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Research Article

Matrix Variate Pareto Distribution of the Second Kind

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We generalize the univariate Pareto distribution of the second kind to the matrix case and give its derivation using matrix variate gamma distribution. We study several properties such as cumulative distribution function, marginal distribution of submatrix, triangular factorization, moment generating function, and expected values of the Pareto matrix. Some of these results are expressed in terms of special functions of matrix arguments, zonal, and invariant polynomials.

1. Introduction

The Lomax distribution, also called the Pareto distribution of the second kind is given by the p.d.f.

$$\frac{\beta}{\lambda}\left(1+\frac{v}{\lambda}\right)^{-(\beta+1)}, \quad v>0, \tag{1.1}$$

where shape parameter $\beta > 0$ and location parameter $\lambda > 0$. The Lomax distribution, named after Lomax, is a heavy-tail probability distribution often used in business, economics, and actuarial modeling. The standard Pareto Distribution of the second kind has $\lambda = 1$ with the p.d.f.

$$\beta(1+v)^{-(\beta+1)}, \quad v>0,\ \beta>0. \tag{1.2}$$

Although a wealth of results on Pareto distribution is available in the literature (see Johnson et al. [1]) nothing appears to have been done to define and study matrix variate Pareto distribution.

Therefore, in this paper, we define matrix variate Pareto distribution and study several of its properties.

We will use the following standard notations (cf. Gupta and Nagar [2]). Let $A = (a_{ij})$ be an $m \times m$ matrix. Then, A^T denotes the transpose of A; $\operatorname{tr}(A) = a_{11} + \cdots + a_{mm}$; $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$; $\det(A)$ = determinant of A; $\|A\|$ = norm of A; $A > 0$ means that A is symmetric positive definite and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A > 0$. The submatrices $A^{(\alpha)}$ and $A_{(\alpha)}$, $1 \leq \alpha \leq m$, of the matrix A are defined as $A^{(\alpha)} = (a_{ij})$, $1 \leq i, j \leq \alpha$, and $A_{(\alpha)} = (a_{ij})$, $\alpha \leq i, j \leq m$, respectively.

The multivariate gamma function which is frequently used in multivariate statistical analysis is defined by

$$\begin{aligned}\Gamma_m(a) &= \int_{X>0} \operatorname{etr}(-X) \det(X)^{a-(m+1)/2} dX \\ &= \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right), \quad \operatorname{Re}(a) > \frac{m-1}{2}.\end{aligned} \tag{1.3}$$

The multivariate generalization of the beta function is given by

$$\begin{aligned}B_m(a,b) &= \int_0^{I_m} \det(X)^{a-(m+1)/2} \det(I_m - X)^{b-(m+1)/2}\, dX \\ &= \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} = B_m(b,a),\end{aligned} \tag{1.4}$$

where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(b) > (m-1)/2$. Further, by using the matrix transformation $X = (I_m + Y)^{-1} Y$ in (1.4) with the Jacobian $J(X \to Y) = \det(I_m + Y)^{-(m+1)}$ one can easily establish the identity

$$B_m(a,b) = \int_{Y>0} \det(Y)^{a-(m+1)/2} \det(I_m + Y)^{-(a+b)}\, dY. \tag{1.5}$$

The beta type 1 and beta type 2 families of distributions are defined by the density functions (Johnson et al. [1]):

$$\{B(\alpha,\beta)\}^{-1} u^{\alpha-1} (1-u)^{\beta-1}, \quad 0 < u < 1, \tag{1.6}$$

$$\{B(\alpha,\beta)\}^{-1} v^{\alpha-1} (1+v)^{-(\alpha+\beta)}, \quad v > 0, \tag{1.7}$$

respectively, where $\alpha > 0$, $\beta > 0$, and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.8}$$

Recently, Cardeño et al. [3] have defined and studied the family of beta type 3 distributions. A random variable w is said to follow a beta type 3 distribution if its density function is given by

$$2^{\alpha}\left\{B(\alpha,\beta)\right\}^{-1}w^{\alpha-1}(1-w)^{\beta-1}(1+w)^{-(\alpha+\beta)}, \quad 0<w<1. \tag{1.9}$$

If a random variable u has the p.d.f. (1.6), then we will write $u \sim B1(\alpha,\beta)$, and if the p.d.f. of the random variable v is given by (1.7), then $v \sim B2(\alpha,\beta)$. The distribution given by the density (1.9) will be designated by $w \sim B3(\alpha,\beta)$. The matrix variate generalizations of (1.6), (1.7), and (1.9) are defined as follows (Gupta and Nagar [2, 4, 5]).

Definition 1.1. An $m\times m$ random symmetric positive definite matrix U is said to have a matrix variate beta type 1 distribution with parameters (α,β), denoted as $U \sim B1(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{\det(U)^{\alpha-(m+1)/2}\det(I_m-U)^{\beta-(m+1)/2}}{B_m(\alpha,\beta)}, \quad 0<U<I_m, \tag{1.10}$$

where $\alpha>(m-1)/2$ and $\beta>(m-1)/2$.

Definition 1.2. An $m\times m$ random symmetric positive definite matrix V is said to have a matrix variate beta type 2 distribution with parameters (α,β), denoted as $V \sim B2(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{\det(V)^{\alpha-(m+1)/2}\det(I_m+V)^{-(\alpha+\beta)}}{B_m(\alpha,\beta)}, \quad V>0, \tag{1.11}$$

where $\alpha>(m-1)/2$ and $\beta>(m-1)/2$.

Definition 1.3. An $m\times m$ random symmetric positive definite matrix W is said to have a matrix variate beta type 3 distribution with parameters (α,β), denoted as $W \sim B3(m,\alpha,\beta)$, if its p.d.f. is given by

$$\frac{2^{m\alpha}\det(W)^{\alpha-(m+1)/2}\det(I_m-W)^{\beta-(m+1)/2}}{B_m(\alpha,\beta)\det(I_m+W)^{\alpha+\beta}}, \quad 0<W<I_m, \tag{1.12}$$

where $\alpha>(m-1)/2$ and $\beta>(m-1)/2$.

2. The Density Function

First we define the matrix variate Pareto distribution of the second kind.

Definition 2.1. An $m \times m$ random symmetric positive definite matrix V is said to have a matrix variate Pareto distribution of the second kind, denoted as $V \sim P_m(\beta)$, $\beta > (m-1)/2$, if its p.d.f. is given by

$$\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det(I_m + V)^{-\beta-(m+1)/2}, \quad V > 0. \tag{2.1}$$

Definition 2.2. An $m \times m$ random symmetric positive definite matrix U is said to have a matrix variate Lomax distribution with parameters Λ and β, denoted as $U \sim L_m(\Lambda, \beta)$, if its p.d.f. is given by

$$\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det(\Lambda)^{-(m+1)/2} \det\left(I_m + \Lambda^{-1}U\right)^{-\beta-(m+1)/2}, \quad U > 0, \tag{2.2}$$

where Λ is an $m \times m$ symmetric positive definite matrix and $\beta > (m-1)/2$.

From Definitions 2.1 and 2.2 it is clear that if $V \sim P_m(\beta)$, then for an $m \times m$ symmetric positive definite constant matrix Λ, $\Lambda^{1/2}V\Lambda^{1/2} \sim L_m(\Lambda, \beta)$ and if $U \sim L_m(\Lambda, \beta)$, then $\Lambda^{-1/2}U\Lambda^{-1/2} \sim P_m(\beta)$.

For $m = 1$, the matrix variate Pareto distribution and matrix variate Lomax distribution reduce to their respective univariate forms.

The matrix variate Pareto distribution can be derived by using independent gamma matrices. A random matrix Y is said to have a matrix variate gamma distribution with parameters Ψ (> 0) and κ $(> (m-1)/2)$, denoted by $Y \sim Ga(m, \kappa, \Psi)$, if its p.d.f. is given by

$$\frac{\operatorname{etr}\left(-\Psi^{-1}Y\right) \det(Y)^{\kappa-(m+1)/2}}{\Gamma_m(\kappa) \det(\Psi)^{\kappa}}, \quad Y > 0. \tag{2.3}$$

Theorem 2.3. *Let Y_1 and Y_2 be independent, $Y_1 \sim Ga(m, (m+1)/2, I_m)$ and $Y_2 \sim Ga(m, \beta, I_m)$. Then, $Y_2^{-1/2}Y_1Y_2^{-1/2} \sim P_m(\beta)$.*

Proof. The joint density function of Y_1 and Y_2 is given by

$$\frac{\operatorname{etr}[-(Y_1 + Y_2)] \det(Y_2)^{\beta-(m+1)/2}}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)}, \quad Y_1 > 0,\ Y_2 > 0. \tag{2.4}$$

Transforming $W = Y_2^{-1/2}Y_1Y_2^{-1/2}$ with the Jacobian $J(Y_1 \to W) = \det(Y_2)^{(m+1)/2}$ in the joint density of Y_1 and Y_2, we obtain the joint density of W and Y_2 as

$$\frac{\operatorname{etr}[-(I_m + W)Y_2] \det(Y_2)^{\beta+(m+1)/2-(m+1)/2}}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)}, \quad 0 < W < I_m,\ Y_2 > 0. \tag{2.5}$$

Now, the desired result is obtained by integrating Y_2 using (1.3). □

The cumulative distribution function of V is obtained as

$$\begin{aligned} P(V<\Omega) &= \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]}\int_0^{\Omega}\det(I_m+V)^{-\beta-(m+1)/2}\,dV \\ &= \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]}\det(\Omega)^{(m+1)/2}\int_0^{I_m}\det(I_m+\Omega W)^{-\beta-(m+1)/2}\,dW, \end{aligned} \tag{2.6}$$

where the last line has been obtained by substituting $W=\Omega^{-1/2}V\Omega^{-1/2}$ with the Jacobian $J(V\to W)=\det(\Omega)^{(m+1)/2}$. Now, writing

$$\det(I_m+\Omega W)=\det(I_m+\Omega)\det\Big(I_m-(I_m+\Omega)^{-1}\Omega(I_m-W)\Big), \tag{2.7}$$

the above expression is rewritten as

$$\begin{aligned} P(V<\Omega) = &\frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]}\frac{\det(\Omega)^{(m+1)/2}}{\det(I_m+\Omega)^{\beta+(m+1)/2}} \\ &\times\int_0^{I_m}\det\Big(I_m-(I_m+\Omega)^{-1}\Omega(I_m-W)\Big)^{-\beta-(m+1)/2}dW. \end{aligned} \tag{2.8}$$

Finally, using the integral representation of the Gauss hypergeometric function (Herz [6], Constantine [7], James [8], and Gupta and Nagar [2]), namely,

$$\begin{aligned} {}_2F_1(a,b;c;X) = &\frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_0^{I_m}\det(R)^{a-(m+1)/2} \\ &\times\det(I_m-R)^{c-a-(m+1)/2}\det(I_m-XR)^{-b}dR, \end{aligned} \tag{2.9}$$

where $\operatorname{Re}(a)>(m-1)/2$, $\operatorname{Re}(c-a)>(m-1)/2$, and $X<I_m$, we obtain

$$\begin{aligned} P(V<\Omega) = &\frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m[(m+1)/2]}{\Gamma_m(\beta)\Gamma_m(m+1)}\frac{\det(\Omega)^{(m+1)/2}}{\det(I_m+\Omega)^{\beta+(m+1)/2}} \\ &\times{}_2F_1\left(\frac{m+1}{2},\beta+\frac{m+1}{2};m+1;(I_m+\Omega)^{-1}\Omega\right). \end{aligned} \tag{2.10}$$

The moment generating function of V is derived as

$$M_X(Z)=\frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]}\int_{V>0}\operatorname{etr}(ZV)\det(I_m+V)^{-\beta-(m+1)/2}dV, \tag{2.11}$$

where $Z\ (m \times m) = ((1+\delta_{ij})z_{ij}/2)$. Now, evaluating the above integral, we obtain

$$M_X(Z) = \frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)} \Psi\left(\frac{m+1}{2}, -\beta + \frac{m+1}{2}; -Z\right), \tag{2.12}$$

where the confluent hypergeometric function Ψ, with $m \times m$ symmetric matrix X as argument, is defined by the integral

$$\begin{aligned} \Psi(a, c; X) = \frac{1}{\Gamma_m(a)} \int_{R>0} & \operatorname{etr}(-RX) \det(R)^{a-(m+1)/2} \\ & \times \det(I_m + R)^{c-a-(m+1)/2} dR \end{aligned} \tag{2.13}$$

valid for $\operatorname{Re}(X) > 0$ and $\operatorname{Re}(a) > (m-1)/2$.

3. Properties

In this section, we give several properties of the matrix variate Pareto distribution of the second kind defined in the previous section.

Theorem 3.1. *Let $V \sim P_m(\beta)$ and let A be an $m \times m$ constant nonsingular matrix. Then, the density of $X = AVA^T$ is*

$$\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det\left(AA^T\right)^{\beta} \det\left(AA^T + X\right)^{-\beta-(m+1)/2}, \quad X > 0. \tag{3.1}$$

Theorem 3.2. *Let $V \sim P_m(\beta)$ and let H be an $m \times m$ orthogonal matrix, whose elements are either constants or random variables distributed independent of V. Then, the distribution of V is invariant under the transformation $V \to HVH^T$. Further, if H is a random matrix, then V and H are distributed independently.*

Theorem 3.3. *If $V \sim P_m(\beta)$, then, $V^{-1} \sim B2(m, \beta, (m+1)/2)$, $(2I_m + V)^{-1}V \sim B3(m, (m+1)/2, \beta)$ and $(I_m + 2V)^{-1} \sim B3(m, \beta, (m+1)/2)$. Further, if $W \sim B3(m, (m+1)/2, \beta)$, then $2(I_m - W)^{-1}W \sim P_m(\beta)$.*

Theorem 3.4. *Let $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, where V_{11} is a $q \times q$ matrix. Define $V_{11\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}$ and $V_{22\cdot 1} = V_{22} - V_{21}V_{11}^{-1}V_{12}$. If $V \sim P_m(\beta)$, then (i) V_{11} and $V_{22\cdot 1}$ are independent, $V_{11} \sim B2(q, (m+1)/2, \beta - (m-q)/2)$ and $V_{22\cdot 1} \sim P_{m-q}(\beta)$; (ii) V_{22} and $V_{11\cdot 2}$ are independent, $V_{22} \sim B2(m-q, (m+1)/2, \beta - q/2)$ and $V_{11\cdot 2} \sim P_q(\beta)$.*

Proof. From the partition of V, we have

$$\det(I_m + V) = \det(I_q + V_{11}) \det\left(I_{m-q} + V_{22\cdot 1} + V_{21}V_{11}^{-1}\left(I_q + V_{11}\right)^{-1}V_{12}\right). \tag{3.2}$$

Now, making the transformation $V_{11} = V_{11}$, $X = V_{21}V_{11}^{-1/2}$ and $V_{22\cdot 1} = V_{22} - V_{21}V_{11}^{-1}V_{12} = V_{22} - XX^T$ with the Jacobian $J(V_{11}, V_{22}, V_{21} \to V_{11}, V_{22\cdot 1}, X) = \det(V_{11})^{(m-q)/2}$ in the density of V, we get the joint density of V_{11}, $V_{22\cdot 1}$, and X as

$$\begin{aligned}&\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det(V_{11})^{(m-q)/2} \det(I_m + V_{11})^{-\beta-(m+1)/2} \\ &\qquad \times \det\left(I_{m-q} + V_{22\cdot 1} + X(I_m + V_{11})^{-1}X^T\right)^{-\beta-(m+1)/2}.\end{aligned} \tag{3.3}$$

Further, transforming $Y = (I_{m-q} + V_{22\cdot 1})^{-1/2}X(I_q + V_{11})^{-1/2}$ with the Jacobian $J(X \to Y) = \det(I_{m-q} + V_{22\cdot 1})^{q/2}\det(I_q + V_{11})^{(m-q)/2}$, the joint density of V_{11}, $V_{22\cdot 1}$, and Y is derived as

$$\begin{aligned}&\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det(V_{11})^{(m-q)/2} \det(I_m + V_{11})^{-\beta-(q+1)/2} \\ &\qquad \times \det(I_{m-q} + V_{22\cdot 1})^{-\beta-(m-q+1)/2} \det\left(I_{m-q} + YY^T\right)^{-\beta-(m+1)/2},\end{aligned} \tag{3.4}$$

where $V_{11} > 0$, $V_{22\cdot 1} > 0$ and $Y \in \mathbb{R}^{(m-q)\times q}$. From the above factorization, it is clear that V_{11} and $V_{22\cdot 1}$ are independent, $V_{11} \sim B2(q, (m+1)/2, \beta - (m-q)/2)$ and $V_{22\cdot 1} \sim P_{m-q}(\beta)$. The second part is similar. $\square$

Theorem 3.5. *Let A be a $q \times m$ constant matrix of rank q $(\le m)$. If $V \sim P_m(\beta)$, then*

$$\begin{aligned}&\left[\left(AA^T\right)^{-1/2} AV^{-1}A^T\left(AA^T\right)^{-1/2}\right]^{-1} \sim P_q(\beta), \\ &\left(AA^T\right)^{-1/2} AVA^T\left(AA^T\right)^{-1/2} \sim B2\left(q, \frac{m+1}{2}, \beta - \frac{m-q}{2}\right).\end{aligned} \tag{3.5}$$

Proof. Write $A = M(I_q, 0)G$, where M is a $q \times q$ nonsingular matrix and G is an $m \times m$ orthogonal matrix. Now,

$$\begin{aligned}\left(AV^{-1}A^T\right)^{-1} &= \left(M(I_q \;\; 0)GV^{-1}G^T(I_q \;\; 0)^T M^T\right)^{-1} \\ &= \left(M^T\right)^{-1}\left[(I_q \;\; 0)Y^{-1}\begin{pmatrix} I_q \\ 0\end{pmatrix}\right]^{-1} M^{-1} \\ &= \left(M^T\right)^{-1}\left(Y^{11}\right)^{-1} M^{-1},\end{aligned} \tag{3.6}$$

where $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = GVG^T \sim P_m(\beta)$, Y_{11} is a $q \times q$ matrix, and $Y^{11} = (Y_{11} - Y_{12}Y_{22}^{-1}Y_{21})^{-1} = Y_{11\cdot 2}^{-1}$. From Theorem 3.4, $Y_{11\cdot 2} \sim P_q(\beta)$, and Theorem 3.1, $Z = (M^T)^{-1}Y_{11\cdot 2}M^{-1}$ has the p.d.f. proportional to

$$\frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det\left(MM^T\right)^{-\beta} \det\left(\left(MM^T\right)^{-1} + Z\right)^{-\beta-(m+1)/2}, \quad Z > 0. \tag{3.7}$$

Now, noting that $MM^T = AA^T$ and making the transformation $S = (AA^T)^{1/2}Z(AA^T)^{1/2}$ with the Jacobian $J(Z \to S) = \det(AA^T)^{-(m+1)/2}$ in the above density, we get the desired result. The proof of the second part is similar. □

From Theorem 3.5, it is clear that if $V \sim P_m(\beta)$ and $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}^T\mathbf{a}(\mathbf{a}^T V^{-1}\mathbf{a})^{-1} \sim P(\beta)$. Further, if $\mathbf{y}(m \times 1)$ is a random vector, independent of W, and $P(\mathbf{y} \neq \mathbf{0}) = 1$, then it follows that $\mathbf{y}^T\mathbf{y}(\mathbf{y}^T V^{-1}\mathbf{y})^{-1} \sim P(\beta)$ and $(\mathbf{y}^T\mathbf{y})^{-1}(\mathbf{y}^T V\mathbf{y}) \sim B2((m+1)/2, \beta - (m-1)/2)$.

From the above results, it is straightforward to show that if $\mathbf{c}(m \times 1)$ is a nonzero constant vector or a random vector independent of V with $P(\mathbf{c} \neq \mathbf{0}) = 1$, then

$$\begin{gathered} \frac{\mathbf{c}^T V^{-1}\mathbf{c}}{\mathbf{c}^T(V^{-1} + I_m)\mathbf{c}} \sim B1(\beta, 1), \\ \frac{\mathbf{c}^T\mathbf{c}}{\mathbf{c}^T(V^{-1} + I_m)\mathbf{c}} \sim B1(1, \beta), \\ \frac{\mathbf{c}^T V^{-1}\mathbf{c}}{\mathbf{c}^T\mathbf{c}} \sim B2(\beta, 1), \\ \frac{\mathbf{c}^T V\mathbf{c}}{\mathbf{c}^T\mathbf{c}} \sim B2\left(\frac{m+1}{2}, \beta - \frac{m-1}{2}\right). \end{gathered} \tag{3.8}$$

The expectation of V, $E(V)$, can easily be obtained from the above result. For any fixed $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c} \neq \mathbf{0}$,

$$E\left[\frac{\mathbf{c}^T V\mathbf{c}}{\mathbf{c}^T\mathbf{c}}\right] = E(v), \tag{3.9}$$

where $v \sim B2((m+1)/2, \beta - (m-1)/2)$. Hence, for all $\mathbf{c} \in \mathbb{R}^m$,

$$\mathbf{c}^T E(V)\mathbf{c} = \mathbf{c}^T\mathbf{c}E(v) = \frac{m+1}{2\beta - m - 1}\mathbf{c}^T\mathbf{c}, \quad \beta > \frac{m+1}{2} \tag{3.10}$$

which implies that

$$E(V) = \frac{m+1}{2\beta - m - 1}I_m, \qquad \beta > \frac{m+1}{2}. \tag{3.11}$$

Theorem 3.6. *If $V \sim P_m(\beta)$ and $V = WW^T$, where $W = (w_{ij})$ is a lower triangular matrix with positive diagonal elements, then $w_{11}, \ldots, w_{mm}$ are all independent, $w_{ii}^2 \sim B2((m+2-i)/2, \beta-(m-i)/2)$, $i = 1, \ldots, m$.*

Proof. Making the transformation $V = WW^T$ with the Jacobian $J(V \to W) = 2^m \prod_{i=1}^m w_{ii}^{m+1-i}$ in (2.1), the density of W is derived as

$$2^m \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det\left(I_m + WW^T\right)^{-\beta-(m+1)/2} \prod_{i=1}^m w_{ii}^{m+1-i}, \tag{3.12}$$

where $w_{ii} > 0$, $i = 1, \ldots, m$ and $-\infty < w_{ij} < \infty$ for $1 \le j < i \le m$. Now, partition W as

$$W = \begin{pmatrix} w_{11} & \mathbf{0} \\ \mathbf{w} & W_{22} \end{pmatrix}, \tag{3.13}$$

where $\mathbf{w}$ is an $(m-1) \times 1$ vector and W_{22} is an $(m-1) \times (m-1)$ lower triangular matrix. Then

$$\begin{aligned} \det\left(I_m + WW^T\right) &= \det\begin{pmatrix} 1+w_{11}^2 & w_{11}\mathbf{w}^T \\ w_{11}\mathbf{w} & I_{m-1} + \mathbf{w}\mathbf{w}^T + W_{22}W_{22}^T \end{pmatrix} \\ &= \left(1+w_{11}^2\right)\det\left(I_{m-1} + W_{22}W_{22}^T\right) \\ &\quad \times \left[1 + \frac{1}{1+w_{11}^2}\mathbf{w}^T\left(I_{m-1} + W_{22}W_{22}^T\right)^{-1}\mathbf{w}\right]. \end{aligned} \tag{3.14}$$

Now, make the transformation

$$\mathbf{y} = \frac{1}{\left(1+w_{11}^2\right)^{1/2}}\left(I_{m-1} + W_{22}W_{22}^T\right)^{-1/2}\mathbf{w} \tag{3.15}$$

with the Jacobian $J(\mathbf{w} \to \mathbf{y}) = (1+w_{11}^2)^{(m-1)/2}\det(I_{m-1} + W_{22}W_{22}^T)^{1/2}$ in (3.12) to get the joint density of w_{11}, W_{22}, and $\mathbf{y}$ as

$$\begin{aligned} &2^m \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} w_{11}^m \left(1+w_{11}^2\right)^{-\beta-1} \\ &\quad \times \det\left(I_{m-1} + W_{22}W_{22}^T\right)^{-\beta-m/2} \prod_{i=2}^m w_{ii}^{m+1-i}\left(1+\mathbf{y}^T\mathbf{y}\right)^{-\beta-(m+1)/2}. \end{aligned} \tag{3.16}$$

From the above factorization, it is clear that w_{11}, W_{22}, and $\mathbf{y}$ are all independent, $w_{11}^2 \sim B2((m+1)/2, \beta-(m-1)/2)$ and the density of W_{22} is proportional to

$$\det\left(I_{m-1} + W_{22}W_{22}^T\right)^{-\beta-m/2} \prod_{i=2}^m w_{ii}^{m+1-i} \tag{3.17}$$

which has the same form as the density (3.12) with m replaced by $m-1$. Repeating the argument given above on the density function of W_{22}, we observe that $w_{22}^2 \sim B2(m/2, \beta - (m-2)/2)$ and is independent of $w_{33}, \ldots, w_{mm}$. Continuing further with the same argument, we get the desired result. □

Corollary 3.7. *If $V \sim P_m(\beta)$, then the distribution of $\det(V)$ is the same as the distribution of the product of m independent beta type 2 variables, that is, $\det(V) \sim \prod_{i=1}^m v_i$ where $v_i \sim B2((m+2-i)/2, \beta - (m-i)/2)$, $i = 1, \ldots, m$.*

Corollary 3.8. *If $V \sim P_m(\beta)$, then*

$$\frac{\det\left(V^{(1)}\right)}{\det\left(V^{(0)}\right)}, \frac{\det\left(V^{(2)}\right)}{\det\left(V^{(1)}\right)}, \ldots, \frac{\det\left(V^{(m)}\right)}{\det\left(V^{(m-1)}\right)} \quad \left(\det\left(V^{(0)}\right) \equiv 1\right) \tag{3.18}$$

are independently distributed. Further, for $i = 1, \ldots, m$, $\det(V^{(i)})/\det(V^{(i-1)}) \sim B2((m+2-i)/2, \beta - (m-i)/2)$.

Theorem 3.9. *If $V \sim P_m(\beta)$ and $V = WW^T$, where $W = (w_{ij})$ is an upper triangular matrix with positive diagonal elements, then $w_{11}, \ldots, w_{mm}$ are all independent, $w_{ii}^2 \sim B2((i+1)/2, \beta - (i-1)/2)$, $i = 1, \ldots, m$.*

Proof. Making the transformation $V = WW^T$ with the Jacobian $J(V \to W) = 2^m \prod_{i=1}^m w_{ii}^i$ in (2.1), the density of W is derived as

$$2^m \frac{\Gamma_m[\beta + (m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} \det\left(I_m + WW^T\right)^{-\beta-(m+1)/2} \prod_{i=1}^m w_{ii}^i, \tag{3.19}$$

where $w_{ii} > 0$, $i = 1, \ldots, m$ and $-\infty < w_{ij} < \infty$ for $1 \le i < j \le m$. Now, partition W as

$$W = \begin{pmatrix} W_{11} & \mathbf{w} \\ \mathbf{0} & w_{mm} \end{pmatrix}, \tag{3.20}$$

where $\mathbf{w}$ is an $(m-1) \times 1$ vector and W_{11} is an $(m-1) \times (m-1)$ upper triangular matrix. Then

$$\begin{aligned} \det\left(I_m + WW^T\right) &= \det\begin{pmatrix} I_{m-1} + \mathbf{w}\mathbf{w}^T + W_{11}W_{11}^T & w_{mm}\mathbf{w} \\ w_{mm}\mathbf{w}^T & 1 + w_{mm}^2 \end{pmatrix} \\ &= \left(1 + w_{mm}^2\right) \det\left(I_{m-1} + W_{11}W_{11}^T\right) \\ &\quad \times \left[1 + \frac{1}{1 + w_{mm}^2} \mathbf{w}^T \left(I_{m-1} + W_{11}W_{11}^T\right)^{-1} \mathbf{w}\right]. \end{aligned} \tag{3.21}$$

Now, make the transformation

$$\mathbf{y} = \frac{1}{\left(1 + w_{mm}^2\right)^{1/2}} \left(I_{m-1} + W_{11}W_{11}^T\right)^{-1/2} \mathbf{w} \tag{3.22}$$

with the Jacobian $J(\mathbf{w} \to \mathbf{y}) = (1+w_{mm}^2)^{(m-1)/2} \det(I_{m-1}+W_{11}W_{11}^T)^{1/2}$ in (3.19) to get the joint density of W_{11}, $\mathbf{y}$, and w_{mm} as

$$2^m \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]} w_{mm}^m \left(1+w_{mm}^2\right)^{-\beta-1} \times \det\left(I_{m-1}+W_{11}W_{11}^T\right)^{-\beta-m/2} \prod_{i=1}^{m-1} w_{ii}^i \left(1+\mathbf{y}^T\mathbf{y}\right)^{-\beta-(m+1)/2}. \tag{3.23}$$

From the above factorization, it is clear that W_{11}, w_{mm}, and $\mathbf{y}$ are all independent, $w_{mm}^2 \sim B2((m+1)/2, \beta-(m-1)/2)$ and the density of W_{11} is proportional to

$$\det\left(I_{m-1}+W_{11}W_{11}^T\right)^{-\beta-m/2} \prod_{i=1}^{m-1} w_{ii}^i \tag{3.24}$$

which has the same form as the density (3.19) with m replaced by $m-1$. Repeating the argument given above on the density function of W_{11}, we observe that $w_{m-1,m-1}^2 \sim B2(m/2, \beta-(m-2)/2)$ and is independent of $w_{m-2,m-2}, \ldots, w_{11}$. Continuing further with the same argument, we get the desired result. □

Corollary 3.10. *If $V \sim P_m(\beta)$, then the distribution of $\det(V)$ is the same as the distribution of the product of m independent beta type 2 variables, that is, $\det(V) \sim \prod_{i=1}^m v_i$ where $v_i \sim B2((i+1)/2, \beta-(i-1)/2)$, $i=1,\ldots,m$.*

Corollary 3.11. *If $V \sim P_m(\beta)$, then*

$$\frac{\det(V_{(1)})}{\det(V_{(2)})}, \frac{\det(V_{(2)})}{\det(V_{(3)})}, \ldots, \frac{\det(V_{(m)})}{\det(V_{(m+1)})} \quad \left(\det(V_{(m+1)}) \equiv 1\right) \tag{3.25}$$

are independently distributed. Further, for $i=1,\ldots,m$, $\det(V_{(i)})/\det(V_{(i+1)}) \sim B2((i+1)/2, \beta-(i-1)/2)$.

We conclude this section by deriving moments of $\det(V)$ and $\det(I_m+V)^{-1}$.

Theorem 3.12. *Let $V \sim P_m(\beta)$, then*

$$E\left[\frac{\det(V)^r}{\det(I_m+V)^s}\right] = \frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m[(m+1)/2+r]\Gamma_m(\beta+s-r)}{\Gamma_m[\beta+(m+1)/2+s]\Gamma_m[(m+1)/2]\Gamma_m(\beta)}, \tag{3.26}$$

where $\operatorname{Re}(r) > -1$ *and* $\operatorname{Re}(\beta+s) > (m-1)/2$.

Proof. By definition

$$E\left[\frac{\det(V)^r}{\det(I_m+V)^s}\right]=\frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[(m+1)/2]}\int_{V>0}\frac{\det(V)^{(m+1)/2+r-(m+1)/2}\,dV}{\det(I_m+V)^{\beta+(m+1)/2+s}}. \tag{3.27}$$

Now, evaluating the above integral using (1.5), we get the result. □

Corollary 3.13. *If $V\sim P_m(\beta)$, then*

$$\begin{gathered}E\left[\det(V)^r\right]=\frac{\Gamma_m[(m+1)/2+r]\Gamma_m(\beta-r)}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)},\quad \operatorname{Re}(\beta-r)>\frac{m-1}{2},\\ E\left[\det(I_m+V)^{-s}\right]=\frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m(\beta+s)}{\Gamma_m[\beta+(m+1)/2+s]\Gamma_m(\beta)},\quad \operatorname{Re}(\beta+s)>\frac{m-1}{2}.\end{gathered} \tag{3.28}$$

By writing multivariate gamma functions in terms of ordinary gamma functions, expressions $E[\det(V)^r]$ and $E[\det(I_m+V)^{-s}]$ can be simplified as

$$E\left[\det(V)^r\right]=\prod_{j=1}^{m}\frac{\Gamma[(m+1)/2+r-(j-1)/2]\Gamma(\beta-r-(j-1)/2)}{\Gamma[(m+1)/2-(j-1)/2]\Gamma(\beta-(j-1)/2)}, \tag{3.29}$$

$$E\left[\det(I_m+V)^{-s}\right]=\prod_{j=1}^{m}\frac{\Gamma[\beta+(m+1)/2-(j-1)/2]\Gamma[\beta+s-(j-1)/2]}{\Gamma[\beta+(m+1)/2+s-(j-1)/2]\Gamma[\beta-(j-1)/2]}. \tag{3.30}$$

Substituting $r,s=1,2$, the first and second order moments of $\det(V)$ and $\det(I_m+V)^{-1}$ are calculated as

$$\begin{gathered}E[\det(V)]=\frac{(2)_m}{(2\beta-m-1)_m},\quad \beta>\frac{m+1}{2},\\ E\left[\det(V)^2\right]=\frac{(2)_m(4)_m}{(2\beta-m-1)_m(2\beta-m-3)_m},\quad \beta>\frac{m+3}{2},\\ E\left[\det(I_m+V)^{-1}\right]=\frac{(2\beta-m+1)_m}{(2\beta+2)_m},\quad \beta>\frac{m-1}{2},\\ E\left[\det(I_m+V)^{-2}\right]=\frac{(2\beta-m+1)_m(2\beta-m+3)_m}{(2\beta+2)_m(2\beta+4)_m},\quad \beta>\frac{m-1}{2},\end{gathered} \tag{3.31}$$

where the Pochhammer notation $(a)_k$ is defined by $(a)_k=a(a+1)\cdots(a+k-1)$, $k=1,2,\ldots$ with $(a)_0=1$.

4. Results Involving Zonal and Invariant Polynomials

Let $C_\kappa(X)$ be the zonal polynomial of an $m \times m$ symmetric matrix X corresponding to the partition $\kappa = (k_1, \ldots, k_m), k_1 + \cdots + k_m = k$, $k_1 \geq \cdots \geq k_m \geq 0$. Then, for small values of k, explicit formulas for $C_\kappa(X)$ are available as (James [8])

$$\begin{aligned}
C_{(1)}(X) &= \operatorname{tr}(X), \\
C_{(2)}(X) &= \frac{1}{3}\left[(\operatorname{tr} X)^2 + 2\operatorname{tr}\left(X^2\right)\right], \\
C_{(1^2)}(X) &= \frac{2}{3}\left[(\operatorname{tr} X)^2 - \operatorname{tr}\left(X^2\right)\right], \\
C_{(3)}(X) &= \frac{1}{15}\left[(\operatorname{tr} X)^3 + 6(\operatorname{tr} X)\left(\operatorname{tr} X^2\right) + 8\operatorname{tr}\left(X^3\right)\right], \\
C_{(2,1)}(X) &= \frac{3}{5}\left[(\operatorname{tr} X)^3 + (\operatorname{tr} X)\left(\operatorname{tr} X^2\right) - 2\operatorname{tr}\left(X^3\right)\right], \\
C_{(1^3)}(X) &= \frac{1}{3}\left[(\operatorname{tr} X)^3 - 3(\operatorname{tr} X)\left(\operatorname{tr} X^2\right) + 2\operatorname{tr}\left(X^3\right)\right].
\end{aligned} \tag{4.1}$$

From the above results, it is straightforward to show that

$$\begin{aligned}
\operatorname{tr}\left(X^2\right) &= C_{(2)}(X) - \frac{1}{2}C_{(1^2)}(X), \\
\operatorname{tr}\left(X^3\right) &= C_{(3)}(X) - \frac{1}{4}C_{(2,1)}(X) + \frac{1}{4}C_{(1^3)}(X), \\
\operatorname{tr}(X)\operatorname{tr}\left(X^2\right) &= C_{(3)}(X) + \frac{1}{6}C_{(2,1)}(X) - \frac{1}{2}C_{(1^3)}(X).
\end{aligned} \tag{4.2}$$

For an ordered partition ρ of r, $\rho = (r_1, \ldots, r_m)$, $r_1 \geq \cdots \geq r_m \geq 0$, $r_1 + \cdots + r_m = r$, $\Gamma_m(a, \rho)$ and $\Gamma_m(a, -\rho)$ are defined by

$$\begin{aligned}
\Gamma_m(a, \rho) &= (a)_\rho \Gamma_m(a), \qquad \Gamma_m(a, 0) = \Gamma_m(a), \\
\Gamma_m(a, -\rho) &= \frac{(-1)^r\, \Gamma_m(a)}{(-a + m + 1/2)_\rho}, \qquad \operatorname{Re}(a) > r_1 + \frac{m-1}{2},
\end{aligned} \tag{4.3}$$

where the *generalized hypergeometric coefficient* $(a)_\rho$ is defined by

$$(a)_\rho = \prod_{i=1}^{m}\left(a - \frac{i-1}{2}\right)_{r_i}. \tag{4.4}$$

Further, $\det(I_m - X)^{-a}$, in terms of zonal polynomials, can be expanded as

$$\det\left(I_m - X\right)^{-a} = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a)_\kappa C_\kappa(X)}{k!}, \qquad \|X\| < 1, \tag{4.5}$$

where $\sum_\kappa$ denotes summation over all ordered partitions κ of k.

For properties and further results, the reader is referred to Constantine [7] and Gupta and Nagar [2].

Lemma 4.1. *Let T be an $m \times m$ arbitrary complex symmetric matrix. Then*

$$\int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m + R)^{-(a+b)} C_\kappa(RT)dR$$
$$= \frac{\Gamma_m(a,\kappa)\Gamma_m(b,-\kappa)}{\Gamma_m(a+b)} C_\kappa(T) \tag{4.6}$$
$$= \frac{(-1)^k \Gamma_m(a,\kappa)\Gamma_m(b)}{(-b+(m+1)/2)_\kappa \Gamma_m(a+b)} C_\kappa(T), \quad \operatorname{Re}(b) > k_1 + \frac{m-1}{2}.$$

Davis [9, 10] introduced a class of polynomials $C_\phi^{\kappa,\lambda}(X,Y)$ of $m \times m$ symmetric matrix arguments X and Y, which are invariants under the transformation $(X,Y) \to (HXH^T, HYH^T)$, $H \in O(m)$. For properties and applications of invariant polynomials, we refer to Davis [9, 10], Chikuse [11], and Nagar and Gupta [12]. Let κ, λ, ϕ, and ρ be ordered partitions of the nonnegative integers k, ℓ, $f = k + \ell$, and r, respectively. Then

$$C_\phi^{\kappa,\lambda}(X,X) = \theta_\phi^{\kappa,\lambda} C_\phi(X), \qquad \theta_\phi^{\kappa,\lambda} = \frac{C_\phi^{\kappa,\lambda}(I_m, I_m)}{C_\phi(I_m)},$$
$$C_\phi^{\kappa,\lambda}(X, I_m) = \theta_\phi^{\kappa,\lambda} \frac{C_\phi(I_m) C_\kappa(X)}{C_\kappa(I_m)}, \tag{4.7}$$

$$C_\kappa^{\kappa,0}(X,Y) \equiv C_\kappa(X), \qquad C_\lambda^{0,\lambda}(X,Y) \equiv C_\lambda(Y),$$
$$C_\kappa(X) C_\lambda(Y) = \sum_{\phi \in \kappa \cdot \lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X,Y), \tag{4.8}$$

where $\phi \in \kappa \cdot \lambda$ denotes that irreducible representation of $Gl(m,R)$, the group of $m \times m$ real invertible matrices, indexed by 2ϕ, appears in the decomposition of the tensor product $2\kappa \otimes 2\lambda$ of the irreducible representation indexed by 2κ and 2λ. Further,

$$\int_0^{I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_\phi^{\kappa,\lambda}(AR, A(I_m - R))\mathrm{d}R$$
$$= \frac{\Gamma_m(t,\kappa)\Gamma_m(u,\lambda)}{\Gamma_m(t+u,\phi)} \theta_\phi^{\kappa,\lambda} C_\phi(A), \tag{4.9}$$

$$\int_0^{I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_\phi^{\kappa,\lambda}(AR, BR)\mathrm{d}R$$
$$= \frac{\Gamma_m(t,\phi)\Gamma_m(u)}{\Gamma_m(t+u,\phi)} C_\phi^{\kappa,\lambda}(A,B). \tag{4.10}$$

From the density of V, we have

$$E[C_\kappa(BV)] = \frac{\Gamma_m[\beta+(m+1)/2]}{\Gamma_m(\beta)\Gamma_m[m+1/2]}\int_{V>0} C_\kappa(VB)\det(I_m+V)^{-\beta-(m+1)/2}\,dV$$
$$= \frac{(-1)^k((m+1)/2)_\kappa}{(-\beta+(m+1)/2)_\kappa}C_\kappa(B), \quad \operatorname{Re}(\beta) > k_1 + \frac{m-1}{2}, \tag{4.11}$$

where the last line has been obtained by using (4.6).

Using results (4.1)–(4.2) on zonal polynomials, it is easy to see that

$$E\left[C_{(1)}(BV)\right] = \frac{m+1}{2\beta-m-1}C_{(1)}(B),$$

$$E[\operatorname{tr}(BV)] = \frac{m+1}{2\beta-m-1}\operatorname{tr}(B), \quad \beta > \frac{m+1}{2},$$

$$E\left[C_{(2)}(BV)\right] = \frac{(m+1)(m+3)}{(2\beta-m-1)(2\beta-m-3)}C_{(2)}(B)$$
$$= \frac{(m+1)(m+3)}{3(2\beta-m-1)(2\beta-m-3)}\left[(\operatorname{tr}B)^2+2\operatorname{tr}\left(B^2\right)\right], \quad \beta > \frac{m+3}{2},$$

$$E\left[C_{(1^2)}(BV)\right] = \frac{m(m+1)}{(2\beta-m-1)(2\beta-m)}C_{(1^2)}(B)$$
$$= \frac{2m(m+1)}{3(2\beta-m-1)(2\beta-m)}\left[(\operatorname{tr}B)^2-\operatorname{tr}\left(B^2\right)\right], \quad \beta > \frac{m+1}{2},$$

$$E\left[C_{(3)}(BV)\right] = \frac{(m+1)(m+3)(m+5)}{(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}C_{(3)}(B)$$
$$= \frac{(m+1)(m+3)(m+5)}{15(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}$$
$$\times\left[(\operatorname{tr}B)^3+6(\operatorname{tr}B)\left(\operatorname{tr}B^2\right)+8\operatorname{tr}\left(B^3\right)\right], \quad \beta > \frac{m+5}{2},$$

$$E\left[C_{(2,1)}(BV)\right] = \frac{m(m+1)(m+3)}{(2\beta-m)(2\beta-m-1)(2\beta-m-3)}C_{(2,1)}(B)$$
$$= \frac{3m(m+1)(m+3)}{15(2\beta-m)(2\beta-m-1)(2\beta-m-3)}$$
$$\times\left[(\operatorname{tr}B)^3+(\operatorname{tr}B)\left(\operatorname{tr}B^2\right)-2\operatorname{tr}\left(B^3\right)\right], \quad \beta > \frac{m+3}{2},$$

$$\begin{aligned}
E\left[C_{(1^3)}(BV)\right] &= \frac{(m-1)m(m+1)}{(2\beta-m)(2\beta-m+1)(2\beta-m-1)}C_{(1^3)}(B)\\
&= \frac{(m-1)m(m+1)}{3(2\beta-m+1)(2\beta-m)(2\beta-m-1)}\\
&\quad\times\left[(\operatorname{tr}B)^3-3(\operatorname{tr}B)\left(\operatorname{tr}B^2\right)+2\operatorname{tr}\left(B^3\right)\right],\quad \beta>\frac{m+1}{2},\\
E\left[\operatorname{tr}(BV)^2\right] &= E\left[C_{(2)}(BV)\right]-\frac{1}{2}E\left[C_{(1^2)}(BV)\right]\\
&= \frac{2\beta(m+1)}{(2\beta-m)(2\beta-m-1)(2\beta-m-3)}(\operatorname{tr}B)^2\\
&\quad+\frac{(m+1)[(m+2)(2\beta-m-1)+2]}{(2\beta-m)(2\beta-m-1)(2\beta-m-3)}\left(\operatorname{tr}B^2\right),\quad \beta>\frac{m+3}{2},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
E\left[\operatorname{tr}(BV)^3\right] &= E\left[C_{(3)}(BV)\right]-\frac{1}{4}E\left[C_{(2,1)}(BV)\right]+\frac{1}{4}E\left[C_{(1^3)}(BV)\right]\\
&= \frac{1+m}{(2\beta-m+1)(2\beta-m)(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}\\
&\quad\times\left[\frac{1}{10}\left\{(m-1)m(3+m)(5+m)-4\left(m^3+5m^2+m-5\right)\beta\right.\right.\\
&\qquad\left.+4\left(m^2+3m+10\right)\beta^2\right\}(\operatorname{tr}B)^3\\
&\qquad+\frac{1}{10}\left\{(m-1)m(3+m)(5+m)\right.\\
&\qquad\left.-4(m(3+m)(17+m)-30)\beta+4\left(m^2+33m+60\right)\beta^2\right\}(\operatorname{tr}B)\left(\operatorname{tr}B^2\right)\\
&\qquad+\frac{4}{5}\left\{(m-1)m(3+m)(5+m)+20\beta-m\left(4m^2+25m+29\right)\beta\right.\\
&\qquad\left.\left.+2\left(2m^2+11m+20\right)\beta^2\right\}\operatorname{tr}\left(B^3\right)\right],\quad \beta>\frac{m+5}{2},\\
E\left[\operatorname{tr}(BV)\operatorname{tr}(BV)^2\right] &= E\left[C_{(3)}(BV)\right]+\frac{1}{6}E\left[C_{(2,1)}(BV)\right]-\frac{1}{2}E\left[C_{(1^3)}(BV)\right]\\
&= \frac{1+m}{(2\beta-m+1)(2\beta-m)(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}\\
&\quad\times\left[-\frac{1}{15}\left\{(m-1)m(3+m)(5+m)-2\left(2m^3-5m^2-48m+15\right)\beta\right.\right.\\
&\qquad\left.+4\left(m^2-12m-15\right)\beta^2\right\}(\operatorname{tr}B)^3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{15}\Big\{7(m-1)m(3+m)(5+m)\\
&\qquad +90\beta-2m\Big(14m^2+70m+39\Big)\beta+4\Big(7m^2+21m+45\Big)\beta^2\Big\}\\
&\times(\operatorname{tr}B)\Big(\operatorname{tr}B^2\Big)\\
&+\frac{2}{15}\Big\{(m-1)m(3+m)(5+m)-4(m(3+m)(17+m)-30)\beta\\
&\qquad +4\Big(m^2+33m+60\Big)\beta^2\Big\}\operatorname{tr}\Big(B^3\Big)\Big],\quad \beta>\frac{m+5}{2}.
\end{aligned}
\tag{4.13}
$$

Further, using the invariance of the distribution of V and the above results, one obtains

$$
\begin{aligned}
E(V)&=\frac{m+1}{2\beta-m-1}I_m,\quad \beta>\frac{m+1}{2},\\
E\Big(V^2\Big)&=\frac{(m+1)[4(m+1)\beta-m(m+3)]}{(2\beta-m)(2\beta-m-1)(2\beta-m-3)}I_m,\quad \beta>\frac{m+3}{2},\\
E\Big(V^3\Big)&=\frac{1+m}{10(2\beta-m+1)(2\beta-m)(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}\\
&\quad\times\Big[(m-1)m(3+m)(5+m)\Big(m^2+m+8\Big)\\
&\qquad -4\Big(m^5+6m^4+29m^3+96m^2+28m-40\Big)\beta\\
&\qquad +4\Big(m^4+4m^3+51m^2+104m+80\Big)\beta^2\Big]I_m,\quad \beta>\frac{m+5}{2},\\
E\Big[\Big(\operatorname{tr}V^2\Big)V\Big]&=-\frac{1+m}{15(2\beta-m+1)(2\beta-m)(2\beta-m-1)(2\beta-m-3)(2\beta-m-5)}\\
&\quad\times\Big[(m-1)m(3+m)(5+m)\Big(m^2-14m-2\Big)\\
&\qquad +2\Big(-2m^5+33m^4+192m^3+143m^2+114m-120\Big)\beta\\
&\qquad +4\Big(m^4-26m^3-59m^2-156m-120\Big)\beta^2\Big]I_m,\quad \beta>\frac{m+5}{2}.
\end{aligned}
\tag{4.14}
$$

Theorem 4.2. *Let V_1 and V_2 be independent, $V_i \sim P_m(\beta_i)$, $i = 1, 2$. Define $S = V_1 + V_2$ and $R = (V_1 + V_2)^{-1/2} V_2 (V_1 + V_2)^{-1/2}$. Then, the density of S is given by*

$$\begin{aligned}
&\frac{\Gamma_m[\beta_1 + (m+1)/2]\Gamma_m[\beta_2 + (m+1)/2]}{\Gamma_m(\beta_1)\Gamma_m^2[(m+1)/2]\Gamma_m(\beta_2)} \det(S)^{(m+1)/2} \det(I_m + S)^{-(\beta_1+\beta_2+m+1)} \\
&\quad \times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda} \frac{(\beta_1 + (m+1)/2)_\kappa (\beta_2 + (m+1)/2)_\lambda}{k!\ l!} \\
&\quad \times \sum_{\phi \in \kappa\cdot\lambda} \left(\theta_\phi^{\kappa,\lambda}\right)^2 \frac{\Gamma((m+1)/2, \kappa)\Gamma((m+1)/2, \lambda)}{\Gamma(m+1, \phi)} C_\phi\left((I_m + S)^{-1} S\right), \quad S > 0.
\end{aligned} \tag{4.15}$$

Further, the density of R is derived as

$$\begin{aligned}
&\frac{\Gamma_m[\beta_1 + (m+1)/2]\Gamma_m[\beta_2 + (m+1)/2]}{\Gamma_m(\beta_1)\Gamma_m(\beta_2)\Gamma_m^2[(m+1)/2]} \\
&\quad \times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda} \frac{(\beta_1 + (m+1)/2)_\kappa (\beta_2 + (m+1)/2)_\lambda}{k!\ l!} \\
&\quad \times \sum_{\phi \in \kappa\cdot\lambda} \frac{\Gamma(m+1, \phi)\Gamma_m(\beta_1 + \beta_2)}{\Gamma_m(\beta_1 + \beta_2 + m + 1, \phi)} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(R, I_m - R), \quad 0 < R < I_m.
\end{aligned} \tag{4.16}$$

Proof. The joint density of V_1 and V_2 is given by

$$\begin{aligned}
&\frac{\Gamma_m[\beta_1 + (m+1)/2]\Gamma_m[\beta_2 + (m+1)/2]}{\Gamma_m(\beta_1)\Gamma_m(\beta_2)\Gamma_m^2[(m+1)/2]} \\
&\quad \times \det(I_m + V_1)^{-\beta_1-(m+1)/2} \det(I_m + V_2)^{-\beta_2-(m+1)/2}, \quad V_1 > 0,\ V_2 > 0.
\end{aligned} \tag{4.17}$$

Making the transformation $V_2 = S^{1/2} R S^{1/2}$ and $V_1 = S^{1/2}(I_m - R)S^{1/2}$ with the Jacobian $J(V_1, V_2 \to S, R) = \det(S)^{(m+1)/2}$ in (4.17), the joint density of R and S is derived as

$$\begin{aligned}
&\frac{\Gamma_m[\beta_1 + (m+1)/2]\Gamma_m[\beta_2 + (m+1)/2]}{\Gamma_m(\beta_1)\Gamma_m(\beta_2)\Gamma_m^2[(m+1)/2]} \det(S)^{(m+1)/2} \det(I_m + S)^{-(\beta_1+\beta_2+m+1)} \\
&\quad \times \det(I_m - S_1 R)^{-\beta_1-(m+1)/2} \det(I_m - S_1(I_m - R))^{-\beta_2-(m+1)/2},
\end{aligned} \tag{4.18}$$

where $S_1 = (I_m + S)^{-1}S$, $0 < R < I_m$, and $S > 0$. Since, $\|S_1R\| < 1$ and $\|S_1(I_m - R)\| < 1$, using (4.5), we can write

$$\begin{aligned} \det(I_m - S_1R)^{-\beta_1-(m+1)/2} &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\beta_1 + (m+1)/2)_\kappa}{k!}C_\kappa(S_1R), \\ \det(I_m - S_1(I_m - R))^{-\beta_2-(m+1)/2} &= \sum_{l=0}^{\infty}\sum_{\lambda}\frac{(\beta_2 + (m+1)/2)_\lambda}{l!}C_\lambda(S_1(I_m - R)), \end{aligned} \tag{4.19}$$

where κ and λ are the ordered partitions of k and l, respectively. Now, the application of (4.8) yields

$$\begin{aligned} &\det(I_m - S_1R)^{-\beta_1-(m+1)/2}\det(I_m - S_1(I_m - R))^{-\beta_2-(m+1)/2} \\ &\quad = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\beta_1 + (m+1)/2)_\kappa(\beta_2 + (m+1)/2)_\lambda}{k!\,l!} \\ &\quad\quad \times \sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(S_1R, S_1(I_m - R)). \end{aligned} \tag{4.20}$$

Finally, substituting (4.20) in (4.18), the joint density of R and S is obtained as

$$\begin{aligned} &\frac{\Gamma_m[\beta_1 + (m+1)/2]\Gamma_m[\beta_2 + (m+1)/2]}{\Gamma_m(\beta_1)\Gamma_m(\beta_2)\Gamma_m^2[(m+1)/2]}\det(S)^{(m+1)/2}\det(I_m + S)^{-(\beta_1+\beta_2+m+1)} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\beta_1 + (m+1)/2)_\kappa(\beta_2 + (m+1)/2)_\lambda}{k!\,l!} \\ &\quad \times \sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(S_1R, S_1(I_m - R)), \quad 0 < R < I_m,\ S > 0. \end{aligned} \tag{4.21}$$

Now, the integration of R in (4.21) using (4.9) yields the density of S. The density of R is obtained by substituting $S_1 = (I_m + S)^{-1}S$ with the Jacobian $J(S \to S_1) = \det(I - S_1)^{-(m+1)}$ in (4.21) and integrating S_1 by using (4.10). □

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