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Research Article

Regularity Criteria for Hyperbolic Navier-Stokes and Related System

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We prove a regularity criterion for strong solutions to the hyperbolic Navier-Stokes and related equations in Besov space.

1. Introduction

First, we consider the following hyperbolic Navier-Stokes equations [1]:

$$\tau u_{tt} + u_t - \Delta u + \nabla \pi + u \cdot \nabla u + \tau u_t \cdot \nabla u + \tau u \cdot \nabla u_t = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$(u, u_t)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n,\ n \geq 2. \tag{1.3}$$

Here u is the velocity, π is the pressure, and $\tau > 0$ is a small relaxation parameter. We will take $\tau = 1$ for simplicity.

When $\tau = 0$, (1.1) and (1.2) reduce to the standard Navier-Stokes equations. Kozono et al. [2] proved the following regularity criterion:

$$\omega := \operatorname{curl}\ u \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\right). \tag{1.4}$$

Here $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space.

Rack and Saal [1] proved the local well posedness of the problem (1.1)–(1.3). The global regularity is still open. The first aim of this paper is to prove a regularity criterion. We will prove the following theorem.

Theorem 1.1. *Let* $(u_0, u_1) \in H^{s+1} \times H^s$ *with* $s > n/2$, $n \geq 2$ *and* $\operatorname{div} u_0 = \operatorname{div} u_1 = 0$ *in* $\mathbb{R}^n$. *Let* (u, π) *be a unique strong solution to the problem* (1.1)–(1.3). *If* u *satisfies*

$$u, \nabla u, u_t \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}\right), \tag{1.5}$$

then the solution u *can be extended beyond* $T > 0$.

In our proof, we will use the following logarithmic Sobolev inequality [2]:

$$\|u\|_{L^\infty} \leq C\left(1 + \|u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|u\|_{H^s})\right) \tag{1.6}$$

and the following bilinear product and commutator estimates according to Kato and Ponce [3]:

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.7}$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^{s-1} g\right\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \tag{1.8}$$

with $s > 0$, $\Lambda := (-\Delta)^{1/2}$ and $1/p = (1/p_1) + (1/q_1) = (1/p_2) + (1/q_2)$.

Next, we consider the fractional Landau-Lifshitz equation:

$$\partial_t \phi = \phi \times \Lambda^{2\beta}\phi, \tag{1.9}$$

$$\phi(x, 0) = \phi_0(x) \in \mathbb{S}^2, \quad x \in \mathbb{R}^n, \tag{1.10}$$

where $\phi \in \mathbb{S}^2$ is a three-dimensional vector representing the magnetization and β is a positive constant.

When $\beta = 1$, using the standard stereographic projection $\mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$, (1.9) can be rewritten as the derivative Schrödinger equation for $w \in \mathbb{C}$,

$$iw_t + \Delta w + 4\frac{(\nabla w)^2}{1 + |w|^2}\overline{w} = 0. \tag{1.11}$$

Equation (1.9) is also called the Schrödinger map and has been studied by many authors [4–31]. Guo and Han [32] proved the following regularity criterion:

$$\nabla\phi \in L^2(0,T;L^\infty(\mathbb{R}^n)) \tag{1.12}$$

with $n \geq 2$.

When $0 < \beta \leq 1/2$, Pu and Guo [33] show the local well posedness of strong solutions and the blow-up criterion

$$\Lambda^{2\beta}\phi \in L^1(0,T;L^\infty(\mathbb{R}^n)) \tag{1.13}$$

with $n \leq 3$.

We will refine (1.13) as follows.

Theorem 1.2. *Let $0 < \beta \leq 1/2$. Let m be an integer such that $2m > (n+1)/2$ for any $n \geq 1$. Let $\Lambda^\beta\phi_0 \in H^{2m}$ and $\phi_0 \in \mathbb{S}^2$ and ϕ be a local smooth solution to the problem* (1.9) *and* (1.10). *If ϕ satisfies*

$$\Lambda^{2\beta}\phi \in L^1\left(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)\right) \tag{1.14}$$

for some finite $T > 0$, then the solution ϕ can be extended beyond $T > 0$.

2. Proof of Theorem 1.1

Since (u,π) is a local smooth solution, we only need to prove a priori estimates.

First, testing (1.1) by u and using (1.2), we see that

$$\begin{aligned}
&\frac{d}{dt}\int\left(\frac{1}{2}u^2 + uu_t\right)dx + \int|\nabla u|^2 dx \\
&\quad= \int u_t^2 dx + \int u\cdot\nabla u\cdot u_t dx \\
&\quad\leq \int u_t^2 dx + \frac{1}{2}\|\nabla u\|_{L^\infty}\left(\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2\right).
\end{aligned} \tag{2.1}$$

Testing (1.1) by $4u_t$ and using (1.2), we find that

$$\begin{aligned}&\frac{d}{dt}\int\left(2u_t^2+2|\nabla u|^2\right)dx+4\int u_t^2dx\\&\quad=-4\int(u\cdot\nabla u+u_t\cdot\nabla u)u_tdx\\&\quad\leq C\|\nabla u\|_{L^\infty}\left(\|u\|_{L^2}^2+\|u_t\|_{L^2}^2\right).\end{aligned}\tag{2.2}$$

Applying Λ^s to (1.1), testing by $\Lambda^s u_t$ and using (1.2), (1.7), (1.8), and (1.6), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\left|\Lambda^{s+1}u\right|^2+|\Lambda^s u_t|^2\right)dx+\int|\Lambda^s u_t|^2dx\\&\quad=-\sum_i\int\Lambda^s\partial_i(u_iu)\cdot\Lambda^su_tdx-\int\Lambda^s(u_t\cdot\nabla u)\cdot\Lambda^su_tdx\\&\qquad-\sum_i\int[\Lambda^s\partial_i(u_iu_t)-u_i\partial_i\Lambda^su_t]\Lambda^su_tdx\\&\quad\leq C\|u\|_{L^\infty}\left\|\Lambda^{s+1}u\right\|_{L^2}\|\Lambda^su_t\|_{L^2}\\&\qquad+C\left(\|u_t\|_{L^\infty}\left\|\Lambda^{s+1}u\right\|_{L^2}+\|\nabla u\|_{L^\infty}\|\Lambda^su_t\|_{L^2}\right)\|\Lambda^su_t\|_{L^2}\\&\quad\leq C(\|u\|_{L^\infty}+\|\nabla u\|_{L^\infty}+\|u_t\|_{L^\infty})\left(\left\|\Lambda^{s+1}u\right\|_{L^2}^2+\|\Lambda^su_t\|_{L^2}^2\right)\\&\quad\leq C\left(1+\left(\|u\|_{\dot B^0_{\infty,\infty}}+\|\nabla u\|_{\dot B^0_{\infty,\infty}}+\|u_t\|_{\dot B^0_{\infty,\infty}}\right)\log\left(e+\|u\|_{H^{s+1}}^2+\|u_t\|_{H^s}^2\right)\right)\\&\qquad\cdot\left(\|\Lambda^su_t\|_{L^2}^2+\left\|\Lambda^{s+1}u\right\|_{L^2}^2\right).\end{aligned}\tag{2.3}$$

Combining (2.1), (2.2), and (2.3) and using the Gronwall inequality, we conclude that

$$\|u\|_{L^\infty(0,T;H^{s+1})}+\|u_t\|_{L^\infty(0,T;H^s)}\leq C.\tag{2.4}$$

This completes the proof.

3. Proof of Theorem 1.2

Since ϕ is a local smooth solution, we only need to prove a priori estimates. In this section, we denote by $(\cdot,\cdot)$ the standard L^2 scalar product.

First, testing (1.9) by $\Lambda^{2\beta}\phi$ and using $(a \times b) \cdot a = 0$, we see that

$$\frac{1}{2}\frac{d}{dt}\int \left|\Lambda^{\beta}\phi\right|^{2} dx = 0. \tag{3.1}$$

Testing (1.9) by $\Delta^{2m}\Lambda^{2\beta}\phi$ and using $(a \times b) \cdot a = 0$, (1.6) and (1.7), we obtain, with $(1/p) + (1/q) = (1/p_\alpha) + (1/q_\alpha) = (1/\widetilde{p}_\alpha) + (1/\widetilde{q}_\alpha) = 1/2$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left|\Delta^{m}\Lambda^{\beta}\phi\right|^{2} dx \\
&\quad = \left(\phi \times \Lambda^{2\beta}\phi, \Delta^{2m}\Lambda^{2\beta}\phi\right) \\
&\quad = \left(\Delta^{m}\left(\phi \times \Lambda^{2\beta}\phi\right), \Delta^{m}\Lambda^{2\beta}\phi\right) \\
&\quad = \left(\Delta^{m}\phi \times \Lambda^{2\beta}\phi + \sum_{\alpha=1}^{2m-1} C_\alpha D^{2m-\alpha}\phi \times \Lambda^{2\beta}D^{\alpha}\phi, \Delta^{m}\Lambda^{2\beta}\phi\right) \\
&\quad = \left(\Lambda^{\beta}\left(\Delta^{m}\phi \times \Lambda^{2\beta}\phi + \sum_{\alpha=1}^{2m-1} C_\alpha D^{2m-\alpha}\phi \times \Lambda^{2\beta}D^{\alpha}\phi\right), \Delta^{m}\Lambda^{\beta}\phi\right) \\
&\quad \le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2}^{2} + C\left\|\Delta^{m}\phi\right\|_{L^p}\left\|\Lambda^{3\beta}\phi\right\|_{L^q}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2} \\
&\qquad + \sum_{\alpha=1}^{2m-2} C_\alpha\left(\left\|D^{2m-\alpha}\phi\right\|_{L^{p_\alpha}}\left\|\Lambda^{3\beta}D^{\alpha}\phi\right\|_{L^{q_\alpha}} + \left\|D^{2m-\alpha}\Lambda^{\beta}\phi\right\|_{L^{\widetilde{p}_\alpha}}\left\|\Lambda^{2\beta}D^{\alpha}\phi\right\|_{L^{\widetilde{q}_\alpha}}\right)\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2} \\
&\quad \le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2}^{2} \\
&\quad \le C\left(1 + \left\|\Lambda^{2\beta}\phi\right\|_{\dot{B}^{0}_{\infty,\infty}}\log\left(e + \left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2}\right)\right)\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^2}^{2},
\end{aligned} \tag{3.2}$$

which yields

$$\left\|\Lambda^{\beta}\phi(t)\right\|_{H^{2m}} \le C. \tag{3.3}$$

Here we have used the following interesting Gagliardo-Nirenberg inequalities:

$$
\begin{aligned}
\|\Delta^m\phi\|_{L^p} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{1-\theta_0}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{\theta_0} \quad \text{with } p=\frac{2m-\beta}{m-\beta},\ \theta_0=\frac{2m-2\beta}{2m-\beta},\\
\left\|\Lambda^{3\beta}\phi\right\|_{L^q} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{\theta_0}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{1-\theta_0} \quad \text{with } q=\frac{2(2m-\beta)}{\beta},\\
\left\|D^{2m-\alpha}\phi\right\|_{L^{p_\alpha}} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{1-\theta_\alpha}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{\theta_\alpha} \quad \text{with } \theta_\alpha=\frac{2m-\alpha-2\beta}{2m-\beta},\ p_\alpha=\frac{4m-2\beta}{2m-\alpha-2\beta},\\
\left\|\Lambda^{3\beta}D^\alpha\phi\right\|_{L^{q_\alpha}} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{\theta_\alpha}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{1-\theta_\alpha} \quad \text{with } q_\alpha=\frac{4m-2\beta}{\alpha+\beta},\\
\left\|D^{2m-\alpha}\Lambda^\beta\phi\right\|_{L^{\tilde{p}_\alpha}} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{1-\tilde{\theta}_\alpha}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{\tilde{\theta}_\alpha} \quad \text{with } \tilde{\theta}_\alpha=\frac{2m-\alpha-\beta}{2m-\beta},\ \tilde{p}_\alpha=\frac{4m-2\beta}{2m-\alpha-\beta},\\
\left\|\Lambda^{2\beta}D^\alpha\phi\right\|_{L^{\tilde{q}_\alpha}} &\le C\left\|\Lambda^{2\beta}\phi\right\|_{L^\infty}^{\tilde{\theta}_\alpha}\left\|\Delta^m\Lambda^\beta\phi\right\|_{L^2}^{1-\tilde{\theta}_\alpha} \quad \text{with } \tilde{q}_\alpha=\frac{4m-2\beta}{\alpha}.
\end{aligned}
\tag{3.4}
$$

This completes the proof.

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References

[1] R. Rack and J. Saal, "Hyperbolic Navier-Stokes equations I: local well posedness," *Evolution Equations and Control Theory*, vol. 1, no. 1, pp. 195–215, 2012.

[2] H. Kozono, T. Ogawa, and Y. Taniuchi, "The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations," *Mathematische Zeitschrift*, vol. 242, no. 2, pp. 251–278, 2002.

[3] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.

[4] I. Bejenaru, A. Ionescu, C. E. Kenig, and D. Tataru, "Equivariant Schrödinger maps in two spatial dimensions," http://arxiv.org/abs/1112.6122.

[5] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, "Global existence and uniqueness of Schrödinger maps in dimensions $d \ge 4$," *Advances in Mathematics*, vol. 215, no. 1, pp. 263–291, 2007.

[6] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, "On the stability of certain spin models in 2 + 1 dimensions," *Journal of Geometric Analysis*, vol. 21, no. 1, pp. 1–39, 2011.

[7] I. Bejenaru, A. D. Ionescu, C. E. Kenig, and D. Tataru, "Global Schrödinger maps in dimensions $d \ge 2$: small data in the critical Sobolev spaces," *Annals of Mathematics*, vol. 173, no. 3, pp. 1443–1506, 2011.

[8] I. Bejenaru and D. Tataru, "Near soliton evolution for equivariant Schrödinger maps in two spatial dimensions," http://arxiv.org/abs/1009.1608.

[9] I. Bejenaru, "Global results for Schrödinger maps in dimensions $n \ge 3$," *Communications in Partial Differential Equations*, vol. 33, no. 1–3, pp. 451–477, 2008.

[10] I. Bejenaru, "On Schrödinger maps," *American Journal of Mathematics*, vol. 130, no. 4, pp. 1033–1065, 2008.

[11] N. H. Chang, J. Shatah, and K. Uhlenbeck, "Schrödinger maps," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 590–602, 2000.

[12] S. Gustafson and E. Koo, "Global well-posedness for 2D radial Schrödinger maps into the sphere," http://arxiv.org/abs/1105.5659.
[13] S. Gustafson, K. Kang, and T. P. Tsai, "Asymptotic stability of harmonic maps under the Schrödinger flow," *Duke Mathematical Journal*, vol. 145, no. 3, pp. 537–583, 2008.
[14] S. Gustafson, K. Nakanishi, and T. P. Tsai, "Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$," *Communications in Mathematical Physics*, vol. 300, no. 1, pp. 205–242, 2010.
[15] A. D. Ionescu and C. E. Kenig, "Low-regularity Schrödinger maps," *Differential and Integral Equations*, vol. 19, no. 11, pp. 1271–1300, 2006.
[16] A. D. Ionescu and C. E. Kenig, "Low-regularity Schrödinger maps. II. Global well-posedness in dimensions $d \geq 3$," *Communications in Mathematical Physics*, vol. 271, no. 2, pp. 523–559, 2007.
[17] J. Kato, "Existence and uniqueness of the solution to the modified Schrödinger map," *Mathematical Research Letters*, vol. 12, no. 2-3, pp. 171–186, 2005.
[18] J. Kato and H. Koch, "Uniqueness of the modified Schrödinger map in $H^{3/4+\epsilon}(R^2)$," *Communications in Partial Differential Equations*, vol. 32, no. 1–3, pp. 415–429, 2007.
[19] C. E. Kenig, G. Ponce, and L. Vega, "On the initial value problem for the Ishimori system," *Annales Henri Poincaré*, vol. 1, no. 2, pp. 341–384, 2000.
[20] C. E. Kenig and A. R. Nahmod, "The Cauchy problem for the hyperbolic-elliptic Ishimori system and Schrödinger maps," *Nonlinearity*, vol. 18, no. 5, pp. 1987–2009, 2005.
[21] H. McGahagan, "An approximation scheme for Schrödinger maps," *Communications in Partial Differential Equations*, vol. 32, no. 1–3, pp. 375–400, 2007.
[22] F. Merle, P. Raphaël, and I. Rodnianski, "Blow up dynamics for smooth data equivariant solutions to the energy critical Schrödinger map problem," http://arxiv.org/abs/1102.4308.
[23] F. Merle, P. Raphaël, and I. Rodnianski, "Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map," *Comptes Rendus Mathématique*, vol. 349, no. 5-6, pp. 279–283, 2011.
[24] A. Nahmod, J. Shatah, L. Vega, and C. Zeng, "Schrödinger maps and their associated frame systems," *International Mathematics Research Notices*, vol. 2007, no. 21, Article ID rnm088, 29 pages, 2007.
[25] A. Nahmod, A. Stefanov, and K. Uhlenbeck, "On Schrödinger maps," *Communications on Pure and Applied Mathematics*, vol. 56, no. 1, pp. 114–151, 2003.
[26] I. Rodnianski, Y. A. Rubinstein, and G. Staffilani, "On the global well-posedness of the one-dimensional Schrödinger map flow," *Analysis & PDE*, vol. 2, no. 2, pp. 187–209, 2009.
[27] I. Rodnianski and J. Sterbenz, "On the formation of singularities in the critical $O(3)\sigma$-model," *Annals of Mathematics*, vol. 172, no. 1, pp. 187–242, 2010.
[28] P. Smith, "Conditional global regularity of Schrödinger maps: sub-threshold dispersed energy," http://arxiv.org/abs/1012.4048.
[29] P. Smith, "Global regularity of critical Schrödinger maps: subthreshold dispersedenergy," http://arxiv.org/abs/1112.0251.
[30] T. Tao, "Gauges forthe Schrödinger map".
[31] T. Ozawa and J. Zhai, "Global existence of small classical solutions to nonlinear Schrödinger equations," *Annales de l'Institut Henri Poincaré*, vol. 25, no. 2, pp. 303–311, 2008.
[32] B. Guo and Y. Han, "Global regular solutions for Landau-Lifshitz equation," *Frontiers of Mathematics in China*, vol. 1, no. 4, pp. 538–568, 2006.
[33] X. Pu and B. Guo, "Well-posedness for the fractional Landau-Lifshitz equation without Gilbert damping," *Calculus of Variations and Partial Differential Equations*. In press.