Ω, Φ, P) is complete, that Φ_0 contains all the P-null sets in Φ and that Φ_t is right continuous. Let $L_2(J \times \Omega, \mathbb{R}^n)$ be the set of Lebesgue $\times\, \Phi$-measurable functions $z(t, \omega)$ for which $E \int_J z(t, \omega)^2 dt < \infty$. Related to $(\Omega, \Phi, \Phi_t, P)$, let B_t be a vector the n' components of which (denoted B_t^j) are independent one-dimensional Brownian motions all adapted to $\{\Phi_t\}_t$, such that $B(t) - B(s)$ is independent of Φ_s for all s, t, $0 \le s < t$, and is normally distributed with mean 0 and covariances $(t - s)I$, with I being the identity matrix. In applications where the B_t^j's are the entities that can be observed, it is natural to take $\{\Phi_t\}_t$ as the natural filtration generated by the B_t^j's. There are given functions $f(t, x, u)$ and $\overline{\sigma}^j(t, x)$, $j = 1, \ldots, n'$, from $J \times X \times U$ into X ($\overline{\sigma}^j$ independent of u). The following conditions are called the basic assumptions.

(A_1) The functions $f(t, x, u)$ and $\overline{\sigma}^j(t, x)$ have continuous derivatives f_x and $\overline{\sigma}_x^j$ with respect to $x \in X$.

(A_2) The functions f and $\overline{\sigma}^j$ have one-sided limits with respect to t, and f and f_x are, separately, continuous in x and in u.

Write $\overline{\sigma}$ for the $n^* \times n'$-matrix whose columns are $\overline{\sigma}^j$; let $\overline{\sigma}_x^j$ be the matrix with entries $\overline{\sigma}_{x_i}^j$, and write $\sum_j \overline{\sigma}^j dB_t^j = \overline{\sigma} dB_t$. Also, write 1_C for the indicator function of the set C. Let U' be a set of functions $u(t, \omega)$ taking values in U, such that $u(\cdot, \cdot)$, for each t, when restricted to $[0, t] \times \Omega$, is Lebesgue $\times\, \Phi_t$-measurable (i.e., progressively measurable), and such that when $u_i(\cdot, \cdot) \in U'$, $i = 1, \ldots, i^*$, i^* arbitrary, and $\{C_i\}_i$, $i = 1, \ldots, i^*$, is a measurable partition of J, then $\sum_{i=1}^{i^*} 1_{C_i}(t) u_i(t, \omega)$ belongs to U' (so-called switching closedness). We will also assume that U' is $\check{\sigma}$-closed, which means that if $u_n(\cdot, \cdot) \in U'$ and $u(t, \omega)$ is progressively measurable and takes values in U and meas $\{t : u(t, \omega)$ is not equal to $u_n(t, \omega)$ a.s.$\}$ converges to zero when $n \to \infty$, then $u(\cdot, \cdot)$ belongs to U'. Let $|\cdot|_2$ be the L_2-norm on $L_2(\Omega, \Phi, \mathbb{R}^k)$.

The following assumptions are called the global assumptions.

(B_1) πf_x is uniformly continuous in x, uniformly in t, u.

(B_2) For some constant M, for all $u(\cdot, \cdot) \in U'$,

$$\operatorname*{essup}_t \left| f(t, 0, u(t, \omega)) \right|_2 \le M, \qquad \sup_{j,t} \left| \overline{\sigma}^j(t, 0) \right| \le M, \tag{2.1}$$

(B_3) A constant M^+ exists such that for all $(t, x, u) \in J \times X \times U$,

$$\left| f_x(t, x, u) \right| \le M^+, \qquad \left| \overline{\sigma}_x^j(t, x) \right| \le M^+. \tag{2.2}$$

(The symbol $|\cdot|$ is used for the Euclidean norm in any Euclidean space $\mathbb{R}^k$, including $\mathbb{R}^1$, and, applied to matrices, it is the linear operator norm.)

(B_4) One has

$$\pi\overline{\sigma} = 0. \tag{2.3}$$

Let $u(\cdot,\cdot) \in U'$. The strong (unique) solution, continuous in t, of the equation

$$x(t,\omega) = x_0 + \int_0^t f(s, x(s,\omega), u(s,\omega))dt + \int_0^t \overline{\sigma}(s, x(s,\omega))dB_t \tag{2.4}$$

is denoted $x^{u(\cdot,\cdot)}(t,\omega) = x^u(t,\omega)$ and is called a system solution.

Let $a \in X$ (a fixed, $\neq 0$) such that $\pi a = 0$; let $\langle\cdot,\cdot\rangle$ denote scalar product, and consider the problem

$$\max_{u(\cdot,\cdot)\in U'} E\left\langle x^{u(\cdot,\cdot)}(T,\cdot), a\right\rangle, \tag{2.5}$$

subject to

$$\pi x^{u(\cdot,\cdot)}(T,\omega) = \widetilde{y} \text{ a.s.}, \quad \text{where } \widetilde{y} \text{ is fixed in } Y. \tag{2.6}$$

Let $u^*(\cdot,\cdot) \in U'$ be an optimal control in the problem and write $x^{u^*(\cdot,\cdot)}(\cdot,\cdot) = x^*(\cdot,\cdot)$. Let $C(t,s,\omega)$ be the resolvent of the equation

$$q(t) = q_0 + \int_0^t f_x(s, x^*(s,\omega), u^*(s,\omega))q(s,\omega)dt + \sum_j \int_0^t \overline{\sigma}_x^j(s, x^*(s))dB_s^j, \tag{2.7}$$

(so $C(s,s,\omega) = I$, with I being the identity map).

In the subsequent necessary conditions, the following local linear controllability condition (2.10) is needed. Let $L_2^{\text{prog}}(J\times\Omega, Y)$ be the set of progressively measurable functions in $L_2(J\times\Omega, Y)$, and for $\alpha \in (0,\infty]$, let

$$B^\alpha := \left\{ z(\cdot,\cdot) \in L_2^{\text{prog}}(J\times\Omega, Y) : \operatorname{essup}_t |z(t,\cdot)|_2 < \alpha \right\}, \tag{2.8}$$

$$B_\alpha = \left\{ \int_0^T z(s,\cdot)ds : z(\cdot,\cdot) \in B^\alpha \right\}, \tag{2.9}$$

and let co denote convex hull. There exists a number $\alpha > 0$ and a progressively measurable function $\check{z}(t,\omega) : J\times\Omega \to Y$, with $\operatorname{essup}_t|\check{z}(t,\cdot)|_2 < \infty$, and a number $c \in [0,T)$ such that

$$\begin{aligned} 1_{[c,T]}[\check{z}(\cdot,\cdot) + B^\alpha] \subset \operatorname{co}\{ & 1_{[c,T]}\pi\left[f(\cdot, x^*(\cdot,\cdot), \widehat{u}(\cdot,\cdot))\right. \\ & \left. - f(\cdot, x^*(\cdot,\cdot), u^*(\cdot,\cdot))\right] : \widehat{u}(\cdot,\cdot) \in U'\}. \end{aligned} \tag{2.10}$$

Theorem 2.1. *Assume that $u^*(\cdot,\cdot)$ is optimal in problems* (2.5) *and* (2.6)*, that assumptions A and B hold (the basic and global assumptions), and that* (2.10) *is satisfied. Then there exists a number $\Lambda_0 \geq 0$ and a linear functional ν on B_∞, bounded on B_1, such that, for all $u(\cdot,\cdot) \in U'$,*

$$\begin{aligned}&\left\langle \int_0^T \pi C(T,t,\cdot)\left[f(t,x^*(t,\cdot),u(t,\cdot)) - f(t,x^*(t,\cdot),u^*(t,\cdot))\right]dt, \nu \right\rangle \\ &\quad + E\left\langle \int_0^T C(T,t,\cdot)\left[f(t,x^*(t,\cdot),u(t,\cdot)) - f(t,x^*(t,\cdot),u^*(t,\cdot))\right]dt, \Lambda_0 a \right\rangle \\ &\quad \leq 0.\end{aligned} \tag{2.11}$$

Finally, $(\Lambda_0, \nu) \neq 0$.

Remark 2.2. If (2.10) holds for $\check{z}(\cdot,\cdot) = 0$, then $\Lambda_0 \neq 0$ and ν is a continuous linear functional on $L_2(\Omega,\Phi,Y)$.

Remark 2.3. Let $\nu_* := \phi \to \langle \phi, \nu\rangle + \Lambda_0 E\langle \phi, a\rangle$ and let $C(T,t,\cdot)^*$ be the transposed of $C(T,t,\cdot)$. Note that for $t < T$, $C(T,t,\cdot)^*\nu_*|_{L_2(\Phi_t,\Omega)}$ is continuous in $|\cdot|_2$-norm and hence can be represented by an L_2-function $p^-(t,\cdot) \in L_2(\Omega,\Phi_t,\mathbb{R}^{n^*})$ ($p^-(t,\omega)$ progressively measurable and continuous in t). Provided U' has the property that if $u, u' \in U'$ and $C \subset \Omega$ is Φ_t-measurable then $(u1_C + u'(1-1_C))1_{[t,T]} + u1_{[0,t)} \in U'$, we have that, for any $u(\cdot) \in U'$, for a.e. t in $(0,T)$, a.s.

$$\left\langle f(t,x^*(t,\omega),u(t,\omega)) - f(t,x^*(t,\omega),u^*(t,\omega)), p^-(t,\omega)\right\rangle \leq 0 \tag{2.12}$$

(a consequence of (2.11)).

When ν is continuous on $L_2(\Omega,\Phi,Y)$, then

$$\lim_{t\to T} p^-(t,\omega)_j = \Lambda_0 a_j, \quad j > m^*, \tag{2.13}$$

(the limit being an L_2-limit), in fact, when $t \to T$, $p^-(t,\cdot) \to \tilde{v}_*(\cdot)$ in L_2, where $\tilde{v}_*(\omega)$ is the L_2-function representing ν_*.

Assume that Φ_t is the natural filtration generated by B_t. Then the progressively measurable function $p^-(t,\omega)$ satisfies the following condition: on $[0,T)$, there exist $\mathbb{R}^{n^*}$-valued, progressively measurable functions $p(t,\omega)$, $q^j(t,\omega)$, $j = 1,\ldots,n'$, $p(t,\omega)$ continuous in t, such that $E\int_0^t p(t,\cdot)^2 dt < \infty$, $E\int_0^t q^j(t,\cdot)^2 dt < \infty$ for all $t < T$, such that

$$\begin{aligned}dp(t,\omega) &= -p(t,\omega) f_x(t,x^*(t,\omega),u^*(t,\omega))dt \\ &\quad - \sum_j \overline{\sigma}_x^j(t,x^*(t,\omega))q^j(t)dt + \sum_j q^j(t)dB_t^j\end{aligned} \tag{2.14}$$

and such that, for all $t < T$, $p^-(t,w) = p(t,\omega)P$-a.s. In this case, if ν is continuous on $L_2(\Omega,\Phi,Y)$, then the following additional properties hold: $E\int_0^T p(t,\cdot)^2 dt < \infty$, $E\int_0^T q^j(t,\cdot)^2 dt < \infty$, the L_2-limit $\lim_{t\to T} p(t,\cdot)$ exists and equals $\tilde{v}_*(\cdot)$, and $\Pr[(\Lambda_0, \lim_{t\to T} p(t,\omega)) \neq 0] > 0$.

3. Proof of Theorem 2.1

The proof consists of three lemmas and the five proof steps A–E and relies on an "abstract" maximum principle, Corollary I in the appendix.

Let $X' := \mathbb{R}^{n^{**}}$ and let $J' := [0, T'], T' \in (0, 1]$.

Lemma 3.1. *Let $g \in L_2(J' \times \Omega, X')$ be progressively measurable. For any $\varepsilon > 0$ there exists a function $b(t,\omega) := \sum_{j=0}^{j^*} g(s_j,\omega)1_{[s_j,t_{j+1})}(t) \in L_2(J' \times \Omega, X')$, with $t_j \le s_j \le t_{j+1}$, $g(s_j,\cdot) \in L_2(\Omega, \Phi_{s_j}, X')$, such that $\int_J |g(t,\omega) - b(t,\omega)|_2 dt < \varepsilon$.*

Proof. Using Dunford and Schwartz [7, III.11.16 Lemma] yields that $g(t,\cdot) \in L_2(J', L_2(\Omega, \Phi, X'))$ a.e. For each $\varepsilon' > 0$ there exists a function

$$a(t,\omega) = \sum_{j=0}^{j^*} a_j(\omega) 1_{[t_j,t_{j+1})}(t), \qquad t_0 = 0,\ t_{j^*+1} = T',\ t_j < t_{j+1}, \tag{3.1}$$

- $a(t,\omega)$ piecewise constant in t -, $a_j(\cdot) \in L_2(\Omega, \Phi, X')$, such that

$$\int_{J'} |g(t,\cdot) - a(t,\cdot)|_2 dt < \varepsilon'^2. \tag{3.2}$$

Thus, there exists an open set $A \subset J'$, such that $\operatorname{meas}(A) < \varepsilon'$, and $A \supset A_0 := \{t : |g(t,\cdot) - a(t,\cdot)|_2 > \varepsilon'\}$ (note that $\operatorname{meas}(A_0) < \varepsilon'$, otherwise the inequality involving ε'^2 is contradicted). Let $B = \complement A$, and let $s_j := \min B \cap [t_j, t_{j+1})$ if $j \in \Gamma := \{j : B \cap [t_j, t_{j+1}) \neq 0\}$. For $j \in \Gamma$,

$$|a_j(\cdot) - g(s_j,\cdot)|_2 \le \varepsilon', \tag{3.3}$$

so for $j \in \Gamma, t \in B \cap [t_j, t_{j+1})$, we have

$$|g(t,\cdot) - g(s_j,\cdot)|_2 \le |g(t,\cdot) - a_j(\cdot)|_2 + |a_j(\cdot) - g(s_j,\cdot)|_2 \le 2\varepsilon'. \tag{3.4}$$

Define

$$b(t,\cdot) := \sum_{j\in\Gamma} g(s_j,\cdot) 1_{[s_j,t_{j+1})}. \tag{3.5}$$

If $t \in B$ and $t < T'$, then for some j, $t \in [t_j, t_{j+1})$, so for this j, $j \in \Gamma$, $t \in [s_j, t_{j+1})$ and (3.4) yields

$$|g(t,\cdot) - b(t,\cdot)|_2 = |g(t,\cdot) - g(s_j,\cdot)|_2 \le 2\varepsilon'. \tag{3.6}$$

Let $\varepsilon > 0$ be arbitrarily given. Assume now that ε' is so small that

$$\operatorname{meas}(C) < \varepsilon' \Longrightarrow \int_C |g|_2 dt < \frac{\varepsilon}{2} \tag{3.7}$$

(meas(C) = Lebesgue measure of C) and

$$3\varepsilon' + 2\varepsilon'^2 < \frac{\varepsilon}{2}. \tag{3.8}$$

Then, using successively, (3.6), (3.2), (3.3), (3.2), (3.7), and (3.8) yields

$$\begin{aligned}
\int_{J'} \left|g(t,\cdot) - b(t,\cdot)\right|_2 dt &= \int_B \left|g(t,\cdot) - b(t,\cdot)\right|_2 dt + \int_A \left|g(t,\cdot) - b(t,\cdot)\right|_2 dt \\
&\le 2\varepsilon' + \int_A \left|g(t,\cdot) - a(t,\cdot)\right|_2 dt + \int_A |a(t,\cdot) - b(t,\cdot)|_2 dt \\
&\le 2\varepsilon' + \varepsilon'^2 + \int_A \sum_{j\in\Gamma} \left|a_j(\cdot) - g(s_j,\cdot)\right|_2 1_{[s_j,t_{j+1})}(t)dt \\
&\quad + \int_A \sum_{j\in\Gamma} \left|a_j(\cdot)\right|_2 1_{[t_j,s_j)}(t)dt + \int_A \sum_{j\notin\Gamma} \left|a_j(\cdot)\right|_2 1_{[t_j,t_{j+1})}(t)dt \\
&\le 2\varepsilon' + \varepsilon'^2 + \varepsilon' + \int_A |a(t,\cdot)|_2 dt \le 3\varepsilon' + \varepsilon'^2 + \int_A \left|a(t,\cdot) - g(t,\cdot)\right|_2 dt \\
&\quad + \int_A \left|g(t,\cdot)\right|_2 dt \le 3\varepsilon' + \varepsilon'^2 + \varepsilon'^2 + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned} \tag{3.9}$$

□

Lemma 3.2. *Let $g \in L_2(J' \times \Omega, X')$ be progressively measurable and let $\tilde{k} \in (0,1)$. Then for each $\varepsilon > 0$ there exists a set $C \subset J'$ such that for all s*

$$\left|\tilde{k}\int_0^s g(t,\omega)dt - \int_0^s g(t,\omega)1_C(t)dt\right|_2 < \varepsilon \tag{3.10}$$

and such that meas$(C) = \tilde{k}T'$ *(C measurable).*

Proof. Apply Lemma 3.1 to obtain

$$\int_{J'} \left|g(t,\omega) - h(t,\omega)\right|_2 dt < \frac{\varepsilon}{4} \qquad \text{for } h(t,\omega) = \sum_{k=0}^{k_*} a_k(\omega) 1_{[t_k,t_{k+1})}(t), \tag{3.11}$$

where $a_k(\omega) \in L_2(\Omega, \Phi_{t_k}, X')$, $t_k < t_{k+1}, t_0 = 0$, $t_{k_*+1} = T'$. Evidently, we can assume of the t_k's that they satisfy the additional property

$$\left|\int_{t_k}^{t_{k+1}} |g| ds\right|_2 < \frac{\varepsilon}{4}. \tag{3.12}$$

Define

$$\phi(t,\omega) := \sum_{k=1}^{k_*} 1_{[t_k,(1-\tilde{k})t_k+\tilde{k}t_{k+1})}(t). \tag{3.13}$$

Now,

$$\begin{aligned}\int_{t_k}^{t_{k+1}} h(t,\omega)\phi(t,\omega)dt &= \int_{t_k}^{t_{k+1}} a_k(\omega)1_{[t_k,(1-\tilde{k})t_k+\tilde{k}t_{k+1})}(t)dt = \tilde{k}a_k(\omega)(t_{k+1}-t_k)\\ &= \tilde{k}\int_{t_k}^{t_{k+1}} a_k(\omega)dt = \tilde{k}\int_{t_k}^{t_{k+1}} h(t,\omega)dt.\end{aligned} \tag{3.14}$$

Hence, for any given k^*,

$$\begin{aligned}\int_0^{t_{k^*}} h(t,\omega)\phi(t,\omega)dt &= \sum_{k<k^*}\int_{t_k}^{t_{k+1}} h(t,\omega)\phi(t,\omega)dt\\ &= \sum_{k<k^*}\tilde{k}\int_{t_k}^{t_{k+1}} h(t,\omega)dt = \tilde{k}\int_0^{t_{k^*}} h(t,\omega)dt.\end{aligned} \tag{3.15}$$

Moreover, by (3.15) and (3.11),

$$\begin{aligned}&\left|\int_0^{t_{k^*}} g(t,\omega)\phi(t,\omega)dt - \tilde{k}\int_0^{t_{k^*}} g(t,\omega)dt\right|_2\\ &\quad\le \left|\int_0^{t_{k^*}} h(t,\omega)\phi(t,\omega)dt - \tilde{k}\int_0^{t_{k^*}} h(t,\omega)dt\right|_2\\ &\qquad + \left|\int_0^{t_{k^*}} \big(g(t,\omega)-h(t,\omega)\big)\phi(t,\omega)dt\right|_2\\ &\qquad + \tilde{k}\left|\int_0^{t_{k^*}} \big(h(t,\omega)-g(t,\omega)\big)dt\right|_2 < \frac{2\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned} \tag{3.16}$$

Finally, for any given t, if $k^* = k$ is the largest k such that $t_k \le t$, then, by (3.12), a.s.,

$$\left|\int_{t_{k^*}}^{t} g(s,\omega)\phi(s,\omega)ds\right|_2 \le \frac{\varepsilon}{4}, \qquad \left|\tilde{k}\int_{t_{k^*}}^{t} g(s,\omega)ds\right|_2 \le \frac{\varepsilon}{4}. \tag{3.17}$$

The conclusion of Lemma 3.2 then follows from (3.16) and (3.17) and the fact that $\int_0^{T'} \phi(s)ds = \sum_k \tilde{k}(t_{k+1}-t_k) = \tilde{k}T'$. □

Lemma 3.3. *Let $z(t,\omega) \in B^{\infty}$, and let $z(t,\omega)$ be continuous in t, and assume that $|\sup_t |z(t,\omega)||_2 < \infty$. Let $u(t,\omega) \in U'$. When $\delta \downarrow 0$, then*

$$\sup_{\theta\in[0,\delta]} \left|\pi\left\{f_x(t,x^*(t,\omega)+\theta z(t,\omega),u(t,\omega)) - \pi f_x(t,x^*(t,\omega),u(t,\omega))\right\}[z(t,\omega)]\right|_2 \longrightarrow 0 \text{ uniformly in } t. \tag{3.18}$$

Proof. Let an error function $e(d)$ be a nonnegative function on $[0,\infty)$ such that $e(d) \downarrow e(0) = 0$ when $d \downarrow 0$. By uniform continuity of πf_x in x, uniformly in t,u, there exists an increasing error function $e(d)$ such that $|f^i_{x_j}(t,x+z,u) - f^i_{x_j}(t,x,u)| \le e(|z|)$ for all $z \in X$, $i \le m^*$. Suppose, by contradiction, that some $\varepsilon > 0$ exists, such that, for each $k = 1,2,\ldots$, there exist θ_k, t_k such that

$$\left|\left(\pi f_x(t_k,x^*(t_k,\omega)+\theta_k z(t_k,\omega),u(t_k,\omega)) - \pi f_x(t_k,x^*(t_k),u(t_k,\omega))\right)[z(t_k,\omega)]\right|_2 \ge \varepsilon, \tag{3.19}$$

and $\theta_k < 1/k$. Then

$$\begin{aligned}\zeta_k(\omega) &:= \left|\left(\pi f_x(t_k,x^*(t_k,\omega)+\theta_k z(t_k,\omega),u(t_k,\omega)) - \pi f_x(t_k,x^*(t_k),u(t_k,\omega))\right)[z(t_k,\omega)]\right| \\ &\le n^{*2} e(\theta_k|z(t_k,\omega)|)|z(t_k,\omega)| \le n^{*2} e\left(\theta_k \sup_t |z(t,\omega)|\right)\sup_t |z(t,\omega)|.\end{aligned} \tag{3.20}$$

Now, $\sup_t |z(t,\omega)| < \infty$ a.s. by the L_2-assumption on $z(\cdot,\cdot)$ in the Lemma. So $\zeta_k(\omega)$ converges a.s. to zero. Moreover, by (2.2), $|\zeta_k(\omega)| \le 2M^+\theta_\kappa|z(t_k,\omega)| \le 2M^+\sup_t|z(t,\omega)|$, the last function being an L_2-function. By dominated convergence, $|\zeta_k(\cdot)|_2 \to 0$ when $k \to \infty$, and a contradiction of (3.19) is obtained. □

(A) Growth Properties

Without loss of generality, from now on, let $x_0 = 0$, $T = 1$. Let $\pi^i := x \to x_i$, the ith component of $x \in X$. For $x(\cdot,\cdot) \in L_2(J \times \Omega, \mathbb{R}^n)$, let

$$\|x(\cdot,\cdot)\|_2 := \operatorname{essup}_t |x(t,\cdot)|_2, \qquad \|x(\cdot,\cdot)\|_2^* = \left|\left(\sup_t |x(t,\cdot)|\right)\right|_2 \tag{3.21}$$

(where of course $\|x(\cdot,\cdot)\|_2 \le \|x(\cdot,\cdot)\|_2^*$). For any $u(\cdot,\cdot)$, $u'(\cdot,) \in U'$, let $q^{u',u}(t,\omega)$ be the solution of

$$\begin{aligned}q^{u',u}(t,\omega) &= \int_0^t f_x(s,x^u(s,\omega),u(s,\omega))q^{u',u}ds + \sum_j \int_0^t \overline{\sigma}^j_x(s,x^u(s,\omega))q^{u',u}dB^j_s \\ &\quad + \int_0^t \left(f(s,x^u(s,\omega),u'(s,\omega)) - f(s,x^u(s,\omega),u(s,\omega))\right)ds.\end{aligned} \tag{3.22}$$

(By general existence results and (2.1) and (2.2), $q^{u',u}$, as well as x^u, see (2.1), do exist, with both being unique (strong) solutions.)

By (2.1) $\int_0^t f(s,0,u(s,\omega))^2 ds \leq M^2$ and $\sum_j \int_0^t \overline{\sigma}^j(s,0)^2 ds \leq n'M^2$. By (A.3) in the appendix, (a consequence of Gronwall's inequality), and (2.2), with $\check{z}(t,\omega) = 0$, $\check{h} = f(t,x,u(t,\omega))$, $\sigma_* = \overline{\sigma}$, $\check{y} = x^u(t,\omega)$, $\alpha(t,\omega) = -f(s,0,u(s,\omega))$, $\alpha^*(t,\omega) = -\overline{\sigma}(t,0)$, and $\beta^*(t,\omega) = \beta(t,\omega) = 0$, we have that, for some constant $\check{D}$ independent of u,

$$\|x^u(\cdot,\cdot)\|_2^* \leq \check{D}(1+n')M =: \widetilde{D}. \tag{3.23}$$

Define $\widehat{X} = \{x(t,\omega) \in L_2(J \times \Omega, X) : \|x(\cdot,\cdot)\|_2 \leq \widetilde{D}\}$. Note that by (2.2), $|f(s,x(s,\omega),u(s,\omega))|_2 \leq |f(s,0,u(s,\omega))|_2 + M^+|x(s,\omega)|_2$. Then, when $x(\cdot,\cdot)$ belong to $\widehat{X}$, by (2.1) and (3.23), we get the following inequality: for all s,

$$\left|f(s,x(s,\omega),u(s,\omega))\right|_2 \leq \check{D} := M + M^+\widetilde{D} \tag{3.24}$$

($\check{D}$ independent of u and $x(\cdot,\cdot) \in \widehat{X}$). For two Φ-measurable functions $\phi(\omega)$ and $\psi(\omega)$, let $\phi(\cdot) \neq \psi(\cdot)$ mean that $\Pr[\omega : \phi(\omega) \neq \psi(\omega)] > 0$. Define $H^{u',u} = \{(t,\omega) : u'(t,\omega) \neq u(t,\omega)\}$ and define $H_{u',u} = \{t : u'(t,\cdot) \neq u(t,\cdot)\}$. Then a.s. $1_{H_{u',u}} \geq 1_{H^{u',u}}$.

Let $u'', u \in U'$. Using (2.4), (2.2), and (A.3) in the appendix, (with $\check{z} = x^u$, $\check{y} = x^{u''}$, $\beta^* = \beta = 0$, $\check{h} = f(t,x,u''(t,\omega))$, $\sigma_* = \overline{\sigma}$, $\check{y}_0 = \check{z}_0 = 0$, $\alpha = f(s,x^u(s,\omega),u(s,\omega)) - f(s,x^u(s,\omega),u''(s,\omega))$, and $\alpha^* = 0$), for some constant D independent of u'' and u, we get

$$\begin{aligned}
&\left|\left(\sup_t \left|x^{u''}(\cdot,\cdot) - x^u(\cdot,\cdot)\right|\right)\right|_2 \\
&\quad \leq D\int_0^1 \left|1_{H_{u'',u}}\left(f\left(\check{s},x^u(\check{s},\omega),u''(\check{s},\omega)\right) - f(\check{s},x^u(\check{s},\omega),u(\check{s},\omega))\right)\right|_2 d\check{s} \\
&\quad \leq 2D\check{D}\int_0^1 1_{H_{u'',u}} dt.
\end{aligned} \tag{3.25}$$

(the last inequality by (3.24).)

Let $u', u \in U'$, $k \in (0,1]$. As explained below, we have

$$\begin{aligned}
\left\|q^{u',u}(\cdot,\cdot)\right\|_2^* &\leq D\left|\int_0^1 \left|f\left(t,x^u(t,\omega),u'(t,\omega)\right) - f(t,x^u(t,\omega),u(t,\omega))\right| dt\right|_2 \\
&\leq 2D\check{D}\int_0^1 1_{H_{u',u}} dt,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\sup_{s\leq t}\left|q^{u'',u}(s,\cdot) - kq^{u',u}(s,\cdot)\right|_2 &\leq D\sup_{s\leq t}\left|\int_0^s \left(f\left(\check{s},x^u(\check{s},\omega),u''(\check{s},\omega)\right)\right.\right. \\
&\quad \left.\left. -(1-k)f(\check{s},x^u(\check{s},\omega),u(\check{s},\omega)) - kf\left(\check{s},x^u(\check{s},\omega),u'(\check{s},\omega)\right)\right)d\check{s}\right|_2
\end{aligned} \tag{3.27}$$

(D independent of u'', u', u, k.) The inequalities (3.27) and (3.26) follow from (A.2) and (A.3), respectively, in the appendix, (together with (2.2) and (3.22)), for $\beta^* = \beta = 0$, $\check{z} = q^{u'',u} - kq^{u',u}$, $\check{y} = 0$, $\check{h}(t,x,\omega) = f_x(t,x^u(t,\omega),u^u(t,\omega))[x]$, $\sigma_*^j(t,x,\omega) = \overline{\sigma}_x^j(t,x^u(t,\omega))[x]$, $\alpha^* = 0$, and

$$\begin{aligned}\alpha(t,\omega) := {} & f\left(t,x^u(t,\omega),u''(t,\omega)\right) - f(t,x^u(t,\omega),u(t,\omega)) \\ & - k\left[f\left(t,x^u(t,\omega),u'(t,\omega)\right) - f(t,x^u(t,\omega),u(t,\omega))\right]\end{aligned} \tag{3.28}$$

(and, for (3.26) for $u'' = u$, $k = 1$).

Similarly, for $\tilde{u}$, u'', u', $u \in U'$,

$$\begin{aligned}&\sup_{s\le t}\left|q^{\tilde{u},u}(s,\cdot) - \left(kq^{u',u}(s,\cdot) + (1-k)q^{u'',u}(s,\cdot)\right)\right|_2 \\ &\quad \le D\sup_{s\le t}\Big|\int_0^s \Big[f(\check{s},x^u(\check{s},\omega),\tilde{u}(\check{s},\omega)) \\ &\qquad\qquad -\left\{kf\left(\check{s},x^u(\check{s},\omega),u'(\check{s},\omega)\right) + (1-k)f\left(\check{s},x^u(\check{s},\omega),u''(\check{s},\omega)\right)\right\}\Big]d\check{s}.\end{aligned} \tag{3.29}$$

Define

$$\check{\sigma}(u,u') := \int_J 1_{H_{u,u'}}(t)dt. \tag{3.30}$$

Define also

$$I_i := \left(1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right], \quad i = 0,1,\ldots, \tag{3.31}$$

$$\sigma^*(u',u) := \sup_i 2^{i+1}\int_{I_i} 1_{H_{u',u}}(t)dt. \tag{3.32}$$

We need to prove that

$$\sup_i 2^i\left|\int_{I_i} \pi\dot{x}^{u'}(t,\cdot) - \pi\dot{x}^u(t,\cdot)dt\right|_2 \longrightarrow 0 \quad \text{when } u' \longrightarrow u \text{ in } \sigma^*\text{-metric.} \tag{3.33}$$

This follows from (3.24), (3.25), (2.2), and (3.32), because, in a shorthand notation,

$$\begin{aligned}&2^i\left|\int_{I_i}\left\{\pi f\left(t,x^{u'},u'\right) - \pi f(t,x^u,u)\right\}dt\right|_2 \\ &\quad \le 2^i\left|\int_{I_i}\left|\pi f\left(t,x^{u'},u'\right) - \pi f(t,x^u,u')\right|dt\right|_2 \\ &\qquad + 2^i\left|\int_{I_i}\left\{\pi f(t,x^u,u') - \pi f(t,x^u,u)\right\}dt\right|_2\end{aligned}$$

$$\begin{aligned}
&\leq 2^i \int_{I_i} M^+ \left\| x^{u'} - x^u \right\|_2 dt + 2 \cdot 2^i \breve{D} \int_{I_i} 1_{H_{u',u}} dt \\
&\leq \frac{M^+ \left\| x^{u'} - x^u \right\|_2}{2} + \breve{D}\sigma^*(u', u).
\end{aligned} \tag{3.34}$$

(B) Properties of the "Linear" Perturbations $q^{u',u}$

Let $\varepsilon'' > 0$ be arbitrarily given, let k be any number in $(0,1]$, and let u'', $u \in U'$. Let us first prove the following consequence of Lemma 3.2. (We drop writing $\cdot$ for ω.) For all s,

$$\begin{aligned}
\Bigg| \int_0^s 1_{I_m} \{ &[f(t, x^u(t), u_k(t)) - f(t, x^u(t), u(t))] \\
&- k[f(t, x^u(t), u''(t)) - f(t, x^u(t), u(t))] \} \Bigg|_2 < \frac{\varepsilon'' k}{2^{m+1}},
\end{aligned} \tag{3.35}$$

where u_k on I_m, $m = 0, 1, 2, \ldots$, is defined by $u_k := u'' 1_{C^m} + u(1 - 1_{C^m})$, with the sets C^m being as follows. They are obtained by replacing J' by I_m, (hence $[0, s]$ by $I_m \cap [0, s]$), and ε by $\varepsilon'' k / 2^{m+1}$ in (3.10), that is, in Lemma 3.2, and denoting the corresponding subset C by $C^m \subset I_m$, with g in Lemma 3.2 being equal to $f(t, x^u(t), u''(t)) - f(t, x^u(t), u(t))$. Here, meas $(C^m) = k \operatorname{meas}(I_m) = k/2^{m+1}$, so $\breve{\sigma}(u_k, u) \leq \sigma^*(u_k, u) \leq k$.

Let $u' \in U'$. Because (3.35) holds for some C^m when u is replaced by u', we get that for some $u_k \in U'$, for all s,

$$\begin{aligned}
&\Bigg| \int_0^s 1_{I_m} \Bigg\{ [f(t, x^u(t), u_k(t)) - f(t, x^u(t), u(t))] \\
&\qquad - \int_0^s [k f(t, x^u(t), u''(t)) + (1-k) f(t, x^u(t), u'(t)) - f(t, x^u(t), u(t))] \Bigg\} dt \Bigg|_2 \\
&\leq \frac{\varepsilon'' k}{2^{m+1}}.
\end{aligned} \tag{3.36}$$

From (3.35) and (3.27) it follows that, for any t,

$$\begin{aligned}
&\left|q^{u_k,u}(t) - kq^{u'',u}(t)\right|_2 \\
&\quad \le D \sup_{s\le t}\left|\int_0^s \sum_{m=0}^{\infty} 1_{I_m}\{[f(\check{s}, x^u(\check{s}), u_k(\check{s})) - f(\check{s}, x^u(\check{s}), u(\check{s}))\right. \\
&\qquad\qquad \left. -k[f(\check{s}, x^u(\check{s}), u''(\check{s})) - f(\check{s}, x^u(\check{s}), u(\check{s}))]\}d\check{s}\right|_2 \\
&\quad \le D\varepsilon'' k \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} = D\varepsilon'' k, \quad \text{with } \sigma^*(u_k, u) \le k,
\end{aligned} \tag{3.37}$$

and similarly, from (3.36) and (3.29) it follows that

$$\sup_t \left|q^{u_k,u}(t) - \left[kq^{u'',u}(t) + (1-k)q^{u',u}(t)\right]\right|_2 \le D\varepsilon'' k. \tag{3.38}$$

From (3.38) it follows, in a shorthand notation, that

$$2^i \left|\int_{I_i} \left\{\pi \dot{q}^{u_k,u} - \left(k\pi \dot{q}^{u'',u} + (1-k)\pi \dot{q}^{u',u}\right)\right\} dt\right|_2 \le \frac{M^+ D\varepsilon'' k}{2} + \frac{\varepsilon'' k}{2}. \tag{3.39}$$

To see this, note that

$$\begin{aligned}
&\left\|\pi f_x(t, x^u, u) q^{u_k,u} - \pi f_x(t, x^u, u)\left[kq^{u'',u} + (1-k)q^{u',u}\right]\right\|_2 \\
&\quad \le M^+ \left\|q^{u_k,u} - \left[kq^{u'',u} + (1-k)q^{u',u}\right]\right\|_2 \le M^+ D\varepsilon'' k,
\end{aligned} \tag{3.40}$$

(by (2.2) and (3.38)), so

$$2^i \left|\int_{I_i} \pi f_x(t, x^u, u) q^{u_k,u} - \pi f_x(t, x^u, u)\left[kq^{u'',u} + (1-k)q^{u',u}\right] dt\right|_2 \le \frac{M^+ D\varepsilon'' k}{2}. \tag{3.41}$$

Note also that $\pi \dot{q}^{u_k,u} = \pi f_x(t, x^u, u) q^{u_k,u} + \pi f(t, x^u, u_k) - \pi f(t, x^u, u)$ and

$$\begin{aligned}
k\pi \dot{q}^{u'',u} + (1-k)\pi \dot{q}^{u',u} &= \pi f_x(t, x^u, u)\left[kq^{u'',u} + (1-k)q^{u',u}\right] \\
&\quad + k\pi f(t, x^u, u'') + (1-k)\pi f(t, x^u, u') - \pi f(t, x^u, u).
\end{aligned} \tag{3.42}$$

Then (3.39) follows from (3.36) and (3.41).

If $u^i \in U'$, $i = 1, \ldots, i^*$, i^* arbitrary, and $k^i \in (0,1)$, $\sum_i k^i = 1$, then for any $\varepsilon^* > 0$, it is easily seen that we can obtain, for some $\hat{u} \in U'$, that

$$2^i \left| \int_{I_i} \left\{ \pi \dot{q}^{\hat{u},u}(t,\cdot) - \sum_{i=1}^{i^*} k^i \pi \dot{q}^{u^i,u}(t,\cdot) \right\} dt \right|_2 \leq \varepsilon^*. \tag{3.43}$$

(For $i^* = 3$, by (3.39), we can first obtain a control $\check{u}$ such that $2^i \left| \int_{I_i} \{\pi \dot{q}^{\check{u},u} - \sum_{i=1}^{2} k^i/(k^1+k^2)\pi \dot{q}^{u^i,u}\} dt \right|_2 \leq \varepsilon^*/2$, and then by (3.39) we can obtain a control $\hat{u}$ such that $2^i \left| \int_{I_i} \{\pi \dot{q}^{\hat{u},u} - [(k^1+k^2)\pi \dot{q}^{\check{u},u} + k^3 \pi q^{u^3,u}]\} dt \right|_2 \leq \varepsilon^*/2$, hence $2^i \left| \int_{I_i} \{\pi \dot{q}^{\hat{u},u} - [(k^1+k^2)(\sum_{i=1}^{2} k^i/(k^1+k^2)\pi \dot{q}^{u^i,u}) + k^3 \pi q^{u^3,u}]\} dt \right|_2 \leq \varepsilon^*/2 + (k^1+k^2)\varepsilon^*/2 \leq \varepsilon^*$. Continuing this argument, we get (3.43) for general i^*.)

Evidently, we can obtain for any $\varepsilon^* > 0$, for some $\hat{u} \in U'$, that both

$$\text{(3.43) holds and } \left| q^{\hat{u}}(1,\cdot) - \left(\sum_{i=1}^{i^*} k^i q^{u^i,u}(1,\cdot) \right) \right|_2 \leq \varepsilon^*. \tag{3.44}$$

Finally, let the number $c' \in [0,1)$ satisfy $2M^+ D\check{D}(1-c') \leq \alpha/4$, $c' \geq c$ (for c and α, see (2.10)), and let $u \in U^{**} := \{u \in U' : \sigma^*(u,u^*) \leq \alpha/16M^+ D\check{D}\}$. We want to prove the inequality (shorthand notation)

$$\left\| \pi f(\cdot, x^*, \hat{u}) - \pi f(\cdot, x^*, u^*) - \left[\pi f_x(\cdot, x^u, u) q^{\hat{u},u} + \pi f(\cdot, x^u, \hat{u}) - \pi f(\cdot, x^u, u^*) \right] \right\|_2 \leq \frac{\alpha}{2} \tag{3.45}$$

whenever $\hat{u}$ equals u on $[0, c')$.

Now, $|\pi f_x(t, x^u, u) q^{\hat{u},u}|_2 \leq M^+ |q^{\hat{u},u}|_2 \leq 2M^+ D\check{D}(1-c') \leq \alpha/4$, see (3.26). Next, for $u \in U^{**}$, $\|x^u - x^*\|_2 \leq \alpha/8M^+$ when $\check{\sigma}(u,u^*) \leq \sigma^*(u,u^*) \leq \alpha/16M^+ D\check{D}$, see (3.25), so $\|\pi f(t,x^*,\hat{u}) - \pi f(t,x^*,u^*) - [\pi f(t,x^u,\hat{u}) - \pi f(t,x^u,u^*)]\|_2 \leq 2M^+ \|x^u - x^*\|_2 \leq \alpha/4$.

Using the two inequalities involving $\alpha/4$, we get (3.45). And from this property and (2.10) it easily follows that (shorthand notation), for all $u \in U^{**}$,

$$\begin{aligned} 1_{[c',1]} \left[\check{z}(t,\cdot) + B^{\alpha/2} \right] \subset \operatorname{clco} \Big\{ & 1_{[c',1]} \Big[\pi f_x(t,x^u,u) q^{\hat{u},u} + \pi f(t,x^u,\hat{u}) \\ & - \pi f(t,x^u,u^*) \Big] : \hat{u} \in U' \Big\} \end{aligned} \tag{3.46}$$

(cl = closure in $\|\cdot\|_2$-norm). To see this, apply Lemma 11.1 in Seierstad [8]. (Intuitively this lemma says that if a ball is contained in the closed convex hull of a set, and the elements of the set are slightly perturbed then a slightly smaller ball is contained in the closed convex hull of the set of perturbed elements.)

(C) Relations between Exact and Linear Perturbations

Let u'', u be given elements in U'. Let $\varepsilon > 0$ be arbitrarily given. Define $\varepsilon' = \varepsilon / \max\{K^+, \overline{K}\}$ where the constants K^+ and $\overline{K}$ are specified in the proof below. Let us first prove that for any $r > 0$, small enough, for any $\delta \in (0, r]$, there exists a $u' \in U'$ such that

$$\sup_t \left| \delta^{-1} \left(x^{u'}(t,\cdot) - x^u(t,\cdot) - \delta q^{u'',u}(t,\cdot) \right) \right|_2 \le \overline{K} \varepsilon' \delta \le \varepsilon, \qquad \sigma^*(u', u) \le \delta. \tag{3.47}$$

Write $\overline{\sigma}_x q dB_t = \sum_j \overline{\sigma}_x^j q dB_t^j$. Define, in a shorthand notation,

$$\begin{gathered} \xi^1(t,\omega) = \delta^{-1} \left\{ f\left(t, x^u + \delta q^{u'',u}, u\right) - f(t, x^u, u) - f_x(t, x^u, u) \delta q^{u'',u} \right\}, \\ \xi_1(t) = \left[E\left(\int_0^t \xi^1(s) ds \right)^2 \right]^{1/2}, \\ \xi_2(t) := \left\{ E\left(\int_0^t \left[\delta^{-1} \left\{ \overline{\sigma}\left(t, x^u + \delta q^{u'',u}\right) - \overline{\sigma}(t, x^u) - \overline{\sigma}_x(t, x^u) \delta q^{u'',u} \right\} \right] dB_t \right)^2 \right\}^{1/2}. \end{gathered} \tag{3.48}$$

There exists a $\delta' \in (0, \varepsilon']$ such that

$$\xi_i(t) \le \varepsilon', \quad i = 1, 2, \text{ when } \delta \in (0, \delta'], \text{ uniformly in } t, \tag{3.49}$$

by Lemmas B and C in the appendix.

In (3.35) let $k = \delta$, $\varepsilon'' = \varepsilon'$ and let $u' = u_\delta = u_k$ (so $\check{\sigma}(u', u) \le \sigma^*(u', u) \le \delta$). We will prove (3.47) for this u'.

Let

$$\xi^3(t) := \left[\delta^{-1} \left\{ f\left(t, x^u + \delta q^{u'',u}, u'\right) - f(t, x^u, u) - f_x(t, x^u, u) \delta q^{u'',u} - f(t, x^u, u') + f(t, x^u, u) \right\} \right], \tag{3.50}$$

and let $\xi_3(t) := [E(\int_0^t \xi^3(s) ds)^2]^{1/2}$. On $\complement H_{u',u}$, $\xi^3(t) = \xi^1(t)$, while, by (2.2),

$$\begin{aligned} & 1_{H_{u',u}} \left| \xi^3(t) - \xi^1(t) \right| \\ & \quad \le 1_{H_{u',u}} \delta^{-1} \left\{ f\left(t, x^u + \delta q^{u'',u}, u'\right) \right. \\ & \qquad \left. - f\left(t, x^u + \delta q^{u'',u}, u\right) - f(t, x^u, u') + f(t, x^u, u) \right\} \\ & \quad \le 1_{H_{u',u}} 2M^+ \left| q^{u'',u} \right|. \end{aligned} \tag{3.51}$$

So, by (3.49), uniformly in t,

$$\begin{aligned}
\xi_3(t) &= \left[E\left(\int_0^t \left\{\xi^1(s) + \xi^3(s) - \xi^1(s)\right\} ds\right)^2\right]^{1/2} \\
&\le \left[E\left(\int_0^t \xi^1(s) ds\right)^2\right]^{1/2} + \left[E\left(\int_0^t 1_{H_{u',u}} \left\{\xi^3(s) - \xi^1(s)\right\} ds\right)^2\right]^{1/2} \\
&\le \varepsilon' + \left[E\left(\int_0^t 1_{H_{u',u}} 2M^+ \left|q^{u'',u}\right| ds\right)^2\right]^{1/2} \le \varepsilon' + 4M^+ D\breve{D}\delta \le \widetilde{K}\varepsilon',
\end{aligned} \tag{3.52}$$

where $\widetilde{K} := 1 + 4M^+ D\breve{D}$ (recall $\operatorname{meas}(H_{u',u}) \le \delta$, and $\|q^{u'',u}\|_2 \le 2D\breve{D}$; see (3.26)).

Consider now

$$\begin{aligned}
\delta^{-1}\left(x^{u'} - x^u - \delta q^{u'',u}\right) = \int_0^t \delta^{-1}\Big\{ & f\left(s, x^{u'}, u'\right) - f(s, x^u, u) - f_x(s, x^u, u)\delta q^{u'',u} \\
& - \delta f\left(s, x^u, u''\right) + \delta f(s, x^u, u)\Big\} ds \\
& + \int_0^t \delta^{-1}\left[\overline{\sigma}\left(s, x^{u'}\right) - \overline{\sigma}(s, x^u) - \overline{\sigma}_x(s, x^u)\delta q^{u'',u}\right] dB_s.
\end{aligned} \tag{3.53}$$

By (A.2) in the appendix, Lemma A, for some constants $\breve{D} > 0$, $\overline{K}^* > 0$,

$$\left\|\delta^{-1}\left(x^{u'} - x^u - \delta q^{u'',u}\right)\right\|_2 \le \breve{D}\left[\xi_3(t) + \xi_2(t) + \varepsilon'\right] \le \breve{D}\overline{K}^*\varepsilon', \tag{3.54}$$

To see this, in Lemma A let $\check{z} = x^{u'}$, let $\check{h} = f(t, x, u')$, let $\check{y} = x^u + \delta q^{u'',u}$, let $\alpha^* = \alpha = 0$, $\sigma_* = \overline{\sigma}$, and let

$$\begin{gathered}
\beta = \left[f(s, x^u, u) - f\left(s, x^u + \delta q^{u'',u}, u'\right) + f_x(s, x^u, u)\delta q^{u'',u} + \delta f\left(s, x^u, u''\right) - \delta f(s, x^u, u)\right], \\
\beta^* = \overline{\sigma}(s, x^u) - \overline{\sigma}\left(s, x^u + \delta q^{u'',u}\right) + \overline{\sigma}_x(s, x^u)\delta q^{u'',u}.
\end{gathered} \tag{3.55}$$

Note that

$$\begin{aligned}\check{y}(t) = &\int_0^t \left[f(s,x^u,u) - f\left(s,x^u+\delta q^{u'',u},u'\right)\right]ds + \int_0^t \left[\overline{\sigma}(s,x^u) - \overline{\sigma}\left(s,x^u+\delta q^{u'',u}\right)\right]dB_s \\ &+ \int_0^t \left[f_x(s,x^u,u)\delta q^{u'',u} + \delta f(s,x^u,u'') - \delta f(s,x^u,u)\right]ds \\ &+ \int_0^t \overline{\sigma}_x(s,x^u)\delta q^{u'',u}dB_s + \int_0^t f\left(s,x^u+\delta q^{u'',u},u'\right)ds + \int_0^t \overline{\sigma}\left(s,x^u+\delta q^{u'',u}\right)dB_s.\end{aligned} \tag{3.56}$$

Hence, we have

$$\check{y} = \int_0^t \beta ds + \int_0^t \beta^* dB_s + \int_0^t \check{h}(s,\check{y})ds + \int_0^t \sigma_*(s,\check{y})dB_s. \tag{3.57}$$

Observe that $|\int_0^t \delta^{-1}\beta^* dB_s|_2 \le \xi_2(t)$ (see definition subsequent to (3.47)), and that $\beta = -\delta\xi^3 + \delta f(s,x^u,u'') - \delta f(s,x^u,u) - (f(s,x^u,u') - f(s,x^u,u))$, so $|\int_0^t \delta^{-1}\beta|_2 \le \xi_3(t) + \varepsilon'$, because $u' = u_\delta = u_k$ satisfies (3.35) for $\varepsilon' = \varepsilon''$. Using also (3.49), (3.52) yields (3.54) for $\overline{K}^* = \tilde{K} + 2$; hence, (3.47) for $\overline{K} = \check{D}\overline{K}^*$.

Next, let us prove that for any $r > 0$ small enough, for any $\delta \in (0,r]$, there exists a $u' = u_\delta \in U'$ such that

$$\text{(3.47) holds, and } 2^i\left|\int_{I_i} \delta^{-1}\left[\pi\dot{x}^{u'} - \pi\dot{x}^u - \delta\pi\dot{q}^{u'',u}\right]ds\right|_2 \le \varepsilon. \tag{3.58}$$

When $\delta' \in (0,\varepsilon']$ is small enough, then $|\pi\xi^1(t)|_2 \le \varepsilon'$, uniformly in t when $\delta \in (0,\delta']$, by (3.26) and Lemma 3.3 above (ξ^1 defined subsequent to (3.47), we use that $\xi^1 = \delta^{-1}\{\int_0^1[f_x(t,x^u+\theta\delta q^{u'',u},u) - f_x(t,x^u,u)]\delta q^{u'',u}d\theta\}$). Moreover, by (3.47), $\|x^{u'} - x^u - \delta q^{u'',u}\|_2 \le \overline{K}\varepsilon'\delta$, so, by (2.2),

$$\left\|\pi f\left(\cdot,x^u+\delta q^{u'',u},u\right) - \pi f\left(\cdot,x^{u'},u\right)\right\|_2 \le M^+\overline{K}\varepsilon'\delta. \tag{3.59}$$

Hence, using the definition of ξ^1 referred to and $|\pi\xi^1(t)|_2 \le \varepsilon'$, we get $\|\xi^4\|_2 \le \varepsilon' + M^+\overline{K}\varepsilon'$, where

$$\xi^4(t) = \delta^{-1}\left\{\pi f\left(t,x^{u'},u\right) - \pi f(t,x^u,u)\right\} - \pi f_x(t,x^u,u)q^{u'',u}. \tag{3.60}$$

Let

$$\xi^5(t) := \delta^{-1}\left\{\pi f\left(t,x^{u'},u'\right) - \pi f(t,x^u,u) - \pi f_x(t,x^u,u)\delta q^{u'',u} - \pi f\left(t,x^u,u'\right) + \pi f(t,x^u,u)\right\}. \tag{3.61}$$

Then $\xi^5(t) = \xi^4(t)$ on $\complement H_{u',u}$. On $H_{u',u}$, by (2.2), (3.25), and (3.26), for all t,

$$\left|\xi^5(t)\right|_2 \leq M^+\delta^{-1}\left|x^{u'} - x^u\right|_2 + M^+\left|q^{u'',u}\right|_2 \leq \delta^{-1}M^+2D\check{D}\check{\sigma}(u',u) + M^+2D\check{D} \leq 4M^+D\check{D}, \quad (3.62)$$

as $\check{\sigma}(u',u) \leq \delta$. Hence, using the inequalities for $|\xi^5|_2$ and $\|\xi^4\|_2$ above and Jensen's inequality,

$$\begin{aligned} 2^i\left[E\left(\int_{I_i}\xi^5(t)dt\right)^2\right]^{1/2} &\leq 2^i\left[E\left(\int_J 1_{I_i}1_{\complement H_{u',u}}\xi^4(t)dt\right)^2\right]^{1/2} \\ &\quad + 2^i\left[E\left(\int_J 1_{I_i}1_{H_{u',u}}\xi^5(t)dt\right)^2\right]^{1/2} \\ &\leq \left(\frac{1}{2}\right)\left[\varepsilon' + M^+\overline{K}\varepsilon'\right] + 2M^+D\check{D}\sigma^*(u',u). \end{aligned} \quad (3.63)$$

Finally, recalling that $\sigma^*(u',u) \leq \delta$, for

$$\xi^6(t) := \delta^{-1}\left\{\pi f\left(t,x^{u'},u'\right) - \pi f(t,x^u,u) - \pi f_x(t,x^u,u)\delta q^{u'',u} - \pi\delta f\left(t,x^u,u''\right) + \pi\delta f(t,x^u,u)\right\}, \quad (3.64)$$

we have that

$$\begin{aligned} &2^i\left[E\left(\int_{I_i}\xi^5(t) - \xi^6(t)dt\right)^2\right]^{1/2} \\ &\leq 2^i\left[E\left(\int_{I_i}\delta^{-1}\left[\delta\pi f\left(t,x^u,u''\right) - \delta\pi f(t,x^u,u) - \left(\pi f\left(t,x^u,u'\right) - \pi f(t,x^u,u)\right)\right]dt\right)^2\right]^{1/2} \\ &\leq \frac{\varepsilon'}{2}, \end{aligned} \quad (3.65)$$

by (3.35). Hence, using $\sigma^*(u',u) \leq \delta \leq \varepsilon'$ and (3.63),

$$\begin{aligned} 2^i\left|\int_{I_i}\xi^6(t)dt\right|_2 &\leq 2^i\left|\int_{I_i}\xi^5(t)dt\right|_2 + 2^i\left|\int_{I_i}\left(\xi^6(t) - \xi^5(t)\right)dt\right|_2 \\ &\leq \left(\frac{1}{2}\right)\left[\varepsilon' + M\overline{K}\varepsilon'\right] + 2M^+D\check{D}\sigma^*(u',u) + \frac{\varepsilon'}{2} \\ &= \left(1 + \frac{M\overline{K}}{2} + 2M^+D\check{D}\right)\varepsilon', \end{aligned} \quad (3.66)$$

so for $K^+ := (1 + M\overline{K}/2 + 2M^+D\check{D})$, (3.58) has been proved.

(D) Continuity of $u \longrightarrow q^{u',u}$ at u^*

Define $\delta(t) := q^{u',u} - q^{u',u^*}$. Let us first prove that

$$\|\delta(\cdot,\cdot)\|_2 \longrightarrow 0, \quad \text{when } u \longrightarrow u^* \text{ in } \sigma^*\text{-metric.} \tag{3.67}$$

Now,

$$\begin{aligned}
\delta(t) &= \int_0^t \left\{ f_x(s,x^u,u)q^{u',u} + f(s,x^u,u') - f(s,x^u,u) \right\} ds \\
&\quad - \int_0^t \left[f_x(s,x^*,u^*)q^{u',u^*} + f(s,x^*,u') - f(s,x^*,u^*) \right] ds \\
&\quad + \int_0^t \left\{ \overline{\sigma}_x(s,x^u)q^{u',u} - \overline{\sigma}_x\left(s,x^{u^*}\right)q^{u',u^*} \right\} dB_s \\
&= \int_0^t \left[f_x(s,x^u,u)q^{u',u} - f_x(s,x^u,u)q^{u',u^*} \right] ds + \int_0^t \left[f_x(s,x^u,u) - f_x(s,x^u,u^*) \right] q^{u',u^*} ds \\
&\quad + \int_0^t \left[f_x(s,x^u,u^*) - f_x(s,x^*,u^*) \right] q^{u',u^*} ds + \int_0^t \left[f(s,x^u,u') - f(s,x^*,u') \right] ds \\
&\quad + \int_0^t \left[f(s,x^*,u^*) - f(s,x^u,u^*) \right] ds + \int_0^t 1_{H_{u,u^*}} \left[f(s,x^u,u^*) - f(s,x^u,u) \right] ds \\
&\quad + \sum_j \int_0^t \left[\overline{\sigma}_x^j(s,x^u)q^{u',u} - \overline{\sigma}_x^j(s,x^u)q^{u',u^*} \right] dB_s^j + \sum_j \int_0^t \left[\overline{\sigma}_x^j(s,x^u) - \overline{\sigma}_x^j(s,x^*) \right] q^{u',u^*} dB_s^j.
\end{aligned} \tag{3.68}$$

Using Ito's isometry, Jensen's inequality, and the algebraic inequality $\left(\sum_{i=1}^N |a_i|\right)^2 \le N \sum_j a_i^2$, then for some number k (only dependent on the number of addends)

$$\begin{aligned}
E|\delta(t)|^2 &\le k \int_0^t M^{+2} E\left| q^{u',u} - q^{u',u^*} \right|^2 ds \\
&\quad + k \left| \int_0^t 1_{H_{u,u^*}} 2M^{+2} E\left| q^{u',u^*} \right|^2 ds + kE \int_0^t \left| \left\{ f_x(s,x^u,u^*) - f_x(s,x^*,u^*) \right\} q^{u',u^*} \right|^2 ds \right. \\
&\quad + k \int_0^t 2M^{+2} E|x^u - x^*|^2 ds + k \left| \int_0^t 1_{H_{u,u^*}} E\left| f(s,x^u,u^*) - f(s,x^u,u) \right|^2 ds \right. \\
&\quad + k \sum_j \int_0^t M^{+2} E\left(q^{u',u} - q^{u',u^*} \right)^2 ds + k \sum_j E \int_0^t \left[\left\{ \overline{\sigma}_x^j(s,x^u) - \overline{\sigma}_x^j(s,x^*) \right\} q^{u',u^*} \right]^2 ds.
\end{aligned} \tag{3.69}$$

Hence,

$$E\delta(t)^2 \le k(1+n')M^{+2}\int_0^t E\delta(s)^2 ds + \int_0^t \gamma(s)ds, \tag{3.70}$$

where

$$\begin{aligned}\gamma(t) := {} & k1_{H_{u,u^*}}2M^{+2}E\left|q^{u',u^*}\right|^2 + kE\left[\left\{f_x(t,x^u,u^*) - f_x(t,x^*,u^*)\right\}q^{u',u^*}\right]^2 \\ & + 2kM^{+2}E|x^u - x^*|^2 + k1_{H_{u,u^*}}E\left[f(t,x^u,u^*) - f(t,x^u,u)\right]^2 \\ & + k\sum_j E\left[\left\{\overline{\sigma}_x^j(t,x^u) - \overline{\sigma}_x^j(t,x^*)\right\}q^{u',u^*}\right]^2.\end{aligned} \tag{3.71}$$

Now, $\|x^u(\cdot,\cdot) - x^{u^*}(\cdot,\cdot)\|_2 \to 0$ when $u \to u^*$ in $\check{\sigma}$-metric, see (3.25). Hence, by the Basic assumption A_1, when $u \to u^*$ in $\check{\sigma}$-metric, then for each t, it is easily seen, using Lebesgue's dominated convergence theorem (and, if necessary, Remark M in the appendix), that the terms in curly brackets converge to zero in L_2, by the bound M^+ on f_x and $\overline{\sigma}_x$; see (3.25) and (2.2). Since $E|q^{u',u^*}|_2 \le 2D\check{D}$, then, for each t, the product of the two terms in curly brackets with q^{u',u^*} converge to zero in L_1 when $u \to u^*$. Hence, the expectation of the two products converge to zero when $u \to u^*$ in $\check{\sigma}$-metric. Since $\gamma(\cdot)$ is bounded, by (3.24), (3.25), and (3.26), then $\int_0^1 \gamma(s)ds \to 0$ when $u \to u^*$ in $\check{\sigma}$-metric (by dominated convergence again). By Gronwall's inequality, for some constant D,

$$E\delta(t)^2 \le D\int_0^t \gamma(s)ds \le D\int_0^1 \gamma(s)ds. \tag{3.72}$$

So (3.67) holds.

Next, we want to prove that δ_i satisfies

$$\sup_i 2^i|\delta_i(\cdot)|_2 \longrightarrow 0 \quad \text{when } u \longrightarrow u^* \text{ in } \sigma^*\text{-metric}, \tag{3.73}$$

where

$$\begin{aligned}\delta_i(\omega) := {} & \int_{I_i}\left[\pi f_x(t,x^u(t,\omega),u(t,\omega))q^{u',u} + \pi f(t,x^u,u') - \pi f(t,x^u,u)\right]dt \\ & - \int_{I_i}\left[\pi f_x(t,x^*(t,\omega),u^*(t,\omega))q^{u',u^*} + \pi f(t,x^*,u') - \pi f(t,x^*,u^*)\right]dt.\end{aligned} \tag{3.74}$$

Now, in a shorthand notation,

$$\begin{aligned}\delta_i(\omega) = &\int_{I_i}\left[\pi f_x(t,x^u,u)q^{u',u} - \pi f_x(t,x^u,u)q^{u',u^*}\right]dt + \int_{I_i}\left\{\pi f_x(t,x^u,u) - \pi f_x(t,x^u,u^*)\right\}q^{u',u^*}dt \\ &+ \int_{I_i}\left\{\pi f_x(t,x^u,u^*) - \pi f_x(t,x^*,u^*)\right\}q^{u',u^*}dt + \int_{I_i}\left[\pi f(t,x^u,u') - \pi f(t,x^*,u')\right]dt \\ &+ \int_{I_i}\left[\pi f(t,x^*,u^*) - \pi f(t,x^u,u^*)\right]dt + \int_{I_i} 1_{H_{u,u^*}}\left[\pi f(t,x^u,u^*) - \pi f(t,x^u,u)\right]dt.\end{aligned} \tag{3.75}$$

From this we get that

$$\begin{aligned}|\delta_i|_2 \le &\int_{I_i} M^+\left\|q^{u',u}(\cdot,\cdot) - q^{u',u^*}(\cdot,\cdot)\right\|_2 dt + \int_{I_i} 1_{H_{u,u^*}} 2M^+\left\|q^{u',u^*}(\cdot,\cdot)\right\|_2 dt \\ &+ \left|\int_{I_i}\left\{\pi f_x(t,x^u(t,\cdot),u^*(t,\cdot)) - \pi f_x(x^*(t,\cdot),u^*(t,\cdot))\right\}q^{u',u^*}(t,\cdot)\right|_2 dt \\ &+ \int_{I_i} 2M^+\|x^u(\cdot,\cdot) - x^*(\cdot,\cdot)\|_2 dt \\ &+ \int_{I_i} 1_{H_{u,u^*}}\left|f(t,x^u(t,\cdot),u^*(t,\cdot)) - f(t,x^u(t,\cdot),u(t,\cdot))\right|_2 dt.\end{aligned} \tag{3.76}$$

For some increasing nonnegative error function $e(\cdot) \le 2M^+$, the third integrand is smaller than $e(\sup_t|x^u(t,\omega) - x^*(t,\omega)|)\sup_t|q^{u',u^*}(t,\omega)|$; see B_1 and B_4 in the global assumptions. We then get that

$$\begin{aligned}2^i|\delta_i|_2 \le &\left(\frac{M^+}{2}\right)\left\|q^{u',u} - q^{u',u^*}\right\|_2 + M^+\left\|q^{u',u^*}\right\|_2 2^i\int_{I_i} 1_{H_{u,u^*}} \\ &+ \left|e\left(\sup_t\left|x^u(t,\cdot) - x^{u^*}(t,\cdot)\right|\right)\left(\sup_t\left|q^{u',u^*}(t,\cdot)\right|\right)\right|_2 \cdot 2^i\int 1_{I_i}dt \\ &+ M^+\|x^u(t,\cdot) - x^*(t,\cdot)\|_2 + 2^i\check{D}\int_{I_i} 1_{H_{u,u^*}}dt.\end{aligned} \tag{3.77}$$

Now, when $\sigma^*(u,u^*) \to 0$, $\|q^{u',u}(\cdot,\cdot) - q^{u',u^*}(\cdot,\cdot)\|_2 \to 0$ by (3.67), and $\|x^u(\cdot,\cdot) - x^*(\cdot,\cdot)\|_2^* \to 0$ (see (3.25)). Then the term $e(\sup_t|x^u(t,\omega) - x^{u^*}(t,\omega)|) \to 0$ in P-measure and then also in L_2-norm (by the bound $2M^+$), so $e(\sup_t|x^u(t,\omega) - x^{u^*}(t,\omega)|)\sup_t|q^{u',u^*}(t,\omega)| \to 0$ in L_1-norm, and then also in L_2-norm, as the term is bounded by the L_2-function $2M^+\sup_t|q^{u',u^*}(t,\omega)|$, (see (3.26)). Thus $2^i\delta_i \to 0$, uniformly in i; that is, (3.73) holds.

(E) Final Proof Steps

For $z(\cdot) \in L_2(\Omega, Y)$, define $\Pi_1(z(\cdot)) := E[z(\cdot) \mid \Phi_{1-1/2}]$ and $\Pi_i(z(\cdot)) := E[z(\cdot) \mid \Phi_{1-1/2^i}] - E[z(\cdot) \mid \Phi_{1-1/2^{i-1}}]$, $i > 1$. Define

$$
\begin{aligned}
{}_2|z(\cdot)| &:= \sup_i 2^i |\Pi_i z(\cdot)|_2 \quad \text{for } z(\cdot) \in L_2(\Omega, Y), \\
{}^2|z(\cdot,\cdot)| &:= \sup_i 2^i \left| \int_{I_i} |z(t,\cdot)| dt \right|_2 \quad \text{for } z(\cdot,\cdot) \in B^\infty,
\end{aligned}
\tag{3.78}
$$

for B^∞ see (2.8). Furthermore, let L^2 be the subset of $L_2(\Omega, Y)$ consisting of all element $z(\cdot) \in L_2(\Omega, Y)$ such that ${}_2|z(\cdot)| := \sup_i 2^i |\Pi_i z(\cdot)|_2 < \infty$, and such that $z(\cdot) = \lim_{k\to\infty} \sum_{1\le i\le k} \Pi_i z(\cdot) = \lim_{k\to\infty} E[z(\cdot) \mid \Phi_{1-1/2^k}]$, (limit in $|\cdot|_2$-norm). It is easily seen that elements of the type $\int_J y(t,\omega)dt$, $y(t,\omega)$ progressively measurable, $\|y(\cdot,\cdot)\|_2 < \infty$, precisely make up the set L^2. To see this, using Jensen's inequality three times (and for any $\alpha(t) \ge 0$, that $\operatorname{essup}_t (\alpha(t))^{1/2} = (\operatorname{essup}_t \alpha(t))^{1/2}$, note that for any interval J',

$$
\begin{aligned}
&\left(E\left[E\left[\int_{J'} y(t,\cdot)dt \mid \Phi_{1-1/2^k} \right]\right]^2 \right)^{1/2} \le \left(E\left[E\left[\left(\int_{J'} y(t,\cdot)dt \right)^2 \mid \Phi_{1-1/2^k} \right]\right]\right)^{1/2} \\
&= \left(E\left[\operatorname{meas}(J') \int_{J'} (y(t,\cdot))^2 dt \right]\right)^{1/2} = (\operatorname{meas}(J'))^{1/2} \left(E\left[\int_{J'} y(t,\cdot)^2 dt \right]\right)^{1/2} \\
&\le (\operatorname{meas}(J'))^{1/2} \left(\int_{J'} \operatorname{essup}_t E|y(t,\cdot)|^2 dt \right)^{1/2} = \operatorname{meas}(J') \left(\operatorname{essup}_t E|y(t,\cdot)|^2 \right)^{1/2} \\
&= \operatorname{meas}(J') \|y(\cdot,\cdot)\|_2,
\end{aligned}
\tag{3.79}
$$

so, in particular, $|\Pi_1 \int_J y(t,\cdot)dt|_2 \le \|y(\cdot,\cdot)\|_2$. This yields also, for $j > 1$, that

$$
\begin{aligned}
\left| \Pi_j \int_J y(t,\cdot)dt \right|_2 &= \left| \Pi_j \sum_{0\le i<\infty} \int_{I_i} y(t,\cdot)dt \right|_2 = \left| \sum_{j-1\le i<\infty} \Pi_j \int_{I_i} y(t,\cdot)dt \right|_2 \\
&\le \left| \sum_{j-1\le i<\infty} E\left[\int_{I_i} y(t,\cdot)dt \mid \Phi_{1-1/2^j} \right] \right|_2 + \left| \sum_{j-1\le i<\infty} E\left[\int_{I_i} y(t,\cdot)dt \mid \Phi_{1-1/2^{j-1}} \right] \right|_2 \\
&\le 2 \cdot \sum_{j-1\le i<\infty} \left(\frac{1}{2^{i+1}} \right) \|y(\cdot,\cdot)\|_2 = 2 \cdot \left(\frac{1}{2^{j-1}} \right) \|y(\cdot,\cdot)\|_2,
\end{aligned}
\tag{3.80}
$$

so ${}_2|\int_J y(t,\omega)dt| \le 4\|y(\cdot,\cdot)\|_2 < \infty$. Moreover, similarly,

$$\begin{aligned}
&\left|\int_J y(t,\omega)dt - \sum_{1\le j\le k}\Pi_j\int_J y(t,\omega)dt\right|_2 \\
&\quad = \left|\int_J y(t,\omega)dt - E\left[\int_J y(t,\omega)dt \mid \Phi_{1-1/2^k}\right]\right|_2 \\
&\quad = \left|\int_J y(t,\omega)dt - \int_0^{1-1/2^k} y(t,\omega)dt - E\left[\int_{1-1/2^k}^1 y(t,\omega)dt \mid \Phi_{1-1/2^k}\right]\right|_2 \\
&\quad \le \left(\frac{2}{2^k}\right)\|y(\cdot,\cdot)\|_2,
\end{aligned} \tag{3.81}$$

so $\int_J y(t,\omega)dt$ is an L_2-limit of $\sum_{1\le j\le k}\Pi_j\int_J y(t,\omega)dt$. Hence, $\int_J y(t,\omega)dt \in L^2$. Finally, if $z(\omega) \in L^2$, then $z(\omega) = \int_J \gamma(t,\omega)dt = \lim_{k\to\infty}\int_0^{1-1/2^{k+1}}\gamma(t,\omega)dt$, for

$$\gamma(t,\omega) := 2\sum_{m\ge 1}2^m\Pi_m z(\omega)1_{[1-1/2^m,1-1/2^{m+1})}(t), \tag{3.82}$$

where $\|\gamma(\cdot,\cdot)\|_2 \le 2\cdot{}_2|z(\cdot)|$, $\gamma(\cdot,\cdot)$ is progressively measurable.

Let Θ be the linear map from B^∞ into $L_2(\Omega,Y)$ defined by $z(\cdot,\cdot) \to \int_0^1 z(t,\omega)dt$. In (3.80) we have just proved that Θ has norm ≤ 4 for the norms $\|\cdot\|_2$ and ${}_2|\cdot|_2$ (or for $\|\cdot\|_2 \to {}_2|\cdot|$, as we will express it). We also have that the norm on Θ is ≤ 8 for the norms ${}^2|\cdot| \to {}_2|\cdot|$, as we will see. Let $z(\cdot,\cdot) \in B^\infty$ and define $z_i = \int_{I_i} z(t,\omega)$. Then, by Jensen's inequality,

$$\begin{aligned}
\left|\Pi_1\sum_{i=0}^\infty z_i(\cdot)\right|_2 &\le \sum_{i=0}^\infty|\Pi_1 z_i(\cdot)|_2 \le \sum_{i=0}^\infty\left(E\left[\Pi_1(z_i(\cdot))^2\right]\right)^{1/2} = \sum_{i=0}^\infty\left(E\left[(z_i(\cdot))^2\right]\right)^{1/2} \\
&\le \sum_{i=0}^\infty\left(\frac{1}{2^i}\right)\cdot 2^i\left|\int_{I_i} z(\cdot,\cdot)dt\right|_2 \le 2\cdot{}^2|z(\cdot,\cdot)|,
\end{aligned} \tag{3.83}$$

while for $j > 1$,

$$\begin{aligned}
\left|\Pi_j \sum_{i=0}^{\infty} z_i(\cdot)\right|_2 &= \left|\Pi_j \sum_{i=j-1}^{\infty} z_i(\cdot)\right|_2 \\
&= \left|\sum_{i=j-1}^{\infty} \left\{E[z_i(\cdot) \mid \Phi_{1-1/2^j}] - E\left[z_i(\cdot) \mid \Phi_{1-1/2^{j-1}}\right]\right\}\right|_2 \\
&\le \sum_{i=j-1}^{\infty} \{|z_i(\cdot)|_2 + |z_i(\cdot)|_2\} \le 2\sum_{i=j-1}^{\infty}\left|\int_{I_i} z(\cdot,\cdot)\right|_2 \\
&\le \sum_{i=j-1}^{\infty} \frac{2\cdot{}^2|z(\cdot,\cdot)|}{2^i} = 8\cdot\left(\frac{1}{2^j}\right)\cdot{}^2|z(\cdot,\cdot)|,
\end{aligned} \tag{3.84}$$

so $\sup_j 2^j|\Pi_j \int_J z(t,\cdot)dt|_2 \le 8\cdot{}^2|z(\cdot,\cdot)|$.

Now, for $z \in {}_2B(0,\alpha(1-c')/4) \subset L^2$, (${}_2B(\cdot,\cdot)$ a ball in ${}_2|\cdot|$-norm), we have

$$z = \int_{c'}^{1} \gamma'(s,\cdot)ds, \qquad \gamma'(s,\omega) = 2(1-c')^{-1}\sum_{i=0}^{\infty} 2^i \Pi_i z 1_{[1-(1-c')/2^i,1-(1-c')/2^{i+1})} \in B^{\alpha/2}. \tag{3.85}$$

Hence, by (3.46), for $u \in U^{**}$,

$$\begin{aligned}
&\int_{c'}^{1} \check{z}dt + {}_2B\left(0,\frac{\alpha(1-c')}{4}\right) \\
&\quad \subset \operatorname{clco}\left\{\int_{c'}^{1}\left[\pi f_x(t,x^u,u)q^{\hat{u},u} + \pi f(t,x^u,\hat{u}) - \pi f(t,x^u,u^*)\right]dt : \hat{u} \in U'\right\}
\end{aligned} \tag{3.86}$$

(cl = closure in ${}_2|\cdot|$; note that $\gamma''(\cdot,\cdot) \to \int_{c'}^{1}\gamma''(s,\cdot)ds$ is continuous in $\|\cdot\|_2 \to {}_2|\cdot|$, as shown above). Observe, finally, that when $u \in U'$ satisfies $\sigma^*(u,u^*) \le \alpha(1-c')/64\check{D}$, then, using $|\Theta| \le 8$ (for ${}^2|\cdot| \to {}_2|\cdot|$) and (3.24) yields

$$\begin{aligned}
&{}_2\left|\int_J 1_{[c',1]}\left(\pi f(t,x^u,u) - \pi f(t,x^u,u^*)\right)dt\right| \\
&\quad \le 8\sup_i 2^i\left|\int_{I_i} 1_{[c',1]}\left(\pi f(t,x^u,u) - \pi f(t,x^u,u^*)\right)dt\right|_2 \\
&\quad \le 16\check{D}\sup_i 2^i\int_{I_i} 1_{[c',1]}1_{H_{u,u^*}}dt \le 8\check{D}\sigma^*(u,u^*) \le \frac{\alpha(1-c')}{8}.
\end{aligned} \tag{3.87}$$

Hence, using (3.86) and the last string of inequalities and a simple argument (or Lemma 11.1 in Seierstad [8] again), we get

$$\begin{aligned}\int_{c'}^{1} \check{z}ds + {}_2B\left(0, \frac{\alpha(1-c')}{8}\right) &\subset \operatorname{clco}\left\{\int_{c'}^{1}\left[\pi f_x(t,x^u,u)q^{\hat{u},u} + \pi f(t,x^u,\hat{u}) - \pi f(t,x^u,u)\right]dt : \hat{u} \in U'\right\} \\ &\subset \operatorname{clco}\left\{\int_{0}^{1}\left[\pi f_x(t,x^u,u)q^{\hat{u},u} + \pi f(t,x^u,\hat{u}) - \pi f(t,x^u,u)\right]dt : \hat{u} \in U'\right\} \\ &= \operatorname{clco}\left\{\pi q^{\hat{u},u}(1) : \hat{u} \in U'\right\},\end{aligned} \tag{3.88}$$

(cl = closure in ${}_2|\cdot|$) for $u \in U'$, $\sigma^*(u,u^*) \le \mu^* := \min\{\alpha(1-c')/64\check{D}, \alpha/16M^+D\check{D}\}$ (the last fraction defined in connection with (3.45)).

To obtain the conclusion in Theorem 2.1, we will now apply Corollary I in the appendix, and for this end, we make the following identifications: $\widehat{Y} = L^2$, $\gamma^* = \mu^*$, $\widetilde{K}_* = \widetilde{K}'_* = 1$, $\widetilde{A} = U'$, $\overline{\sigma}^* = \sigma^*$, the norm $|\cdot|$ on $\widehat{Y}$ equal to ${}_2|\cdot|$, $\widetilde{a}' = u'$, $\widetilde{a}^* = u^*$, $\widetilde{a} = u$, $\widetilde{a}^+ = u''$, $\widehat{H}(\widetilde{a}) = \pi\Theta\dot{x}^u$, $\widehat{q}^{\widetilde{a}',\widetilde{a}} = \pi\Theta\dot{q}^{u',u}$, $\check{H}(\widetilde{a}) = E[a\cdot\Theta\dot{x}^u]$, and $\check{q}^{\widetilde{a}',\widetilde{a}} = E[a\cdot\Theta\dot{q}^{u',u}]$. By $|\Theta| \le 8$ for ${}^2|\cdot| \to {}_2|\cdot|$ and (3.58), it follows that the property (A.20) is satisfied, and by $|\Theta| \le 8$ and (3.44) it follows that (A.21) is satisfied, both (A.20) and (A.21) in the manner required in Corollary I. Moreover, for $\check{z}^* := \int_{c'}^{1}\check{z}(t,\cdot)dt$, ${}_2B(\check{z}^*, \alpha(1-c')/8) \subset \operatorname{clco}\pi q^{U',u}(1,\cdot)$, by (3.88) for $u \in \operatorname{cl}B(u^*,\mu^*)$. By (3.33) and (3.25), $\widehat{H}$ and $\check{H}$ are continuous, and by (3.73), (3.67), $a \to \widetilde{q}^{a',a}$ is continuous at $\widetilde{a}^* = u^*$. The required boundedness of $\check{q}^{a',a}$ is satisfied because of (3.26). The space $(U',\check{\sigma})$ is complete by well-known arguments; see Lemma 5.1 in Seierstad [8] and Lemma 1, page 202 in Clarke [9]. Moreover, if u_n is a Cauchy sequence in σ^*, then it is a Cauchy sequence in $\check{\sigma}$. Let u be its $\check{\sigma}$-limit. Then, for all i, $\lim_{m\to\infty} 2^{i+1}\int_{I_i} 1_{H_{u_n,u_m}}dt = 2^{i+1}\int_{I_i} 1_{H_{u_n,u}}dt$. Now, for any $\varepsilon > 0$, for m, $n \ge$ some N, for all i, $2^{i+1}\int_{I_i} 1_{H_{u_n,u_m}}dt \le \varepsilon$, and so also $2^{i+1}\int_{I_i} 1_{H_{u_n,u}}dt \le \varepsilon$. Thus the space (U',σ^*) is complete. Hence all conditions in Corollary I are satisfied. Thus, for some $\Lambda_0 \ge 0$, some nonzero continuous linear functional ν on L^2, $(\Lambda_0,\nu) \ne 0$, for all $u \in U'$, $\Lambda_0 E[a\cdot q^{u,u^*}(1,\cdot)] + \langle \pi q^{u,u^*}(1,\cdot),\nu\rangle \le 0$. Because $q^{u,u^*}(1,\cdot) = \int_0^1 C(1,t,\cdot)[f(t,x^*(t,\cdot),u(t,\cdot)) - f(t,x^*(t,\cdot),u^*(t,\cdot))]dt$, the conclusion in Theorem 2.1 follows.

Proof of Remark 2.2. Note that (2.10) implies ${}_2B(0,\alpha(1-c')/8) \subset \operatorname{clco}\{\pi q^{U',u^*}(1)\}$. See (3.88). If $\Lambda_0 = 0$, the maximum condition (2.11) implies $\langle \operatorname{clco}\{\pi q^{U',u^*}(1)\},\nu\rangle \le 0$ and hence $\langle {}_2B(0,\alpha(1-c')/8),\nu\rangle \le 0 \Rightarrow \nu = 0$, a contradiction. Observe that on $L^2|\cdot|_2 \le {}_2|\cdot|$ (for $v \in L^2$, $|v|_2 = |\sum_{i=1}^{\infty}\Pi_i v|_2 \le \sum_{i=1}^{\infty}(1/2^i)2^i|\Pi_i v|_2 \le {}_2|v|$). If we show for a ${}_2|\cdot|$-dense subset Q of ${}_2B(0,\alpha(1-c')/8)$ that $\langle \overline{q},\nu\rangle \le k|q|_2$ for $\overline{q} \in Q$ for some k independent of $\overline{q}$, then $\langle \pm\overline{q},\nu\rangle \le k|\pm\overline{q}|_2$ for $\overline{q} \in {}_2B(0,\alpha(1-c')/8)$, implying $|\cdot|_2$-continuity on L^2. The maximum condition implies $\langle \pi q^{U',u^*}(1), Eaq^{U',u^*}(1)\},(\nu,\Lambda_0)\rangle \le 0$; hence, for any $\phi \in {}_2B(0,\alpha(1-c')/8) \cap \pi\operatorname{co}q^{U',u^*}(1)$, there exists a $\psi \in \operatorname{co}q^{U',u^*}(1)$ such that $\phi = \pi\psi$ and $\langle \pi\psi,\nu\rangle + \Lambda_0 E\langle \psi,a\rangle \le 0$, so $\langle \phi,\nu\rangle = \langle \pi\psi,\nu\rangle \le -\Lambda_0 E\langle \psi,a\rangle \le \Lambda_0|a||\psi|_2 \le \Lambda_0|a|2D\check{D}$, by (3.26). □

Proofs of Remark 2.3 and the following remark can be found in the appendix.

Remark 3.4 (exact attainability). In Theorem 2.1, drop the assumption that u^* is optimal (and the optimization problem). Then, for each $z(t,\omega) \in \operatorname{int}(\operatorname{clco}\{\pi\dot{q}^{u,u^*} : u \in U'\})$, $z(\cdot,\cdot) \in$

$L_2^{\text{prog}}(J \times \Omega, Y)$, $\|z(\cdot,\cdot)\|_2 < \infty$, (cl and int = interior both corresponding to $\|\cdot\|_2$), for all $r > 0$, for some number $\gamma \in (0, r]$ and some control $u \in U'$, $\pi x^u(T,\omega) = \tilde{y} + \gamma \int_0^T z(t,\omega)dt$ a.s.

Example

Let $t \in [0,1] = J$, let $dX_t^1 = -u^2 dt$, let $dX_t^2 = u + X_t^3$, let $dX_t^3 = \sigma dB_t$, let σ be a nonzero constant, let $u \in \mathbb{R}$, let $X_0^i = 0$, let $i = 1,2,3$, and let us maximize EX_1^1 subject to $X_1^2 = 1$, a.s. This trivial problem was solved in Seierstad [10] using the HJB equation. Let $\{\Phi_t\}_t$ be the natural filtration generated by B_t and let $U'' = \{v(\cdot,\cdot) \in L_2^{\text{prog}}(J \times \Omega, \mathbb{R}) : \|v(\cdot,\cdot)\|_2 \le 1\}$. Let us merely show that the solution presented in Seierstad [10], namely $u^* = 1 - X_t^{*3}$, $(X_t^{*3} = \sigma B_t)$, satisfies the necessary conditions for $U' = U'' + u^*$. (So X_t^{*2} is deterministic and equals t.) Evidently, the conditions in Remark 2.2 are satisfied, so Λ_0 can be put equal to 1, and ν is $|\cdot|_2$-continuous.

The 3×3-matrix C_t satisfies $dC_t = DC_t$ where the 3×3-matrix D consists of the partial derivatives of the drift terms in the three-state equation above, the only nonzero element in D being the element $D_{23} = 1$. Hence, $C_{ii}(t,s) = 1$, $i = 1,2,3$, $C_{ij} = 0$, $i \neq j$, except for $C_{23}(t,s)$ which equals $t - s$. Now, $\nu_* = (1,\nu,0)$, so the maximum condition (2.11) reduces to $\langle \int_0^1 (u - u^*)dt, \nu\rangle - \int_0^1 (u^2 - u^{*2})dt \le 0$, $u = u(\cdot,\cdot) \in U'$. Writing $p^- = (p_1^-, p_2^-, p_3^-)$, $p_1^- \equiv 1$, the time-pointwise version (2.12) of this condition becomes $p_2^-(t,\omega)[u(t,\omega) - u^*(t,\omega)] - (u(t,\omega)^2 - u^*(t,\omega)^2) \le 0$, which evidently requires that $2u^*(t,\omega) = p_2^-(t,\omega)$. Now, for $u^* = 1 - X_t^{*3}$, $p_2^-(t,\omega) = 2 - 2X_t^{*3}$, $\nu = \lim_{t\to 1} p_2^-(t,\omega) = 2 - 2X_1^{*3}$. This p_2^- does satisfy $p_2^-(t,\omega) = (E[(1,\nu,0)C(1,t) \mid \Phi_t])_2 = E[\nu \mid \Phi_t] = 2 - 2X_t^{*3}$. Moreover, $p_3^-(t,\omega) := (E[(1,\nu,0)C(1,t) \mid \Phi_t])_3 = (E[(1-t)\nu \mid \Phi_t])_3 = (1-t)p_2^-(t,\omega)$. (Concerning (2.14), it is evidently satisfied by $q_1 = 0$, $q_2 = -2\sigma$, $q_3 = -(1-t)2\sigma$, $dp_1 = dp_1^- = 0$, $p_1 = p_1^- \equiv 1$, $dp_2 = dp_2^- = -2\sigma dB_t$, $p_2 = p_2^- = 2 - 2\sigma B_t$, $dp_3 = dp_3^- = -p_2^-(t,\omega) + (1-t)dp_2^-(t,\omega) = -p_2(t,\omega) - (1-t)2\sigma dB_t$).

Appendix

The appendix includes a number of well-known results, included for the convenience of the reader. The first one concerns a result on comparison of solutions.

Lemma A. *Assume that $\check{h}(s,x,\omega)$ (an n-vector) and $\sigma_*(s,x,\omega)$, (an $n \times n'$ matrix, with columns σ_*^j, $j = 1,\ldots,n'$) are Lipschitz continuous in $x \in \mathbb{R}^n$ with rank $\check{K}$ and progressively measurable in (s,ω). Assume that six progressively measurable functions $\check{z}(t,\omega)$, $\alpha(t,\omega)$, $\alpha^*(t,\omega)$, $\check{y}(t,\omega)$, $\beta(t,\omega)$, and $\beta^*(t,\omega)$ exist (α^*, β^* $n \times n'$-matrices), satisfying*

$$\begin{aligned}
\check{z}(t,\omega) &= \check{z}_0 + \int_0^t \alpha(s,\omega)ds + \int_0^t \check{h}(s,\check{z}(s,\omega),\omega)ds \\
&\quad + \int_0^t \alpha^*(s,\omega)dB_s + \int_0^t \sigma_*(s,\check{z}(s,\omega),\omega)dB_s, \\
\check{y}(t,\omega) &= \check{y}_0 + \int_0^t \beta(s,\omega)ds + \int_0^t \check{h}(s,\check{y}(s,\omega),\omega)ds \\
&\quad + \int_0^t \beta^*(s,\omega)dB_s + \int_0^t \sigma_*(s,\check{y}(s,\omega),\omega)dB_s,
\end{aligned} \tag{A.1}$$

where $\check{z}_0, \check{y}_0 \in L_2(\Omega, \Phi_0, \mathbb{R}^n)$. *(Assume that the eight integrands belong to* $L_2(J \times \Omega)$*-spaces). Then, for some constant* $\check{D}$,

$$\sup_{s\le t}\left|\check{z}(s)-\check{y}(s)\right|_2 \le \check{D}\left[\left|\check{y}_0-\check{z}_0\right|_2+\sup_{s\le t}\left|\int_0^s \alpha d\check{s}\right|_2+\sup_{s\le t}\left|\int_0^s \beta d\check{s}\right|_2 + \sum_j\left(E\int_0^t\left|\alpha_j^*\right|^2 d\check{s}\right)^{1/2}+\sum_j\left(E\int_0^t\left|\beta_j^*\right|^2 d\check{s}\right)^{1/2}\right], \tag{A.2}$$

(applied to matrices, the index j *indicates columns), and for some constant* $\check{D}^*$,

$$\begin{aligned}
&\left|\left(\sup_{s\le t}\left|\check{z}(s)-\check{y}(s)\right|\right)\right|_2 \\
&\le \check{D}^*\left[\left|\check{y}_0-\check{z}_0\right|_2+\left|\left(\sup_{s\le t}\left|\int_0^s \alpha d\check{s}\right|\right)\right|_2+\left|\left(\sup_{s\le t}\left|\int_0^s \beta d\check{s}\right|\right)\right|_2 \right. \\
&\qquad \left. +\sum_j\left(E\int_0^t\left|\alpha_j^*\right|^2 d\check{s}\right)^{1/2}+\sum_j\left(E\int_0^t\left|\beta_j^*\right|^2 d\check{s}\right)^{1/2}\right] \\
&\le \check{D}\left[\left|\check{y}_0-\check{z}_0\right|_2+\left(E\int_0^t|\alpha|^2 d\check{s}\right)^{1/2}+\left(E\int_0^t|\beta|^2 d\check{s}\right)^{1/2} \right. \\
&\qquad \left. +\sum_j\left(E\int_0^t\left|\alpha_j^*\right|^2 d\check{s}\right)^{1/2}+\sum_j\left(E\int_0^t\left|\beta_j^*\right|^2 d\check{s}\right)^{1/2}\right],
\end{aligned} \tag{A.3}$$

with $\check{D}$ *and* $\check{D}^*$ *being only dependent on* $\check{K}$.

Proof of (A.3). We will use a shorthand notation. Using the algebraic inequality $\left(\sum_{j=1}^N |a_j|\right)^2 \le N\sum_j a_j^2$, then for some positive constant k,

$$\begin{aligned}
\phi(t) := \left|\check{y}(t) - \check{z}(t)\right|^2 &= \Bigg|\check{y}_0 - \check{z}_0 + \int_0^t \check{h}(s,\check{y})ds + \sum_j \int_0^t \sigma_*^j(s,\check{y})dB_s^j + \int_0^t \beta ds \\
&\quad - \int_0^t \alpha ds - \int_0^t \check{h}(s,\check{z})ds - \sum_j \int_0^t \sigma_*^j(s,\check{z})dB_s^j - \sum_j \int_0^t \alpha_j^* dB_s^j + \int_0^t \beta_j^* dB_s^j\Bigg|^2 \\
&\le k\left|\check{y}_0 - \check{z}_0\right|^2 + k\left|\int_0^t \left(\check{h}(s,\check{y}) - \check{h}(s,\check{z})\right)ds\right|^2 + k\sum_j \left|\int_0^t \left\{\sigma_*^j(s,\check{y}) - \sigma_*^j(s,\check{z})\right\}dB_s^j\right|^2 \\
&\quad + k\left|\int_0^t \alpha ds\right|^2 + k\left|\int_0^t \beta ds\right|^2 + k\sum_j \left|\int_0^t \alpha_j^* dB_s^j\right|^2 + k\sum_j \left|\int_0^t \beta_j^* dB_s^j\right|^2.
\end{aligned} \tag{A.4}$$

The Burkholder-Davis-Gundy inequality yields, for a "universal" constant $\widetilde{K}$, that $E\sup_{s\le t}|\int_0^s(\sigma_*^j(\check{y}) - \sigma_*^j(\check{z}))dB_{\check{s}}^j|^2 \le \widetilde{K}\int_0^t E|\sigma_*^j(\check{y}) - \sigma_*^j(\check{z})|^2 d\check{s}$. Similar inequalities hold for the other terms involving B^j. Hence (using also Jensen's inequality) we get

$$\begin{aligned}
\psi(t) := E\left(\sup_{s\le t}\phi(s)\right) &\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + kE\sup_{s\le t} s\int_0^s \left(\check{h}(\check{s},\check{y}) - \check{h}(\check{s},\check{z})\right)^2 d\check{s} \\
&\quad + kE\sum_j \sup_{s\le t}\left|\int_0^s \left(\sigma_*^j(\check{s},\check{y}) - \sigma_*^j(\check{s},\check{z})\right)dB_{\check{s}}^j\right|^2 + kE\sup_{s\le t}\left(\int_0^s \alpha d\check{s}\right)^2 + kE\sup_{s\le t}\left(\int_0^s \beta d\check{s}\right)^2 \\
&\quad + kE\sum_j \sup_{s\le t}\left(\int_0^s \alpha_j^* dB_{\check{s}}^j\right)^2 + kE\sum_j \sup_{s\le t}\left(\int_0^s \beta_j^* dB_{\check{s}}^j\right)^2 \\
&\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\int_0^t E\left|\check{h}(\check{s},\check{y}) - \check{h}(\check{s},\check{z})\right|^2 d\check{s} \\
&\quad + k\widetilde{K}\sum_j \int_0^t E\left|\sigma_*^j(\check{s},\check{y}) - \sigma_*^j(s,\check{z})\right|^2 d\check{s} \\
&\quad + kE\sup_{s\le t}\left(\int_0^s \alpha d\check{s}\right)^2 + kE\sup_{s\le t}\left(\int_0^s \beta d\check{s}\right)^2 + k\widetilde{K}\sum_j E\int_0^t \left|\alpha_j^*\right|^2 d\check{s} + k\widetilde{K}\sum_j E\int_0^t \left|\beta_j^*\right|^2 ds \\
&\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\int_0^t E\check{K}^2\left|\check{y} - \check{z}\right|^2 d\check{s} + k\widetilde{K}n'\int_0^t E\check{K}^2\left|\check{y} - \check{z}\right|^2 d\check{s} \\
&\quad + kE\left(\sup_{s\le t}\left|\int_0^s \alpha d\check{s}\right|\right)^2 + kE\left(\sup_{s\le t}\left|\int_0^s \beta d\check{s}\right|\right)^2 + k\widetilde{K}\sum_j E\int_0^t \left|\alpha_j^*\right|^2 d\check{s} \\
&\quad + k\widetilde{K}\sum_j E\int_0^t \left|\beta_j^*\right|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq kE\left|\check{y}_0-\check{z}_0\right|^2+k\check{K}^2\int_0^t\psi(\check{s})d\check{s}+k\tilde{K}\check{K}^2n'\int_0^t\psi(\check{s})d\check{s}\\
&\quad+kE\left(\sup_{s\leq t}\left|\int_0^s\alpha d\check{s}\right|\right)^2+kE\left(\sup_{s\leq t}\left|\int_0^s\beta d\check{s}\right|\right)^2+k\tilde{K}\sum_jE\int_0^t\left|\alpha_j^*\right|^2d\check{s}\\
&\quad+k\tilde{K}\sum_jE\int_0^t\left|\beta_j^*\right|^2ds.
\end{aligned}\tag{A.5}$$

Note that, by Gronwall's inequality, for any functions $w(t)$, $v(t)$, if $0\leq w(t)\leq v(t)+\int_0^tKw(s)ds$, and $v(t)$ is increasing, then $w(t)\leq v(t)(1+e^{Kt})$. Hence, for $\check{K}^2:=k(1+e^{k\check{K}^2(1+\tilde{K}n')})$,

$$\begin{aligned}
\psi(t)\leq\check{K}^2\Bigg[&E\left|\check{y}_0-\check{z}_0\right|^2+E\left(\sup_{s\leq t}\left|\int_0^s\alpha d\check{s}\right|\right)^2+E\left(\sup_{s\leq t}\left|\int_0^s\beta d\check{s}\right|\right)^2\\
&+\tilde{K}\sum_jE\int_0^t\left|\alpha_j^*\right|^2d\check{s}+\tilde{K}\sum_jE\int_0^t\left|\beta_j^*\right|^2ds\Bigg].
\end{aligned}\tag{A.6}$$

Using the fact that the square root of a sum of positive numbers is less than or equal the sum of square roots of the numbers, we get

$$\begin{aligned}
\left|\left(\sup_{s\leq t}\left|y(s)-z(s)\right|\right)\right|_2\leq\check{K}\Bigg[&\left|\check{y}_0-\check{z}_0\right|_2+\left|\left(\sup_{s\leq t}\left|\int_0^s\alpha d\check{s}\right|\right)\right|_2\\
&+\left|\left(\sup_{s\leq t}\left|\int_0^s\beta d\check{s}\right|\right)\right|_2+\sum_j\left(\tilde{K}\int_0^tE\left|\alpha_j^*\right|^2d\check{s}\right)^{1/2}\\
&+\sum_j\left(\tilde{K}\int_0^tE\left|\beta_j^*\right|^2ds\right)^{1/2}\Bigg].
\end{aligned}\tag{A.7}$$

Note that $\sup_{s\leq t}|\int_0^s\alpha d\check{s}|\leq\sup_{s\leq t}\int_0^s|\alpha|d\check{s}\leq\int_0^t|\alpha|d\check{s}$, and that $|\int_0^t|\alpha|d\check{s}|_2\leq\int_0^t|\alpha|_2d\check{s}$. Using this for the term containing α, and a similar argument for the term containing β, then (A.3) follows. □

Proof of (A.2). Using Ito's isometry,

$$E\left(\int_0^t \sigma_*^j(s,\check{y}) - \sigma_*^j(s,\check{z})dB_s^j\right)^2 \le E\int_0^t \left(\sigma_*^j(s,\check{y}) - \sigma_*^j(s,\check{z})\right)^2 ds,$$
$$E\left(\int_0^t \alpha_j^* dB_s^j\right)^2 = E\int_0^t \left(\alpha_j^*\right)^2 ds, \tag{A.8}$$
$$E\left(\int_0^t \beta_j^* dB_s^j\right)^2 = E\int_0^t \left(\beta_j^*\right)^2 ds.$$

Then, again using $\left(\sum_{j=1}^N |a_j|\right)^2 \le N\sum_j a_j^2$ and Jensen's inequality, for some positive constant k,

$$\begin{aligned}
\gamma(t) := \sup_{s\le t} E\phi(s) &\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\sup_{s\le t} Es\int_0^s \left(\check{h}(\check{y}) - \check{h}(\check{z})\right)^2 d\check{s} \\
&\quad + k\sum_j \sup_{s\le t} E\left|\int_0^s \left(\sigma_*^j(\check{y}) - \sigma_*^j(\check{z})\right)dB_{\check{s}}^j\right|^2 + k\sup_{s\le t} E\left(\int_0^s \alpha d\check{s}\right)^2 + k\sup_{s\le t} E\left(\int_0^s \beta d\check{s}\right)^2 \\
&\quad + k\sum_j \sup_{s\le t} E\left(\int_0^s \alpha_j^* dB_{\check{s}}^j\right)^2 + k\sum_j \sup_{s\le t} E\left(\int_0^s \beta_j^* dB_{\check{s}}^j\right)^2 \\
&\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\sup_{s\le t}\left|\int_0^s E\left|\check{h}(\check{y}) - \check{h}(\check{z})\right|^2 d\check{s}\right| \\
&\quad + k\sum_j \sup_{s\le t}\int_0^s E\left|\sigma_*^j(\check{y}) - \sigma_*^j(\check{z})\right|^2 d\check{s} + k\sup_{s\le t} E\left(\int_0^s \alpha d\check{s}\right)^2 + k\sup_{s\le t} E\left(\int_0^s \beta d\check{s}\right)^2 \\
&\quad + k\sum_j \sup_{s\le t} E\int_0^s \left|\alpha_j^*\right|^2 d\check{s} + k\sum_j \sup_{s\le t} E\int_0^s \left|\beta_j^*\right|^2 ds \\
&\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\int_0^t E\check{K}^2\left|\check{y} - \check{z}\right|^2 d\check{s} + kn'\int_0^t E\check{K}^2\left|\check{y} - \check{z}\right|^2 d\check{s} \\
&\quad + k\sup_{s\le t} E\left(\int_0^s \alpha d\check{s}\right)^2 + k\sup_{s\le t} E\left(\int_0^s \beta d\check{s}\right)^2 + k\sum_j E\int_0^t \left|\alpha_j^*\right|^2 d\check{s} + k\sum_j E\int_0^t \left|\beta_j^*\right|^2 ds \\
&\le kE\left|\check{y}_0 - \check{z}_0\right|^2 + k\check{K}^2\int_0^t \gamma(\check{s})d\check{s} + k\check{K}^2 n'\int_0^t \gamma(\check{s})d\check{s} \\
&\quad + k\left(\sup_{s\le t} E\left|\int_0^s \alpha d\check{s}\right|\right)^2 + k\sup_{s\le t} E\left(\int_0^s \beta d\check{s}\right)^2 + k\sum_j E\int_0^t \left|\alpha_j^*\right|^2 d\check{s} + k\sum_j E\int_0^t \left|\beta_j^*\right|^2 ds.
\end{aligned} \tag{A.9}$$

Thus, for $\check{K}^{*2} = k(1 + e^{k\check{K}^2(1+n')})$,

$$\sup_{s\le t} E\left|\check{y}(t) - \check{z}(t)\right|^2 \le \check{K}^* \left[E\left|\check{y}_0 - \check{z}_0\right|^2 + \sup_{s\le t} E\left(\int_0^s \alpha d\check{s}\right)^2 \right.$$
$$\left. + \sup_{s\le t} E\left(\int_0^s \beta d\check{s}\right)^2 + E\sum_j \int_0^t \left|\alpha_j^*\right|^2 d\check{s} + E\sum_j \int_0^t \left|\beta_j^*\right|^2 ds \right], \tag{A.10}$$

so (A.2) follows. □

Simple results on Gâteaux derivatives appear in the next two lemmas.

Lemma B. *For each t, $x(t,\omega) \to \int_0^t \overline{\sigma}(s, x(s,\omega))dB_s (x(\cdot,\cdot) \in L_2^{\text{prog}}(J \times \Omega, X))$ has, in the norm $|\cdot|_2$, a bounded linear Gâteaux derivative, which in "direction" $z(\cdot,\cdot)$, $(z(\cdot,\cdot) \in L_2^{\text{prog}}(J \times \Omega, X))$, equals $\int_0^t \overline{\sigma}_x(s, x(s,\omega))z(s,\omega)dB_s$. The derivative is uniform in t; see the first inequality below.*

Proof. By Ito's isometry,

$$E\left|\int_0^t \delta^{-1}[\overline{\sigma}(s, x(s,\omega) + \delta z(s,\omega)) - \overline{\sigma}(s, x(s,\omega)) - \overline{\sigma}_x(s, x(s,\omega))\delta z(s,\omega)]dB_s\right|^2$$
$$= E\int_0^t \left\{\delta^{-1}[\overline{\sigma}(s, x(s,\omega) + \delta z(s,\omega)) - \overline{\sigma}(s, x(s,\omega))] - \overline{\sigma}_x(s, x(s,\omega))z(s,\omega)\right\}^2 ds \tag{A.11}$$
$$=: \beta^{**}(t) \le \beta^{**}(1).$$

The term in curly brackets converges to zero for each (s,ω) and is smaller than the $L_1(J \times \Omega, X)$-function $(2M^+)^2 z(s,\omega)^2$. Hence, Lebesgue's dominated convergence theorem gives that $\beta^{**}(1) \to 0$ when $\delta \to 0$. □

Lemma C. *For each t, each $u(\cdot,\cdot) \in U'$, $x(t,\omega) \to \int_0^t f(s, x(s,\omega), u(s,\omega))ds (x(\cdot,\cdot) \in L_2^{\text{prog}}(J \times \Omega, X))$ has, in the norm $|\cdot|_2$, a bounded linear Gâteaux derivative, which in "direction" $z(\cdot,\cdot)$, $(z(\cdot,\cdot) \in L_2^{\text{prog}}(J \times \Omega, X))$, equals $\int_0^t f_x(s, x(s,\omega), u(s,\omega))z(t,\omega)ds$. In fact, for each $z(\cdot,\cdot)$, for*

$$\beta^*(t) := \left|\int_0^t \delta^{-1}\{f(s, x(s,\omega) + \delta z(s,\omega), u(s,\omega)) - f(s, x(s,u), u(s,\omega))\}\right.$$
$$\left. - f_x(s, x(s,\omega), u(s,\omega))z(s,\omega)ds\right|_2,$$
$$\beta^{**}(t) := E\int_0^t \left|\delta^{-1}\{f(s, x(s,\omega) + \delta z(s,\omega), u(s,\omega)) - f(s, x(s,u), u(s,\omega))\}\right.$$
$$\left. - f_x(s, x(s,\omega), u(s,\omega))z(s,\omega)\right|^2 ds, \tag{A.12}$$

then $0 \le \beta^*(t) \le \beta^{**}(t)^{1/2} \le \beta^{**}(1)^{1/2} \to 0$ *when* $\delta \to 0$.

Proof. Jensen's inequality yields the inequality $\beta^*(t)^2 \le \beta^{**}(t)$. The remaining arguments are as in the preceding proof. □

Below, on product spaces, maximum norms (= maximum of norms) and maximum metrics are used. In the sequel, the following entities are used:

$$\begin{aligned}&Y \text{ is a normed space, } A \text{ is a complete pseudo-metric space with}\\&\text{pseudo-metric } \rho, \text{ and } a^* \text{ is a given element in } A. \text{ The function } H(a)\\&\text{from } A \text{ into } Y \text{ is continuous.}\end{aligned} \tag{A.13}$$

Theorem D (attainability). *Let the entities in* (A.13) *be given. Let positive numbers* K, $\widehat{\mu}$, μ', μ, $\mu \in (0,1)$ *and an element* $\widehat{z}^*$ *in* Y *be given. Assume that the following properties hold for all* $a \in \operatorname{cl} B(a^*, \widehat{\mu})$*: for all* $\widehat{v} \in Y$ *with* $|\widehat{v} - \widehat{z}^*| = \mu'$*, for all* $r > 0$*, a* $(a', \delta) \in A \times (0, r]$ *exists, such that*

$$\left|H(a') - H(a) - \delta\widehat{v}\right| \le \frac{(1-\mu)\delta\mu'|\widehat{v}|}{(|\widehat{z}^*| + \mu')}, \qquad \rho(a', a) \le \delta K|\widehat{v}|. \tag{A.14}$$

Then, for all $z \in \operatorname{cl} B(H(a^*), \mu\mu'\widehat{\mu}/4K(|\widehat{z}^*|+\mu'))$*, there exists a pair* $(a, \alpha) \in \operatorname{cl} B(a^*, \widehat{\mu}\gamma/2) \times [0, \widehat{\mu}\gamma/2K(|\widehat{z}^*| + \mu')]$*, such that* $z + \alpha\widehat{z}^* = H(a)$*, where* $\gamma := 4K(|\widehat{z}^*| + \mu')|H(a^*) - z|/\mu\mu'\widehat{\mu} \le 1$.

Corollary E. *Assume that* $w := \inf\{|\widehat{v}| : |\widehat{v} - \widehat{z}^*| = \mu'\} > 0$*. Then, in* (A.14)*, evidently* $\rho(a', a) \le \delta K|\widehat{v}|$ *can be replaced by the stronger inequality* $\rho(a', a) \le \delta K w$.

(On the other hand, when $w > 0$, then $\rho(a', a) \le \delta K|\widehat{v}| \Rightarrow \rho(a', a) \le \delta K' w$ for $K' = (|\widehat{z}^*| + \mu')K/w$).

Central ideas in the proof of Theorem D stem from the proof of the multifunction inverse function theorem Theorem 4, page 431, in Aubin and Ekeland [11].

Proof of Theorem D. The property (A.14) also holds for $\widehat{v}$ in the set $B^* := \{\lambda\widetilde{v} : \lambda > 0, \widetilde{v} \in Y, |\widetilde{v} - \widehat{z}^*| = \mu'\}$. To see this, let $\widehat{v}' \in B^*$ and let $r > 0$. Then $\widehat{v}' = \lambda\widehat{v}$ for some $\lambda > 0$, some $\widehat{v}$ such that $|\widehat{v} - \widehat{z}^*| = \mu'$. Now, for all $a \in \operatorname{cl} B(a^*, \widehat{\mu})$, there exists a pair (a', δ), $0 < \delta \le r\lambda$, such that the inequalities in (A.14) hold. From these inequalities, for $\delta' := \delta/\lambda \in (0, r]$, using $\delta'\widehat{v}' = \delta\widehat{v}$, it follows that $|H(a') - H(a) - \delta'\widehat{v}'| \le (1-\mu)\delta'\mu'|\widehat{v}'|/(|\widehat{z}^*| + \mu')$ and $\rho(a', a) \le \delta' K|\widehat{v}'|$. Hence, (A.14) holds for $\widehat{v}' \in B^*$.

Below, write $|\widehat{z}^*| + \mu' =: \kappa$. The following lemma is needed in the proof.

Lemma F. *Let* $z \in \operatorname{cl} B(H(a^*), \mu\mu'\widehat{\mu}/4K\kappa)$*. Assume that the pair* $(a_1, \lambda_1) \in \operatorname{cl} B(a^*, \widehat{\mu}/2) \times [-\widehat{\mu}/2K\kappa, 0]$ *minimizes*

$$(a, \lambda) \longrightarrow |H(a) + \lambda\widehat{z}^* - z| + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\{\rho(a, a_1), |\lambda - \lambda_1|K\kappa\} \tag{A.15}$$

in $\operatorname{cl} B(a^*, \widehat{\mu}) \times [-\widehat{\mu}/K\kappa, 0]$*. Then* $|H(a_1) + \lambda_1\widehat{z}^* - z| = 0$.

Proof of Lemma F. By contradiction, assume $|\hat{z}| > 0$, $\hat{z} := H(a_1) + \lambda_1\hat{z}^* - z$. The vector $\hat{v} := \hat{z}^* - \mu'\hat{z}/|\hat{z}|$ satisfies $|\hat{v} - \hat{z}^*| = \mu'$, so $|\hat{z}|\hat{v} = |\hat{z}|\hat{z}^* - \mu'\hat{z}$ belongs to B^*. Hence, by the extended property (A.14), there exist an $a' \in A$ and a $\delta \le \hat{\mu}/(2K\kappa|\hat{z}|), \delta \in (0, 1/\mu']$, such that

$$\left|H(a') - H(a_1) - \delta\left(|\hat{z}|\hat{z}^* - \mu'\hat{z}\right)\right| \le \frac{(1-\mu)\delta\mu'\left|\left(|\hat{z}|\hat{z}^* - \mu'\hat{z}\right)\right|}{\kappa} \le (1-\mu)\delta\mu'|\hat{z}|. \tag{A.16}$$

Moreover, $\rho(a', a_1) \le \delta K|(|\hat{z}|\hat{z}^* - \mu'\hat{z})| \le \hat{\mu}/2$, (use the first inequality for δ), which implies $a' \in \operatorname{cl} B(a^*, \hat{\mu})$. Define $\lambda' = \lambda_1 - \delta|\hat{z}| \in [-\hat{\mu}/K\kappa, 0]$ $(\delta|\hat{z}| \le \hat{\mu}/2K\kappa)$. Then, using (A.16), $\delta\mu' \le 1$, and the definition of $\hat{z}$, we get

$$\begin{aligned}
\left|H(a') + \lambda'\hat{z}^* - z\right| &= \left|-z + H(a_1) + H(a') - H(a_1) + \lambda_1\hat{z}^* - \delta|\hat{z}|\hat{z}^*\right| \\
&\le \left|-z + H(a_1) + \delta\left(|\hat{z}|\hat{z}^* - \mu'\hat{z}\right) + \lambda_1\hat{z}^* - \delta|\hat{z}|\hat{z}^*\right| + (1-\mu)\delta\mu'|\hat{z}| \\
&= \left|-z + H(a_1) + \lambda_1\hat{z}^* - \delta\mu'\hat{z}\right| + (1-\mu)\delta\mu'|\hat{z}| \\
&= \left|\hat{z} - \delta\mu'\hat{z}\right| + (1-\mu)\delta\mu'|\hat{z}| \le (1-\delta\mu')|\hat{z}| + (1-\mu)\delta\mu'|\hat{z}| = (1-\mu\delta\mu')|\hat{z}|.
\end{aligned} \tag{A.17}$$

Using $|H(a') + \lambda'\hat{z}^* - z| \le (1-\mu\delta\mu')|\hat{z}|$ and $\lambda' - \lambda_1 = -\delta|\hat{z}|$ yields

$$\begin{aligned}
&\left|-z + H(a') + \lambda' z^*\right| + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\left\{\rho(a', a_1), \left|\lambda' - \lambda_1\right|K\kappa\right\} \\
&\quad \le (1-\mu\delta\mu')|\hat{z}| + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\left\{\delta K\left|\left(|\hat{z}|\hat{z}^* - \mu'\hat{z}\right)\right|, \delta|\hat{z}|K\kappa\right\} \\
&\quad \le (1-\mu\delta\mu')|\hat{z}| + \frac{\mu\delta\mu'|\hat{z}|}{2} < |\hat{z}| \\
&\quad = |H(a_1) + \lambda_1\hat{z}^* - z| + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\left\{\rho(a_1, a_1), |\lambda_1 - \lambda_1|K\kappa\right\},
\end{aligned} \tag{A.18}$$

a contradiction of the optimality of (a_1, λ_1).

Continued Proof of the Theorem. Let $z \in \operatorname{cl} B(H(a^*), \mu\mu'\hat{\mu}/4K\kappa)$, let γ be as in the conclusion of the theorem, and let $\phi(a, \lambda) := |H(a) + \lambda\hat{z}^* - z|$. Note that $\phi(a^*, 0) := |H(a^*) - z| \le \gamma\mu\mu'\hat{\mu}/4K\kappa$. Let the distance between (a, λ) and (a'', λ'') be $(\mu\mu'/2K\kappa)\max\{\rho(a, a''), |\lambda - \lambda''|K\kappa\}$ in the complete space $\operatorname{cl} B(a^*, \hat{\mu}) \times [-\hat{\mu}/K\kappa, 0]$. By Aubin and Ekeland ([11, Theorem 1, page 255]) (Ekeland's variational principle), there exists a $(a_1, \lambda_1) \in \operatorname{cl} B(a^*, \hat{\mu}) \times [-\hat{\mu}/K\kappa, 0]$ such that

$$\begin{gathered}
\phi(a_1, \lambda_1) \le \phi(a, \lambda) + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\left\{\rho(a, a_1), |\lambda - \lambda_1|K\kappa\right\} \quad \forall (a, \lambda) \in \operatorname{cl} B(a^*, \hat{\mu}) \times \left[\frac{-\hat{\mu}}{K\kappa}, 0\right], \\
\phi(a_1, \lambda_1) + \left(\frac{\mu\mu'}{2K\kappa}\right)\max\left\{\rho(a_1, a^*), |\lambda_1 - 0|K\kappa\right\} \le \phi(a^*, 0) \le \frac{\mu\mu'\hat{\mu}\gamma}{4K\kappa},
\end{gathered} \tag{A.19}$$

which gives $\rho(a_1, a^*) \leq \widehat{\mu}\gamma/2$, $|\lambda_1| \leq \widehat{\mu}\gamma/2K\kappa$. By Lemma F, $|-z + H(a_1) + \lambda_1\widehat{z}^*| = 0$, so $z + \alpha\widehat{z}^* = H(a_1)$, for $\alpha = -\lambda_1 \in [0, \widehat{\mu}\gamma/2K\kappa]$. □

Below, $q^{a',a}$ is a sort of Gâteaux derivative at a of $H(\cdot)$.

Corollary G. *Let $\widetilde{\mu} > 0$, let Y be a normed space, let A be a complete pseudometric space with metric σ, let a^* be a given element in A, and let $H(a) : A \to Y$ be continuous. Assume also the existence of a function $q^{a',a}$ from $A \times A$ into Y and positive constants $\widetilde{K}$ and $\widetilde{K}'$ such that, for each $a \in \operatorname{cl} B(a^*, \widetilde{\mu})$, for all $r > 0$, all $\varepsilon > 0$, all $a^+ \in A$, there exists a pair (a', δ), $a' \in \operatorname{cl} B(a, \widetilde{K}'\delta)$, $\delta \in (0, r]$ such that*

$$\left|H(a') - H(a) - \delta q^{a^+,a}\right| \leq \varepsilon\widetilde{K}\delta. \tag{A.20}$$

Assume also that for all $a \in A$,

$$\operatorname{co} q^{A,a} \subset \operatorname{cl} q^{A,a}. \tag{A.21}$$

Assume that $a \to q^{\check{a},a}$ is continuous at a^* for any $\check{a} \in A$. Assume finally that b is an interior point in $\operatorname{clco} q^{A,a^*}$, and that, for some $\varepsilon > 0$, some $z^* \in Y$, $B(z^*, \varepsilon) \subset \operatorname{clco} q^{A,a}$ for all $a \in \operatorname{cl} B(a^*, \widetilde{\mu})$. Then, for some $\widehat{\gamma} > 0$ and some $a \in A$, $H(a^*) + \widehat{\gamma}b = H(a)$.

Proof. Write $\widetilde{Q}^a = \operatorname{clco} q^{A,a}$, and let $B(b, \alpha) \subset \widetilde{Q}^{a^*}$ for some $\alpha > 0$. Then, for some $\kappa > 0$, $-\kappa z^* \in B(0, \alpha) \subset \widetilde{Q}^{a^*} - b$. Define $B_z = \operatorname{co}\{z, B(z^*, \varepsilon) + b\}$. Evidently, b is an interior point in B_z if $z = -\kappa z^* + b$. Then b is an interior point in B_z even if $z = -\kappa z^* + b$ is only an approximate equality; in fact there exist positive numbers ρ^* and ξ such that $B(b, \xi) \subset B_z$ for all $z \in \operatorname{cl} B(-\kappa z^* + b, \rho^*)$. Because $-\kappa z^* \in \widetilde{Q}^{a^*} - b$, by (A.21) there exists a $\widetilde{a} \in A$, such that $|b - \kappa z^* - q^{\widetilde{a},a^*}| < \rho^*/2$. By the continuity assumption on $q^{a',a}$ in the corollary, for $\beta > 0$ small enough, $|q^{\widetilde{a},a} - q^{\widetilde{a},a^*}| \leq \rho^*/2$ for $a \in B(a^*, \beta)$. We assume $\beta \leq \widetilde{\mu}$, $\beta \leq \widetilde{K}'/2$. Evidently, $|b - \kappa z^* - q^{\widetilde{a},a}| < \rho^*$. Hence, $q^{\widetilde{a},a} \in B(-\kappa z^* + b, \rho^*)$ for all $a \in B(a^*, \beta)$. Thus, for $a \in \operatorname{cl} B(a^*, \beta)$, $B(b, \xi) \subset B_{q^{\widetilde{a},a}} := \operatorname{co}\{q^{\widetilde{a},a}, \operatorname{cl} B(z^*, \varepsilon) + b\} \subset \operatorname{co}\{q^{\widetilde{a},a}, \widetilde{Q}^a + b\} \subset \widetilde{Q}^a + [0,1]b$, because $q^{\widetilde{a},a} \in \widetilde{Q}^a$ and $\widetilde{Q}^a$ is convex. Hence, $B(0, \xi) \subset \widetilde{Q}^a - [0,1]b$, $a \in \operatorname{cl} B(a^*, \beta)$. It follows that if $\widehat{v} \in Y$, $|\widehat{v}| = \varsigma = \xi/2$, then, for any $a \in \operatorname{cl} B(a^*, \beta)$, by (A.21), for some $a^+ \in A$, $\gamma \in [0,1]$,

$$\left|\widehat{v} - \left(q^{a^+,a} - \gamma b\right)\right| < \frac{\varsigma}{4}. \tag{A.22}$$

By (A.20), for $\varepsilon = \varsigma(1/4\widetilde{K})$, for some $a' \in A$, and some arbitrarily small $\delta \in (0, 1/2)$,

$$\left|H(a') - H(a) - \delta q^{a^+,a}\right| \leq \varepsilon\widetilde{K}\delta = \left(\frac{1}{4}\right)\delta\varsigma, \qquad \sigma(a', a) \leq \widetilde{K}'\delta. \tag{A.23}$$

Now, by (A.22), $|\delta q^{a^+,a} - \delta(\widehat{v} + \gamma b)| \leq \varsigma\delta/4$. Then, by (A.23),

$$\left|H(a') - H(a) - \delta(\widehat{v} + \gamma b)\right| \leq \left(\frac{1}{2}\right)\delta\varsigma = \left(\frac{1}{2}\right)\delta|\widehat{v}|. \tag{A.24}$$

$(a' \in \operatorname{cl} B(a, \widetilde{K}'\delta)$, $\gamma \in [0,1])$. In Theorem D, replace $H(a)$ by $H(a) - \lambda b$, a by (a, λ), a^* by $(a^*, 0)$, and A by $A \times [0,1]$ and let $\widehat{\mu} = \beta\varsigma$, let $\mu' = \varsigma$, let $\widehat{z}^* = 0$, let $\mu = 1/2$, let $K = \widetilde{K}'$, and let $\rho((a'', \lambda''), (a, \lambda)) = \max\{\varsigma\sigma(a'', a), \widetilde{K}'\varsigma|\lambda'' - \lambda|\}$. Then the conditions in Theorem D are satisfied when, in (A.14), (a, a', δ) is replaced by $((a, \lambda), (a', \lambda'), \delta)$, a' as just constructed, $\lambda' = \delta\gamma + \lambda \in [0,1]$ ($\delta < 1/2$, for $(a, \lambda) \in \operatorname{cl} B((a^*, 0), \widehat{\mu})$, $\sigma(a, a^*) \le \beta$, and $\lambda \le 1/2$ as $\widehat{\mu} = \beta\varsigma \le \widetilde{K}'\varsigma/2$). Thus, we get that, for all $\theta > 0$ small enough, $H(a^*) + \theta b = H(a) - \widetilde{\gamma} b$ for some $(a, \widetilde{\gamma}) \in A \times [0,1]$, or $H(a^*) + (\widetilde{\gamma} + \theta) b = H(a)$. □

Remark H. If $q^{a^*, a^*} = 0$, then $\widetilde{\gamma} + \theta > 0$ can be taken to be arbitrary small (b can be replaced by $\check{\beta} b$ for any $\check{\beta} \in (0,1]$).

Corollary I. *Let $\widehat{Y}$ be a normed space with norm $\|\cdot\|$, let $\gamma^* > 0$, and let $\widetilde{A}$ be a complete pseudometric space with metric $\overline{\sigma}^*$. Assume that $\widetilde{a}^*$ is a given element in $\widetilde{A}$, let $\widetilde{H}(\widetilde{a}) = (\widehat{H}(\widetilde{a}), \check{H}(\widetilde{a})) : \widetilde{A} \to \widehat{Y} \times \mathbb{R} =: \widetilde{Y}$ be continuous. Let $\widetilde{q}^{\widetilde{a}', \widetilde{a}} = (\widehat{q}^{\widetilde{a}', \widetilde{a}}, \check{q}^{\widetilde{a}', \widetilde{a}}) \in \widehat{Y} \times \mathbb{R}$, $\widetilde{a}'$, $\widetilde{a} \in \widetilde{A}$, with $\widetilde{q}^{\widetilde{a}, \widetilde{a}} = 0$ for all $\widetilde{a}$. Assume that* (A.20) *and* (A.21) *are satisfied for $(H, q^{a', a}, A, Y, a^*, a, a', a^+, \widetilde{\mu}, \sigma, \widetilde{K}, \widetilde{K}')$ replaced by $(\widetilde{H}, \widetilde{q}^{\widetilde{a}', \widetilde{a}}, \widetilde{A}, \widetilde{Y}, \widetilde{a}^*, \widetilde{a}, \widetilde{a}', \widetilde{a}^+, \gamma^*, \overline{\sigma}^*, \widetilde{K}_*, \widetilde{K}'_*)$ and also that $\widetilde{a} \to \widetilde{q}^{\widetilde{a}', \widetilde{a}}$ is continuous at $\widetilde{a}^*$ for any $\widetilde{a}' \in \widetilde{A}$. Assume, for some given $\widehat{y} \in \widehat{Y}$, that $\check{H}(\widetilde{a}^*) = \max_{\widetilde{a} \in \{\widetilde{a} \in \widetilde{A} : \widehat{H}(\widetilde{a}) = \widehat{y}\}} \check{H}(\widetilde{a})$. Assume also, for some $\varepsilon > 0$, some $\check{z}^* \in \widehat{Y}$, that $B(\check{z}^*, \varepsilon) \subset \operatorname{clco} \widehat{q}^{\widetilde{A}, \widetilde{a}}$ for all $\widetilde{a} \in \{\widetilde{a} \in \widetilde{A}, \overline{\sigma}^*(\widetilde{a}, \widetilde{a}^*) \le \gamma^*\}$. Assume, finally, that $M := \sup_{\widetilde{a} \in B(\widetilde{a}^*, \gamma^*)} \sup_{\widetilde{a}' \in \widetilde{A}} |\check{q}^{\widetilde{a}', \widetilde{a}}| < \infty$. Then, for some continuous non zero linear functional $(\widehat{y}^*, \check{\lambda})$ on $\widehat{Y} \times \mathbb{R}$, $\check{\lambda}$ a number ≥ 0, we have $\langle \widehat{q}^{\widetilde{a}', \widetilde{a}^*}, \widehat{y}^* \rangle + \check{\lambda} \check{q}^{\widetilde{a}', \widetilde{a}^*} \le 0$ for all $\widetilde{a}' \in \widetilde{A}$.*

Proof. Define $z^{**} = (\check{z}^*/2, -2M)$, and $\varepsilon^{**} = \min\{\varepsilon/2, 2M\}$. It is easily seen that the following inclusions hold for all $\widetilde{a} \in B(\widetilde{a}^*, \gamma^*)$: $B(z^{**}, \varepsilon^{**}) \subset S = (1/2)[B(\check{z}^*, \varepsilon) \times \{0\}] + (1/2)[\{0\} \times (-8M, 0)] \subset K^{\widetilde{a}} := \operatorname{clco}\{(\widehat{q}^{\widetilde{a}', \widetilde{a}}, \check{q}^{\widetilde{a}', \widetilde{a}} + \gamma) : \widetilde{a}' \in \widetilde{A}, \gamma \in [-8M, 0]\}$ (if necessary, use the proof of Lemma 11.2 in Seierstad [8]). Assume by contradiction that $(0, \zeta)$ belongs to $\operatorname{int} K^{\widetilde{a}^*}$ for some $\zeta > 0$. Define $A = \widetilde{A} \times [-9M, 0]$, and for $a = (\widetilde{a}, \alpha) \in A$, $a' = (\widetilde{a}', \alpha') \in A$, let $q^{a', a} = (\widehat{q}^{\widetilde{a}', \widetilde{a}}, \check{q}^{\widetilde{a}', \widetilde{a}} + \alpha' - \alpha)$ and $\sigma(a, a') = \max\{\overline{\sigma}^*(\widetilde{a}', \widetilde{a}), |\alpha' - \alpha| \widetilde{K}'_*/9M\}$, and let $H(a) = (\widehat{H}(\widetilde{a}), \check{H}(\widetilde{a}) + \alpha)$, $a = (\widetilde{a}, \alpha)$. Then (A.21) and (A.20) are evidently satisfied (the latter for $\alpha' = \delta\alpha^+ + (1 - \delta)\alpha$ when $a^+ = (\widetilde{a}^+, \alpha^+)$). Obviously, $B(z^{**}, \varepsilon^{**}) \subset \operatorname{clco} q^{A, a}$ for each $a \in B(a^*, \widetilde{\mu})$, $\widetilde{\mu} = \min\{M, \gamma^*\}$, where $a^* = (\widetilde{a}^*, 0)$. Hence, by the preceding corollary, for some $\eta > 0$, $\eta(0, \zeta) + H(a^*) = H(a)$ for some $a = (\widetilde{a}, \alpha) \in \widetilde{A} \times [-9M, 0]$. Hence, $\widehat{H}(\widetilde{a}) = \widehat{H}(\widetilde{a}^*) = \widehat{y}$, $\check{H}(\widetilde{a}) + \alpha = \eta\zeta + \check{H}(\widetilde{a}^*)$, or $\check{H}(\widetilde{a}) = \check{H}(\widetilde{a}^*) + \eta\zeta - \alpha > \check{H}(\widetilde{a}^*)$, contradicting optimality. Thus the set $L = \{(0, \zeta) : \zeta > 0\}$ is disjoint from $\operatorname{int} K^{\widetilde{a}^*}$, so the convex set L can be separated from the convex set $\operatorname{int} K^{\widetilde{a}^*}$ by a nonzero continuous linear functional $y^* = (\widehat{y}^*, \check{\lambda})$ such that $\langle K^{\widetilde{a}^*}, y^* \rangle \le 0 \le \langle L, y^* \rangle$, $(0 \in K^{\widetilde{a}^*})$, which implies $\check{\lambda} \ge 0$. □

Remark J. [a nonzero continuous linear functional on L^2 vanishing on all $L_2(\Omega, \Phi_t, \mathbb{R})$, $t < T$]. Let $T = 1$, Φ_t be the natural filtration corresponding to some given B_t. Choose a $v^* \in L^2$ such that $2^j |\Pi_j v^*|_2 = j/(j+1)$. Then ${}_2|v^*| = 1$, so v^* belongs to the ${}_2|\cdot|$-boundary of the ${}_2|\cdot|$-ball ${}_2B(0,1)$. Then for some nonzero continuous linear functional μ on L^2, $\langle \operatorname{cl} {}_2B(0,1), \mu \rangle \le \langle v^*, \mu \rangle$. Let k be any given integer. If $\phi \in L_2(\Phi_{1-1/2^k}, \Omega)$, $|\phi|_2 \le 2^{-(k+1)}/(j+1)$, then $2^j |\Pi_j(\pm\phi)|_2 \le 2^j 2 \cdot 2^{-(k+1)}/(j+1) \le 1/(j+1)$ for $j \le k$ and $2^j |\Pi_j(\pm\phi)|_2 = 0$ for $j > k$, so, $2^j |\Pi_j(v^* \pm \phi)|_2 \le 1$ ($\Rightarrow v^* \pm \phi \in \operatorname{cl} {}_2B(0,1)$). Then the inequality involving μ yields $\langle \pm\phi, \mu \rangle \le 0$, that is, μ vanishes on $L_2(\Omega, \Phi_t, \mathbb{R})$, $t < 1$. To show in detail that such a v^* exists, let $v_i = 1_{M_i} - 1_{\complement M_i}$, and let $M_i = [B_{1-1/2^i} - B_{1-1/2^{i-1}} \in [0, \infty)]$. Then $E[v_i \mid \Phi_{1-1/2^{i-1}}] = P[M_i \mid \Phi_{1-1/2^{i-1}}] - P[\complement M_i \mid \Phi_{1-1/2^{i-1}}] = 0$, so for $j < i$, $\Pi_j v_i = E[v_i \mid \Phi_{1-1/2^j}] - E[v_i \mid \Phi_{1-1/2^{j-1}}] = 0$, and for $j > i$, $\Pi_j v_i = E[v_i \mid \Phi_{1-12^j}] - E[v_i \mid$

$\Phi_{1-1/2^{j-1}}] = v_i - v_i = 0$. Letting $v^* = \sum_{i=1}^{\infty}(1/2^i)(i/(i+1))v_i$, we get $2^j\Pi_j v^* = \Pi_j(j/(j+1))v_j = (j/(j+1))E[1_{M_j} - 1_{\complement M_j} \mid \Phi_{1-1/2^j}] - E[1_{M_j} - 1_{\complement M_j} \mid \Phi_{1-1/2^{j-1}}] = (j/(j+1))(1_{M_j} - 1_{\complement M_j}) = (j/(j+1))v_j$, so $2^j|\Pi_j v^*|_2 = (j/(j+1))|v_j|_2 = j/(j+1)$.

Proof of Remark 3.4. Let $T = 1$, and let $x_0 = 0$. Corollary G will be applied. Let $Y = L^2$, let $\widetilde{\mu} = \mu^*$ (for μ^*, see (3.88)), let $A = U'$, let $\sigma = \sigma^*$, let the norm $|\cdot|$ on Y be equal to ${}_2|\cdot|$, let $u' = a'$, let $u^* = a^*$, let $u = a$, let $a^+ = u''$, let $H(a) = \pi x^u(1)$, let $q^{a',a} = \pi q^{u',u}(1,\cdot)$, let $\widetilde{K} = \widetilde{K}' = 1$, and let $b = \int_J z(t,\cdot)dt$. Recall that (3.88) says that ${}_2B(z^*,\varepsilon) \subset \operatorname{clco} q^{A,a} = \operatorname{clco} \pi q^{U',u}$ for $z^* = \int_J 1_{[c',1]}\check{z}(s,\cdot)ds$, $\varepsilon = (1-c')\alpha/8$. By (3.58), it follows that the property (A.20) is satisfied, and by (3.43), it follows that (A.21) is satisfied. By (3.33) H is continuous, and by (3.73), $a \to q^{a',a}$ is continuous at $a^* = u^*$. Let $B(z(\cdot,,\cdot),\widetilde{\varepsilon}) \subset \operatorname{clco} \pi\dot{q}^{U',u^*}$ (the ball and cl corresponding to $\|\cdot\|_2$). For any $\widetilde{b} \in {}_2B(0,\widetilde{\varepsilon}/2)$, it was shown earlier that there exists a $\widetilde{z} \in L_2^{\text{prog}}(J\times\Omega,Y)$ such that $\widetilde{b} = \int_J z(t,\cdot)dt$, $\|z(\cdot,\cdot)\|_2 \le 2\cdot{}_2|\widetilde{b}| < \widetilde{\varepsilon}$, so ${}_2B(b,\widetilde{\varepsilon}/2) \subset \operatorname{clco} \pi q^{U',u^*}(1)$ (cl corresponding to ${}_2|\cdot|$). Hence all conditions in Corollary G are satisfied and the conclusion in Remark 3.4 follows. □

Proof of Remark 2.3. Let $T = 1$, and let $x_0 = 0$. Note that if $y(\cdot) \in L_2(\Omega,\Phi_{1-1/2^k},\mathbb{R}^{n^*})$, then $\Pi_{k'}y(\cdot) = 0$, for $k' > k$, and hence, $|y(\cdot)|_2 = |\sum_{i=1}^{\infty}\Pi_i y(\cdot)|_2 \le \sum_{i=1}^{k}|\Pi_i y(\cdot)|_2 \le \sum_{i=1}^{k}2^i|\Pi_i y(\cdot)|_2 \le k\cdot{}_2|y(\cdot)|$. On the other hand ${}_2|y(\cdot)| = \sup_i 2^i|\Pi_i y(\cdot)|_2 \le 2^k 2\cdot|y(\cdot)|_2$. Hence, on $L_2(\Omega,\Phi_{1-1/2^k},\mathbb{R}^{n^*})$, the norms $|\cdot|_2$ and ${}_2|\cdot|$ are equivalent. Thus, the spaces $L_2(\Omega,\Phi_t,\mathbb{R}^{n^*})$, $t < 1$, are subspaces of L^2. For $y = y(\cdot) \in L_2(\Omega,\Phi_s,\mathbb{R}^{n^*})$, define

$$\begin{aligned} q^y(t,s,\omega) = y(\omega) &+ \int_s^t f_x(\check{s},x^*(\check{s},\omega),u^*(\check{s},\omega))q^y(\check{s},\omega)d\check{s} \\ &+ \int_s^t \overline{\sigma}_x(\check{s},x^*(\check{s},\omega))q^y(\check{s},\omega)dB_{\check{s}}. \end{aligned} \tag{A.25}$$

Then an application of Lemma A in the appendix, similar to the one yielding (3.27) gives that $\sup_{t,s,t\ge s}|q^y(t,s,\cdot)|_2 \le D|y(\cdot)|_2$ for some constant D independent of $y(\cdot)$. Let $y(\cdot) \in L_2(\Omega,\Phi_{1-1/2^k},\mathbb{R}^{n^*})$ for some k. Then ${}^2|1_{[s,1]}(\cdot)\partial\pi q^y(\cdot,s,\cdot)/\partial t| \le \sup_i 2^i|\int_{I_i}1_{[s,1]}(t)\pi f_x(t,x^*(t,\omega),u^*(t,\omega))q^y(t,s,\omega)dt|_2 \le DM^+|y(\cdot)|_2$. Because $|\Theta| \le 8$ for ${}^2|\cdot| \to {}_2|\cdot|$, see (3.84),

$$\begin{aligned} {}_2\left|\pi q^y(1,s,\cdot) - \pi y(\cdot)\right| &= {}_2\left|\int_0^1\left[1_{[s,1]}(t)\cdot\frac{\partial\pi q^y(t,s,\cdot)}{\partial t}\right]dt\right| \\ &\le 8\cdot{}^2\left|1_{[s,1]}\cdot\frac{\partial\pi q^y(\cdot,s,\cdot)}{\partial t}\right| \le 8DM^+\left|y(\cdot)\right|_2. \end{aligned} \tag{A.26}$$

Now, $\pi C(1,s,\cdot)y(\cdot) = \pi q^y(1,s,\cdot)$, and for some constant γ, $|\langle z(\cdot),\nu\rangle| \le \gamma\cdot{}_2|z(\cdot)|$ for $z(\cdot) \in L^2$, so $|\langle \pi C(1,s,\cdot)y(\cdot) - \pi y(\cdot),\nu\rangle| \le \gamma\cdot{}_2|\pi q^y(1,s,\cdot) - \pi y(\cdot)| \le 8\gamma DM^+|y(\cdot)|_2$ and $|\langle \pi C(1,s,\cdot)y(\cdot),\nu\rangle| \le 8\gamma DM^+|y(\cdot)|_2 + \gamma 2^{k+1}|y(\cdot)|_2$. Hence, for any given $t < 1$, by the $|\cdot|_2$-continuity of $y(\cdot) \to \langle \pi C(1,t,\cdot)y(\cdot),\nu\rangle$ on $L_2(\Omega,\Phi_t,\mathbb{R}^{n^*})$, $t < 1$, see (A.26), and hence $y(\cdot) \to \langle y(\cdot),C(1,t,\cdot)^*\nu_*\rangle$, there exists an $L_2(\Omega,\Phi_t,\mathbb{R}^{n^*})$-function $p^-(t,\omega)$ on Ω such that for any $L_2(\Omega,\Phi_t,\mathbb{R}^{n^*})$-function $\alpha(\omega)$, we have $\langle\alpha(\cdot),C(1,t,\cdot)^*\nu_*\rangle = \int_\Omega\langle\alpha(\omega),p^-(t,\omega)\rangle dP(\omega)$. In fact, the last equality yields that $p^-(t,\omega) = E[C(1,t,\cdot)^*v_* \mid \Phi_t]$.

Let $u(\cdot,\cdot)$ be any given element in U' and let $\beta(t,\omega) = f(t,x^*(t,\omega),u(t,\omega)) - f(t,x^*(t,\omega),u^*(t,\omega))$. Moreover, let $b \in (0,1)$ be an arbitrarily given Lebesgue point of $t \to \beta(t,\cdot) : J \to L_2(\Omega,\Phi,\mathbb{R}^n)$. Then

$$\left|\delta^{-1}\left[\int_b^{b+\delta} C(1,b,\omega)C(b,t,\omega)\beta(t,\omega)dt - \delta C(1,b,\omega)\beta(b,\omega)\right]\right|_2 \longrightarrow 0, \tag{A.27}$$

when $\delta \to 0$. (Here L_2-continuity of $z(\omega) \to C(1,b,\omega)z(\omega)$ on $L_2(\Omega,\Phi_b,\mathbb{R}^{n^*})$ and of $t \to C(b,t,\omega)z(\omega)$, uniformly in $z(\omega)$, $|z(\cdot)|_2 \le 1$ is used.) In fact, b needs only be a Lebesgue point from the right. Replacing $u(\cdot,\cdot)$ by $u(\cdot,\cdot)1_{[b,b+\delta]} + u^*(\cdot,\cdot)(1-1_{[b,b+\delta]})$ in (2.6), we get

$$0 \ge \left\langle \delta^{-1}\int_b^{b+\delta} C(b,t,\omega)\beta(t,\omega)dt, C(1,b,\omega)^* v_* \right\rangle \longrightarrow \int_\Omega \langle \beta(b,\omega), p^-(b,\omega)\rangle dP(\omega). \tag{A.28}$$

(Here L_2-convergence when $t \downarrow b$ of $C(b,t,\omega)\beta(t,\omega)$ to $\beta(b,\omega)$ and the L_2-representation of $C(1,b,\omega)^* v_*|_{L_2(\Omega,\Phi_b,\mathbb{R}^\times)}$ is used). Now, if b is a right Lebesgue point of $t \to \beta(t,\cdot)$, then b is a right Lebesgue point of $t \to \beta(t,\cdot)1_C 1_{[b,1]}$, for any $C \in \Phi_b$. So $\int_\Omega \langle 1_C\beta(b,\omega), p^-(b,\omega)\rangle dP(\omega) \le 0$, for any $C \in \Phi_b$. Hence, for a.e. $b < 1$, a.s., we get $0 \ge \langle \beta(b,\omega), p^-(b,\omega)\rangle$. From this the property (2.12) follows.

Now, let Φ_t be the natural filtration generated by B_t, and let $b \in [0,1)$. Then, consider the pair of equations

$$\begin{aligned} dp(t,\omega) &= -p(t,\omega)f_x(t,x^*(t,\omega),u^*(t,\omega))dt \\ &\quad - \sum_j \sigma_x^j(t,x^*(t,\omega))q^j(t)dt + \sum_j q^j(t)dB_t^j, \\ p(b,\omega) &= p^-(b,\omega). \end{aligned} \tag{A.29}$$

By Theorem 2.2, page 349 in Yong and Zhou [4], there is a unique progressively measurable collection $p(t,\omega)$, $q^j(t,\omega)$, $p(t,\omega)$ continuous in t, satisfying these equations, $|p(\cdot,\cdot)|_2 < \infty$, $|q^j(\cdot,\cdot)|_2 < \infty$, and (by (2.20) in the proof of this theorem) for all $t \le b$, $p(t,\omega)$ equals $p^-(t,\omega)P$-a.s. The uniqueness in particular says that if two pairs (p,q), $(q = q^1,\ldots,q^{n'})$, and $(\hat{p},\hat{q})$ satisfy the pair of equations, then $\Pr[p(t,\omega) = \hat{p}(t,\omega)$ for all $t \in [0,b] = 1$ and $q(t,\omega) = \hat{q}(t,\omega)$ for a.e. $t \in [0,b]] = 1$. We can let $b = b_k \to 1$ when $k \to \infty$, (b_k, $k = 1,2,\ldots$, increasing) and obtain functions $p_k(t,\omega)$, $q_k(t,\Omega)$ defined on $[0,b_k]$. For $k' < k$, by the fact that $p_{b_{k'}}(b_{k'},\omega) = p^-(b_{k'},\omega) = p_{b_k}(b_{k'},\omega)$ a.s. and uniqueness, we have that $(p_{k'}(t,\omega),q_{k'}(t,\omega)) = (p_k(t,\omega),q_k(t,\omega))$ on $[0,k']$ in the sense just stated. Then, evidently, there exists a unique pair $(p(t,\omega),q(t,\omega))$ on $[0,1)$ satisfying (A.29), with $p(t,\omega)$ a.s. equal to $E[C(1,t,\cdot)^* v_* \mid \Phi_t]$ for any $t \in [0,1)$.

Let $\hat{\pi} = (x_1,\ldots,x_{n^*}) \to (x_{m^*+1},\ldots,x_{n^*})$ and let b_k, $k = 1,2,\ldots$, be an increasing sequence with $\lim_k b_k = 1$. Assume that v is $L_2(\Omega,\Phi,\mathbb{R}^n)$-continuous. For $\phi \in L_2(\Omega,\Phi,\mathbb{R}^n)$, it is easily seen, using the appendix, Lemma A, that $C(1,b_k,\cdot)\phi = q^\phi(1,b_k,\cdot) \to C(1,1,\cdot)\phi = \phi$ in L_2, uniformly in ϕ, $|\phi|_2 \le 1$. Hence, $\langle \phi, p^-(b_k,\cdot)\rangle = \langle C(1,b_k,\cdot)\phi, v_*\rangle \to \langle \phi, v_*\rangle$, uniformly in ϕ, $|\phi|_2 \le 1$, and thus $p^-(b_k,\cdot) \to \tilde{v}_*(\cdot)$ in L_2, where $\tilde{v}_*(\omega)$ is the L_2-function representing v_*. □

Remark K. Let us imagine that some of the states x_i, $i > m^*$ are required to be softly constrained; that is, $Ex_i(T) = \overline{x}_i$ for $\overline{x}_i$ given, $i \in I^* \subset \{m^* + 1, \ldots, n^*\}$. Then Theorem 2.1 would hold for $\Lambda_0 a$ in (2.11) replaced by $\Lambda_0 a + b$, with b being some vector in $\mathbb{R}^{n^*}$ for which $b_i = 0$ for $i \notin I^*$, with $(\Lambda_0, b, \nu) \neq 0$. To obtain this result, only a slight modification of the proof is needed (all approximation tools needed are worked out, what is needed is a change in the separation argument).

Remark L. Assume that $\overline{\sigma} = \overline{\sigma}(t, x, u)$. Then, a least when first and second order derivatives of $\overline{\sigma}$ and f with respect to x and u exist and are continuous and bounded and U' is a closed convex subset of $\{u \in L_2^{\text{prog}}(J \times \Omega, \mathbb{R}^{k^*}) : \|u(\cdot,\cdot)\|_2 < \infty\}$ for some k^*, the following necessary condition, based on weak variations, holds: for some linear functional ν on B_∞, bounded on B_1 and some number $\Lambda_0 \geq 0$, for all $w(\cdot,\cdot) \in U' - u^*$,

$$\langle \pi q^w(T), \nu \rangle + \Lambda_0 E\langle q^w(T), a \rangle \leq 0, \tag{A.30}$$

where q^w is the solution of

$$\begin{aligned} dq^w(t,\omega) &= f_x(t, x^*(t,\omega), u^*(t,\omega)) q^w(t,\omega) dt + f_u(t, x^*(t,\omega), u^*(t,\omega)) w(t,\omega) dt \\ &\quad + \sum_j \left[\overline{\sigma}_x^j(t, x^*(t,\omega), u^*(t,\omega)) q^w(t,\omega) + \overline{\sigma}_u^j(t, x^*(t,\omega), u^*(t,\omega)) w(t,\omega) \right] dB_t^j. \end{aligned} \tag{A.31}$$

Moreover, $(\Lambda_0, \nu) \neq 0$.

We then need the linear controllability condition: for some $\check{z}(\cdot,\cdot) \in B^\infty$, for some $\alpha > 0$, some $c \in [0, T)$,

$$1_{[c,T]} \check{z}(\cdot,\cdot) + B^\alpha \subset \left\{ 1_{[c,T]} \left[\pi f_u(\cdot, x^*(\cdot,\cdot), u^*(\cdot,\cdot)) w(\cdot,\cdot) \right] : w(\cdot,\cdot) \in U' - u^* \right\}. \tag{A.32}$$

A moderate modification of the above proof works in this case. That, however, is another story.

Remark M. If $h(x,\omega) : X \times \Omega \to X'$ (X, X' Euclidean spaces) is continuous in x and Φ-measurable in ω, $\sup_x |h(x,\omega)| \leq \alpha(\omega)$, $\alpha(\cdot) \in L_2(\Omega, \Phi, R)$, and $x_n(\omega) \to x(\omega)$ in P-measure, ($x_n(\omega)$ and $x(\omega)$ Φ-measurable), then $h(x_n(\cdot), \omega) \to h(x(\omega), \omega)$ in L_2. This result, which is a special case of Krasnoselskii's theorem (see page 20 in Aubin and Ekeland [11]), can be proved as follows. By contradiction, assume for some $\varepsilon > 0$ and for some subsequence n_j that $|h(x_{n_j}(\omega), \omega) - h(x(\omega), \omega)|_2 > \varepsilon$ for all j. A subsequence $x_{n_{j_k}}(\omega) =: x^k(\omega)$ converges a.s. to $x(\omega)$. Then, by continuity, $h(x^k(\omega), \omega) \to h(x(\omega), \omega)$ a.s. and even in L_2, by Lebesgue is dominated convergence theorem. A contradiction has been obtained.

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