International Scholarly Research Network
ISRN Algebra
Volume 2012, Article ID 782953, 8 pages
doi:10.5402/2012/782953

Research Article

Amenability of the Restricted Fourier Algebras

Massoud Amini[1] and Marzieh Shams Yousefi[2]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran*
[2] *Department of Pure Mathematics, Faculty of Mathematical Science, University of Guilan, Rasht 1841, Iran*

Correspondence should be addressed to Massoud Amini, mamini@modares.ac.ir

Received 7 March 2012; Accepted 2 May 2012

Academic Editors: V. K. Dobrev, K. Fujii, M. Ladra, and M. Przybylska

We discuss amenability of the restricted Fourier-Stieltjes algebras on inverse semigroups. We show that, for an E-unitary inverse semigroup, amenability of the restricted Fourier-Stieltjes algebra is related to the amenability of an associated Banach algebra on a Fell bundle.

1. Introduction and Preliminaries

An inverse semigroup S is a discrete semigroup such that, for each $s \in S$, there is a unique element $s^* \in S$ such that

$$ss^*s = s, \qquad s^*ss^* = s^*. \tag{1.1}$$

One can show that $s \mapsto s^*$ is an involution on S [1]. Set E consisting of idempotents of S, elements of the form ss^*, $s \in S$, is a commutative sub-semigroup of S [1]. There is a natural order $\leq$ on E defined by $e \leq f$ if and only if $ef = e$. The semigroup algebra

$$\ell^1(S) = \left\{ f : S \longrightarrow \mathbb{C} : \sum_{s\in S} |f(s)| < \infty \right\} \tag{1.2}$$

is a Banach algebra under convolution

$$f * g(x) = \sum_{st=x} f(s)g(t) \tag{1.3}$$

and norm $\|f\|_1 = \sum_{s\in S}|f(s)|$. Although $\ell^1(S)$ has some common features with the group algebra $\ell^1(G)$, there are certain technical difficulties when one tries to do things on inverse semigroups similar to the group case, and some well-known properties of the group algebra $\ell^1(G)$ break down for inverse semigroups. For instance, $\ell^1(G)$ is a Banach $*$-algebra under the canonical involution $\tilde{f}(x) = \overline{f(x^{-1})}$. This is important when one constructs the enveloping C^*-algebra of $\ell^1(G)$ or studies the automatic continuity properties of characters and homomorphisms. For an inverse semigroup S, the natural involution on $\ell^1(S)$ is $\tilde{f}(x) = \overline{f(x^*)}$. This is an isometry on $\ell^1(S)$ but does not satisfy $(f * g)^{\sim} = \tilde{g} * \tilde{f}$. On the other hand, unlike the group case, $\ell^1(S)$ does not have a bounded approximate identity (this happens when E fails to satisfy the condition (D_k) of Duncan and Namioka [2] for any positive integer k, a Brandt semigroup on an infinite index set is a concrete example). Finally, the left regular representation λ of an inverse semigroup loses its connection with positive definite functions. This is because the crucial equality

$$\langle \lambda(x^*)\xi, \eta\rangle = \langle \xi, \lambda(x)\eta\rangle \tag{1.4}$$

for $x \in S$ and $\xi, \eta \in \ell^2(S)$ fails in general. This makes it difficult to study positive definite functions [3] and Fourier (Fourier-Stieltjes) algebras on semigroups [4].

The first author and Medghalchi introduced and studied the notion of restricted semigroup algebra in [5, 6] to overcome such difficulties. They showed that, if the convolution product on $\ell^1(S)$ is appropriately modified, one gets a Banach $*$-algebra $\ell^1_r(S)$, called the restricted semigroup algebra with an approximate identity (not necessarily bounded). In the new convolution product, positive definite functions fit naturally with a restricted version of the left regular representation λ_r. The idea is that one requires the homomorphism property of representations to hold only for those pairs of elements in the semigroup whose range and domain match. This is quite similar to what is done in the context of groupoids, but the representation theory of groupoids is much more involved [6].

The basic idea of the restricted semigroup algebra is to consider the associated groupoid of an inverse semigroup S. Given $x, y \in S$, the restricted product of x, y is xy if $x^*x = yy^*$, and undefined, otherwise. The set S with its restricted product forms a groupoid, which is called the associated groupoid of S and is denoted by S_a. If we adjoin a zero element 0 to this groupoid, and put $0^* = 0$, we get an inverse semigroup S_r with the multiplication rule

$$x \bullet y = \begin{cases} xy, & \text{if } x^*x = yy^*, \\ 0, & \text{otherwise,} \end{cases} \tag{1.5}$$

for $x, y \in S \cup \{0\}$, which is called the restricted semigroup of S. A restricted representation $\{\pi, \mathcal{H}_\pi\}$ of S is a pair consisting of a Hilbert space $\mathcal{H}_\pi$ and a map $\pi : S \to \mathcal{B}(\mathcal{H}_\pi)$ into the algebra $\mathcal{B}(\mathcal{H}_\pi)$ of bounded operators on $\mathcal{H}_\pi$ such that $\pi(x^*) = \pi(x)^*$ for $x \in S$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy), & \text{if } x^*x = yy^*, \\ 0, & \text{otherwise,} \end{cases} \tag{1.6}$$

for $x, y \in S$. Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| \leq 1$. It is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$ consists of all $\pi \in \Sigma(S_r)$ with

$\pi(0) = 0$ [5]. One of the central concepts in the analytic theory of inverse semigroup is the left regular representation $\lambda : S \to \mathcal{B}(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y), & \text{if } xx^* \geq yy^*, \\ 0, & \text{otherwise}, \end{cases} \tag{1.7}$$

for $\xi \in \ell^2(S), x, y \in S$. The restricted left regular representation $\lambda_r : S \to \mathcal{B}(\ell^2(S))$ is defined in [5] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y), & \text{if } xx^* = yy^*, \\ 0, & \text{otherwise}, \end{cases} \tag{1.8}$$

for $\xi \in \ell^2(S), x, y \in S$. The main objective of [5] is to change the convolution product on the semigroup algebra to restore the relation between positive definite functions and left regular representation [6].

Throughout this paper, S is an inverse semigroup. For each $f, g \in \ell^1(S)$, define

$$(f \bullet g)(x) = \sum_{x^*x=yy^*} f(xy)g(y^*) \tag{1.9}$$

and $\widetilde{f}(x) = \overline{f(x^*)}$, for $x \in S$. Then, $\ell_r^1(S) := (\ell^1(S), \bullet, \tilde{\ })$ is a Banach $*$-algebra with an approximate identity [5]. The restricted left regular representation λ_r lifts to a faithful representation $\widetilde{\lambda}$ of $\ell_r^1(S)$. We call the completion $C^*_{\lambda_r}(S)$ of $\ell_r^1(S)$ in the C^*-norm $\|\cdot\|_{\lambda_r} := \|\widetilde{\lambda}_r(\cdot)\|$ the restricted reduced C^*-algebra of S and its completion $C^*_r(S)$ in the C^*-norm $\|\cdot\|_{\Sigma_r} := \sup\{\|\widetilde{\pi}(\cdot)\|, \pi \in \Sigma(S_r)\}$ the restricted full C^*-algebra of S. The dual space of the C^*-algebra $C^*_r(S)$ is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by $B_{r,e}(S)$. The closure of the set of finitely support functions in $B_{r,e}(S)$ is called the restricted Fourier algebra and is denoted by $A_{r,e}(S)$ [6].

In this paper, we discuss the amenability of the restricted Fourier and Fourier-Stieltjes algebras on inverse semigroups. We show that, for an E-unitary inverse semigroup, the restricted Fourier algebra is amenable if and only if its maximal homomorphic group image is abelian by finite (i.e., it has an abelian subgroup of finite index). We refer the readers to [5, 6] for more details about restricted semigroup algebra, restricted semigroup C^*-algebra, restricted positive definite functions, and restricted Fourier and Fourier-Stieltjes algebras.

A bounded complex valued function $u : S \to \mathbb{C}$ is called positive definite if for all positive integers n and all $c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in S$ we have

$$\sum \sum_{i,j}^{n} \overline{c_i} c_j u(x_i^* x_j) \geq 0, \tag{1.10}$$

and it is called restricted positive definite if for all positive integers n and all $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in S$ we have

$$\sum \sum_{i,j}^{n} \overline{c_i} c_j (\lambda_r(x_i)u)(x_j) \geq 0. \tag{1.11}$$

The sets of all positive definite and restricted positive definite functions on S are denoted by $P(S)$ and $P_r(S)$, respectively. Positive definite functions are usually considered on unital semigroups. Of course, one can always adjoin a unit 1 to an inverse semigroup T with $1^* = 1$. But extending a positive definite function on T to one on $T^1 = T \cup \{1\}$ is not always possible. We denote all extendable restricted positive definite functions by $P_{r,e}(S)$ which are exactly those $u \in P_r(S)$ such that $\tilde{u} = u$, and there exists a constant $c > 0$ such that for all $n \geq 1$, $x_1, \ldots, x_n \in S$, and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\left|\sum_{i=1}^{n} c_i u(x_i)\right|^2 \leq c \sum_{x_i x_i^* = x_j x_j^*} \overline{c_i} c_j u(x_i^* x_j). \tag{1.12}$$

Then, $P_{r,e}(S) \cong \ell_r^1(S)_+^*$ and $B_{r,e}(S)$ is the linear span of $P_{r,e}(S)$ [5]. Since the restricted Fourier-Stieltjes algebra is the dual space of the restricted semigroup C^*-algebra, it is an ordered Banach algebra in the sense of [7], where the order structure comes from the set of extendable restricted positive definite functions as the positive cone. The same applies to the restricted Fourier algebra.

For each inverse semigroup S, the states on the $*$-algebra $\mathbb{C}S$ (the vector space over S spanned by S with convolution and involution comes from S) are defined by Milan in [8]. A state on a $*$-algebra $\mathcal{A}$ is a positive linear map $\rho : \mathcal{A} \to \mathbb{C}$ such that

$$\sup\left\{|\rho(a)|^2 : a \in \mathcal{A}; \rho(a^* a) \leq 1\right\} = 1. \tag{1.13}$$

If $\mathcal{S}(\mathcal{A})$ is the set of states on $\mathcal{A}$, we know that

$$\|f\|_{C^*(S)} = \sup\left\{\rho(f^* f)^{1/2} : \rho \in \mathcal{S}(C^*(S))\right\} = \sup\left\{\rho(f^* f)^{1/2} : \rho \in \mathcal{S}(\mathbb{C}S)\right\} \tag{1.14}$$

for each f in $\mathbb{C}S$ [8]. When S is a (discrete) group, all these concepts are already discussed by Eymard in [9]. The amenability results for Fourier and Fourier-Stieltjes algebras on groups are surveyed in [10].

2. Amenability and Restricted Weak Containment Property

Working with an inverse semigroup, it is quite natural to go to the maximal group homomorphic image. However, when one deals with an inverse semigroup with zero such as S_r, some modification is necessary. This is because the maximum group homomorphic image of S_r is trivial. To remedy this, we work with maps which are not quite homomorphism. This is the idea of Milan in [8, Section 4].

Definition 2.1. Let S be an inverse semigroup with zero, a grading of S by the group G is a map $\varphi : S \to G \cup \{0\}$ such that $\varphi^{-1}(0) = \{0\}$ and $\varphi(ab) = \varphi(a)\varphi(b)$ provided that $ab \neq 0$.

A Fell bundle over a discrete group G is a collection of closed subspaces $\mathbb{B} = \{B_g\}_{g \in G}$ of a C^*-algebra B, satisfying $B_g{}^* = B_{g^{-1}}$ and $B_g B_h = B_{gh}$ for all g and h in G. The ℓ_1 cross-sectional algebra $\ell_1(\mathbb{B})$ of $\mathbb{B}$ is the Banach $*$-algebra consisting of the ℓ_1 cross-sections of $\mathbb{B}$ under the canonical multiplication, involution, and norm, and the cross-sectional C^*-algebra

$C^*(\mathbb{B})$ of $\mathbb{B}$ is the enveloping C^*-algebra of $\ell_1(\mathbb{B})$ [11]. We also denote the dual space of $C^*(\mathbb{B})$ by $B(\mathbb{B})$.

Next, let us define the Fell bundle arising from a grading φ. For an inverse semigroup S with zero, the algebras $C_0^*(S)$ and $\mathbb{C}_0S$ are the quotients of the algebras $C^*(S)$ and $\mathbb{C}S$ by the (closed) ideal generated by the zero of S. For each $g \in G$, let

$$A_g = \operatorname{span}\{s : \varphi(s) = g\} \text{ inside } \mathbb{C}_0S, \qquad B_g = \overline{A_g} \text{ inside } C_0^*(S). \tag{2.1}$$

By [8, Proposition 3.3], the collection $\mathbb{B} = \{B_g\}_g \in G$ is a Fell bundle for $C_0^*(S)$ and representations of $\mathbb{B}$ are in one-one correspondence with representations of $C_0^*(S)$, and hence $C^*(\mathbb{B})$ is isomorphic to $C_0^*(S)$.

Since S_r is an inverse semigroup with zero, the grading map technique applies to S_r. Let S be an inverse semigroup and G its maximal group homomorphic image with $\varphi : S \to G$, we define the following new product on $G \cup \{0\}$. Put $\varphi(x) \bullet \varphi(y) = \varphi(x \bullet y)$ and $\varphi(s)^* := \varphi(s^*)$. It is easy to see that, with this new multiplication, $G \cup \{0\}$ is an inverse semigroup, which is denoted by G^0.

Now, there is a homomorphism $\varphi_r : S_r \to G^0$, $\varphi_r(s) = \varphi(s)$, $s \in S$, and $\varphi_r(0) = 0$ that induces the natural map $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ defined by

$$\theta\left(\sum_{s\in S_r} \alpha_s s\right) = \sum_{s\in S_r} \alpha_s \varphi_r(s). \tag{2.2}$$

Proposition 2.2. *With the above notation, $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ is a positive map.*

Proof. Let $f = \sum \alpha_s \delta_s$ be a typical element of $\mathbb{C}S_r$. It is enough to show that $\theta(f \bullet f^*)$ is positive for each $f \in \mathbb{C}S_r$. Observe that

$$f \bullet f^* = \left(\sum \alpha_s \delta_s\right) \bullet \left(\sum \alpha_s \delta_s\right)^* = \sum \alpha_s \overline{\alpha_t} \delta_s \bullet \delta_{t^*} = \sum \alpha_s \overline{\alpha_t} \delta_{s\bullet t^*}. \tag{2.3}$$

Hence, we have $\theta(f \bullet f^*) = \sum_{s,t} \alpha_s \overline{\alpha_t} \delta_{\varphi(s)} \bullet \delta_{\varphi(t)^*}$, which is a positive element of $\mathbb{C}G^0$. □

Proposition 2.3. *For each $a \in \mathbb{C}S_r$, $\|\theta a\|_{C^*(G^0)} \le \|a\|_{C^*(S_r)}$.*

Proof. It is enough to check the relation between states on these spaces. It is easy to check that, for each $\rho \in \mathcal{S}(\mathbb{C}G^0)$, $\rho \circ \theta \in \mathcal{S}(\mathbb{C}S_r)$. Indeed, by the previous proposition, $\rho \circ \theta$ is positive, since, for each $a \in \mathbb{C}S_r$, $\theta(a)^* \bullet \theta(a) = \theta(a^* \bullet a)$. It follows that, for each $a \in \mathbb{C}S_r$ with $\rho \circ \theta(a^* \bullet a) \le 1$, we have $\theta(a) \in \mathbb{C}G^0$ with $\rho(\theta(a)^* \bullet \theta(a)) \le 1$. Therefore,

$$\begin{aligned}
\|\theta a\|_{C^*(G^0)} &= \sup\left\{\rho\left((\theta a)^* \bullet \theta a\right)^{1/2} : \rho \in \mathcal{S}\left(\mathbb{C}G^0\right)\right\} \\
&= \sup\left\{(\rho \circ \theta)(a^* \bullet a)^{1/2} : \rho \in \mathcal{S}\left(\mathbb{C}G^0\right)\right\} \\
&\le \sup\left\{(\rho \circ \theta)(a^* \bullet a)^{1/2} : \rho \circ \theta \in \mathcal{S}(\mathbb{C}S_r)\right\} \\
&\le \|a\|_{C^*(S_r)}.
\end{aligned} \tag{2.4}$$

□

Recall that a strongly E^*-unitary inverse semigroup S is an inverse semigroup that admits a grading $\varphi : S \to G \cup \{0\}$ such that $\varphi^{-1}(e)$ is equal to the set of nonzero idempotent of S, where e is the identity of G.

Lemma 2.4. *If S is E-unitary, then S_r is strongly E^*-unitary.*

Proof. Let $\varphi : S \to G$ be the natural epimorphism of S onto its maximal group homomorphic image. Then, $\varphi_r : S_r \to G \cup \{0\}$ with $\varphi_r(0) = 0$ is a grading map and $\varphi_r^{-1}(e) = E$. □

Proposition 2.5. *Let S be an E-unitary inverse semigroup with the maximal group homomorphic image G. Then, the natural map $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ is an isometry.*

Proof. Let $\pi : S_r \to \mathcal{B}(\mathcal{H})$ be a representation of the inverse semigroup S_r such that $\pi(0) = 0$. Clearly, π maps the idempotents of S to projections on $\mathcal{H}$. Since S_r is strongly E^*-unitary, π induces a representation on G^0 defined by $\tilde{\pi}(\varphi(x)) = [\pi(x)]$, for each $x \in S \setminus \{0\}$ and $\tilde{\pi}(0) = 0$, where $[\pi(x)]$ is the equivalence class of $\pi(x)$. Therefore, representations of S lift to representations on the corresponding inverse semigroup G^0.

Now, assume that $a \in \mathbb{C}S_r$ is a hermitian element. Then, $\|a\|_{C^*(S_r)} = \sup_\pi \|\pi(a)\|$. By the definition of the quotient norm for $\tilde{\pi}(\theta(a)) = [\pi(a)]$, for each $\varepsilon > 0$, there is a projection $i \in \mathcal{B}(\mathcal{H})$ such that $\|\pi(a) + i\| \leq \|[\pi(a)]\| + \varepsilon$. But i is positive as an element in the C^*-algebra $\mathcal{B}(\mathcal{H})$ thus $\pi(a) \leq \pi(a) + i$, and therefore $\|\pi(a)\| \leq \|\pi(a) + i\|$. Hence, $\|\tilde{\pi}(\varepsilon(a))\| = \|\pi(a)\|$, and the result follows from Proposition 2.3 and the definition of the C^*-norm. □

Proposition 2.6. *Let $\varphi : S \to G$ be the quotient map of an E-unitary inverse semigroup S onto its maximal group homomorphic image. Then, there exists an isometric isomorphism $\phi : B(\mathbb{B}) \to B_{r,e}(S)$ such that, for each f in $B(\mathbb{B})$, $\phi(f) = f \circ \varphi$, where $\mathbb{B}$ is the Fell bundle for $C_0^*(G^0)$.*

Proof. Let $\varphi_r : S_r \to G^0$ be the induced homomorphism of φ on S_r and θ the natural map defined in Proposition 2.3. Then, Proposition 2.5 says that the natural map $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ extends to an isometric surjection

$$\theta : C^*(S_r) \longrightarrow C^*\left(G^0\right) \tag{2.5}$$

which maps zero to zero. Hence, we have the following map, again denoted by θ :

$$\theta : \frac{C^*(S_r)}{\mathbb{C}\delta_0} \longrightarrow \frac{C^*(G^0)}{\mathbb{C}\delta_0} \tag{2.6}$$

which induces an $*$-homomorphism

$$\tilde{\theta} : C_r^*(S) \longrightarrow C_0^*\left(G^0\right) \tag{2.7}$$

and gives the isometric linear isomorphism $(C^*(G^0)/\mathbb{C}\delta_0)^* \cong B_{r,e}(S)$. By the paragraph before Proposition 2.2, there is an isometric isomorphism of Banach algebras

$$\tilde{\theta}^* : B(\mathbb{B}) \longrightarrow B_{r,e}(S), \tag{2.8}$$

where $\mathbb{B}$ is the Fell bundle for $C_0^*(G^0)$. Note that, for each $u \in C^*(\mathbb{B})^*$ and $\delta_s \in \ell^1(S_r)$, $\widetilde{\theta}^*(u)(\delta_s) = \sum u(x)\theta(\delta_s)(x) = u \circ \varphi(s)$. This means that $\widetilde{\theta}^*(u) = u \circ \varphi$. Therefore, $\phi = \widetilde{\theta}^*$ is the required map. □

Next, we adapt the notion of weak containment property of inverse semigroups [8] to the restricted case.

Definition 2.7. The inverse semigroup S has restricted weak containment property if $C_r^*(S) \cong C_{\lambda_r}^*(S)$.

Proposition 2.8. *For an inverse semigroup S, S_r has weak containment property if and only if S has restricted weak containment property.*

Proof. The result follows from the following isomorphisms [5]:

$$C_r^*(S) \cong \frac{C^*(S_r)}{\mathbb{C}\delta_0}, \qquad C_{\lambda_r}^*(S) \cong \frac{C_\Lambda^*(S_r)}{\mathbb{C}\delta_0}, \tag{2.9}$$

where Λ is the left regular representation of S_r and $C_\Lambda^*(S_r)$ is the completion of $\ell^1(S_r)$ in the norm $\|f\|_\Lambda := \sup \|\Lambda(f)\|$. □

Proposition 2.9. *For an inverse semigroup S, the following three conditions are equivalent:*

(i) $\ell_r^1(S)$ *is amenable,*

(ii) E_S *is finite,*

(iii) $\ell_r^1(S)$ *has a bounded approximate identity.*

Proof. If E_S is finite then so is $E_{S_r} = E_S \cup \{0\}$, hence $\ell^1(S_r)$ is amenable [2], and so is $\ell_r^1(S) \cong \ell^1(S_r)/\mathbb{C}\delta_0$.

Conversely, if $\ell_r^1(S)$ is amenable, then $\ell^1(S_r)/\mathbb{C}\delta_0$ and $\mathbb{C}\delta_0$ are both amenable. Hence, $\ell^1(S_r)$ is amenable, therefore E_{S_r} and so E_S are finite [2]. □

Let $\varphi : S \to G$ be the quotient map of the inverse semigroup S onto its maximal group homomorphic image and $\varphi_r : S_r \to G \cup 0$ a grading of S_r by the group G, let $H_r = \varphi_r^{-1}(e) \cup \{0\}$ and $H = \varphi^{-1}(e)$. There is a conditional expectation

$$\begin{aligned} \varepsilon &: C_0^*(S_r) \cong \frac{C^*(S_r)}{\mathbb{C}\delta_0} \cong C_r^*(S) \longrightarrow C_0^*(H_r) \cong \frac{C^*(H_r)}{\mathbb{C}\delta_0} \cong C_r^*(H), \\ \varepsilon_r &: C_{\lambda,0}^*(S_r) \cong \frac{C_\lambda^*(S_r)}{\mathbb{C}\delta_0} \cong C_{\lambda_r}^*(S) \longrightarrow \overline{\mathbb{C}_0 H_r} = \frac{\overline{\ell^1(H_r)}}{\mathbb{C}\delta_0} = \overline{\ell_r^1(H)}^{\lambda} \subseteq C_{\lambda_r}^*(H). \end{aligned} \tag{2.10}$$

These are extensions of the restriction map $\ell_r^1(S) \to \ell_r^1(H)$, where $H \leq S$. By [8, Theorem 4.2], S_r has weak containment property if and only if ε is faithful and H_r has weak containment property, that is $C^*(H_r) \cong C_\lambda^*(H_r)$. It follows that S has restricted weak containment property if and only if ε is faithful and H has restricted weak containment property. Now, most of the results in [8] extend to the restricted version. In particular, we have the following two results, where, in the latter, the amenability of the Fell bundle is in the sense of Exel [11].

Theorem 2.10. *Let $H = \varphi^{-1}(e) \leq S$ where φ is the quotient map of S onto its maximal group homomorphic image. Then, S has restricted weak containment property if and only if $\varepsilon : C_r^*(S) \to C_r^*(H)$ is faithful and H has restricted weak containment property.*

Corollary 2.11. *For an E-unitary inverse semigroup S, the following three conditions are equivalent:*

(i) *S has restricted weak containment property,*

(ii) *$\varepsilon : C_r^*(S) \to C_r^*(E)$ is faithful,*

(iii) *the Fell bundle of $C_r^*(S)$ is amenable.*

References

[1] M. V. Lawson, *Inverse Semigroups, The Theory of Partial Symmetries*, World Scientific, Singapore, 1998.

[2] J. Duncan and I. Namioka, "Amenability of inverse semigroups and their semigroup algebras," *Proceedings of the Royal Society of Edinburgh A*, vol. 80, no. 3-4, pp. 309–321, 1978.

[3] R. J. Lindahl and P. H. Maserick, "Positive-definite functions on involution semigroups," *Duke Mathematical Journal*, vol. 38, pp. 771–782, 1971.

[4] A. T. M. Lau, "The Fourier-Stieltjes algebra of a topological semigroup with involution," *Pacific Journal of Mathematics*, vol. 77, no. 1, pp. 165–181, 1978.

[5] M. Amini and A. Medghalchi, "Restricted algebras on inverse semigroups. I. Representation theory," *Mathematische Nachrichten*, vol. 279, no. 16, pp. 1739–1748, 2006.

[6] M. Amini and A. Medghalchi, "Restricted algebras on inverse semigroups—part II: positive definite functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 324821, 21 pages, 2011.

[7] M. Shams Yousefi, M. Amini, and F. Sady, "Complete order amenability of the Fourier algebra," *Indian Journal of Pure and Applied Mathematics*, vol. 41, no. 3, pp. 485–504, 2010.

[8] D. Milan, "C^*-algebras of inverse semigroups: amenability and weak containment," *Journal of Operator Theory*, vol. 63, no. 2, pp. 317–332, 2010.

[9] P. Eymard, "L'algèbre de Fourier d'un groupe localement compact," *Bulletin de la Société Mathématique de France*, vol. 92, pp. 181–236, 1964.

[10] N. Spronk, "Amenability properties of Fourier algebras and Fourier-Stieltjes algebras: a survey," in *Proceedings of the International Conference on Banach Algebras*, Banach Center Publications, Będlewo, Poland, July 2009.

[11] R. Exel, "Partial representations and amenable Fell bundles over free groups," *Pacific Journal of Mathematics*, vol. 192, no. 1, pp. 39–63, 2000.