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Research Article

Regularity Criteria for Hyperbolic Navier-Stokes and Related System

Jishan Fan¹ and Tohru Ozawa²

Correspondence should be addressed to Tohru Ozawa, txozawa@waseda.jp

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We prove a regularity criterion for strong solutions to the hyperbolic Navier-Stokes and related equations in Besov space.

1. Introduction

First, we consider the following hyperbolic Navier-Stokes equations [1]:

$$\tau u_{tt} + u_t - \Delta u + \nabla \pi + u \cdot \nabla u + \tau u_t \cdot \nabla u + \tau u \cdot \nabla u_t = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$(u, u_t)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n, n \ge 2.$$
 (1.3)

Here u is the velocity, π is the pressure, and $\tau > 0$ is a small relaxation parameter. We will take $\tau = 1$ for simplicity.

When $\tau=0$, (1.1) and (1.2) reduce to the standard Navier-Stokes equations. Kozono et al. [2] proved the following regularity criterion:

$$\omega := \operatorname{curl} \ u \in L^1\left(0, T; \dot{B}^0_{\infty, \infty}\right). \tag{1.4}$$

Here $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space.

¹ Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China

² Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan

Rack and Saal [1] proved the local well posedness of the problem (1.1)–(1.3). The global regularity is still open. The first aim of this paper is to prove a regularity criterion. We will prove the following theorem.

Theorem 1.1. Let $(u_0, u_1) \in H^{s+1} \times H^s$ with s > n/2, $n \ge 2$ and div $u_0 = \text{div } u_1 = 0$ in \mathbb{R}^n . Let (u, π) be a unique strong solution to the problem (1.1)–(1.3). If u satisfies

$$u, \nabla u, u_t \in L^1(0, T; \dot{B}^0_{\infty, \infty}), \tag{1.5}$$

then the solution u can be extended beyond T > 0.

In our proof, we will use the following logarithmic Sobolev inequality [2]:

$$||u||_{L^{\infty}} \le C \Big(1 + ||u||_{\dot{B}_{\infty,\infty}^0} \log(e + ||u||_{H^s})\Big)$$
 (1.6)

and the following bilinear product and commutator estimates according to Kato and Ponce [3]:

$$\|\Lambda^{s}(fg)\|_{L^{p}} \le C(\|f\|_{L^{p_{1}}} \|\Lambda^{s}g\|_{L^{q_{1}}} + \|\Lambda^{s}f\|_{L^{p_{2}}} \|g\|_{L^{q_{2}}}), \tag{1.7}$$

$$\|\Lambda^{s}(fg) - f\Lambda^{s}g\|_{L^{p}} \le C(\|\nabla f\|_{L^{p_{1}}} \|\Lambda^{s-1}g\|_{L^{q_{1}}} + \|\Lambda^{s}f\|_{L^{p_{2}}} \|g\|_{L^{q_{2}}}), \tag{1.8}$$

with s > 0, $\Lambda := (-\Delta)^{1/2}$ and $1/p = (1/p_1) + (1/q_1) = (1/p_2) + (1/q_2)$. Next, we consider the fractional Landau-Lifshitz equation:

$$\partial_t \phi = \phi \times \Lambda^{2\beta} \phi, \tag{1.9}$$

$$\phi(x,0) = \phi_0(x) \in \mathbb{S}^2, \quad x \in \mathbb{R}^n, \tag{1.10}$$

where $\phi \in \mathbb{S}^2$ is a three-dimensional vector representing the magnetization and β is a positive constant.

When β = 1, using the standard stereographic projection $\mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$, (1.9) can be rewritten as the derivative Schrödinger equation for $w \in \mathbb{C}$,

$$iw_t + \Delta w + 4 \frac{(\nabla w)^2}{1 + |w|^2} \overline{w} = 0.$$
(1.11)

Equation (1.9) is also called the Schrödinger map and has been studied by many authors [4–31]. Guo and Han [32] proved the following regularity criterion:

$$\nabla \phi \in L^2(0,T; L^\infty(\mathbb{R}^n)) \tag{1.12}$$

with $n \ge 2$.

When $0 < \beta \le 1/2$, Pu and Guo [33] show the local well posedness of strong solutions and the blow-up criterion

$$\Lambda^{2\beta}\phi \in L^1(0,T;L^\infty(\mathbb{R}^n)) \tag{1.13}$$

with $n \leq 3$.

We will refine (1.13) as follows.

Theorem 1.2. Let $0 < \beta \le 1/2$. Let m be an integer such that 2m > (n+1)/2 for any $n \ge 1$. Let $\Lambda^{\beta}\phi_0 \in H^{2m}$ and $\phi_0 \in \mathbb{S}^2$ and ϕ be a local smooth solution to the problem (1.9) and (1.10). If ϕ satisfies

$$\Lambda^{2\beta}\phi \in L^1(0,T;\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^n))$$
(1.14)

for some finite T > 0, then the solution ϕ can be extended beyond T > 0.

2. Proof of Theorem 1.1

Since (u, π) is a local smooth solution, we only need to prove a priori estimates. First, testing (1.1) by u and using (1.2), we see that

$$\frac{d}{dt} \int \left(\frac{1}{2}u^{2} + uu_{t}\right) dx + \int |\nabla u|^{2} dx$$

$$= \int u_{t}^{2} dx + \int u \cdot \nabla u \cdot u_{t} dx$$

$$\leq \int u_{t}^{2} dx + \frac{1}{2} ||\nabla u||_{L^{\infty}} \left(||u||_{L^{2}}^{2} + ||u_{t}||_{L^{2}}^{2}\right).$$
(2.1)

Testing (1.1) by $4u_t$ and using (1.2), we find that

$$\frac{d}{dt} \int (2u_t^2 + 2|\nabla u|^2) dx + 4 \int u_t^2 dx
= -4 \int (u \cdot \nabla u + u_t \cdot \nabla u) u_t dx
\leq C \|\nabla u\|_{L^{\infty}} (\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2).$$
(2.2)

Applying Λ^s to (1.1), testing by $\Lambda^s u_t$ and using (1.2), (1.7), (1.8), and (1.6), we have

$$\frac{1}{2} \frac{d}{dt} \int \left(\left| \Lambda^{s+1} u \right|^{2} + \left| \Lambda^{s} u_{t} \right|^{2} \right) dx + \int \left| \Lambda^{s} u_{t} \right|^{2} dx \\
= -\sum_{i} \int \Lambda^{s} \partial_{i} (u_{i} u) \cdot \Lambda^{s} u_{t} dx - \int \Lambda^{s} (u_{t} \cdot \nabla u) \cdot \Lambda^{s} u_{t} dx \\
- \sum_{i} \int \left[\Lambda^{s} \partial_{i} (u_{i} u_{t}) - u_{i} \partial_{i} \Lambda^{s} u_{t} \right] \Lambda^{s} u_{t} dx \\
\leq C \|u\|_{L^{\infty}} \|\Lambda^{s+1} u\|_{L^{2}} \|\Lambda^{s} u_{t}\|_{L^{2}} \\
+ C \left(\|u_{t}\|_{L^{\infty}} \|\Lambda^{s+1} u\|_{L^{2}} + \|\nabla u\|_{L^{\infty}} \|\Lambda^{s} u_{t}\|_{L^{2}} \right) \|\Lambda^{s} u_{t}\|_{L^{2}} \\
\leq C \left(\|u\|_{L^{\infty}} + \|\nabla u\|_{L^{\infty}} + \|u_{t}\|_{L^{\infty}} \right) \left(\|\Lambda^{s+1} u\|_{L^{2}}^{2} + \|\Lambda^{s} u_{t}\|_{L^{2}}^{2} \right) \\
\leq C \left(1 + \left(\|u\|_{\mathcal{B}^{0}_{\infty,\infty}} + \|\nabla u\|_{\mathcal{B}^{0}_{\infty,\infty}} + \|u_{t}\|_{\mathcal{B}^{0}_{\infty,\infty}} \right) \log \left(e + \|u\|_{H^{s+1}}^{2} + \|u_{t}\|_{H^{s}}^{2} \right) \right) \\
\cdot \left(\|\Lambda^{s} u_{t}\|_{L^{2}}^{2} + \|\Lambda^{s+1} u\|_{L^{2}}^{2} \right). \tag{2.3}$$

Combining (2.1), (2.2), and (2.3) and using the Gronwall inequality, we conclude that

$$||u||_{L^{\infty}(0,T;H^{s+1})} + ||u_t||_{L^{\infty}(0,T;H^s)} \le C.$$
(2.4)

This completes the proof.

3. Proof of Theorem 1.2

Since ϕ is a local smooth solution, we only need to prove a priori estimates. In this section, we denote by (\cdot, \cdot) the standard L^2 scalar product.

First, testing (1.9) by $\Lambda^{2\beta}\phi$ and using $(a \times b) \cdot a = 0$, we see that

$$\frac{1}{2}\frac{d}{dt}\int \left|\Lambda^{\beta}\phi\right|^2 dx = 0. \tag{3.1}$$

Testing (1.9) by $\Delta^{2m} \Lambda^{2\beta} \phi$ and using $(a \times b) \cdot a = 0$, (1.6) and (1.7), we obtain, with $(1/p) + (1/q) = (1/p_{\alpha}) + (1/q_{\alpha}) = (1/\widetilde{p}_{\alpha}) + (1/\widetilde{q}_{\alpha}) = 1/2$,

$$\begin{split} &\frac{1}{2}\frac{d}{dt}\int\left|\Delta^{m}\Lambda^{\beta}\phi\right|^{2}dx\\ &=\left(\phi\times\Lambda^{2\beta}\phi,\Delta^{2m}\Lambda^{2\beta}\phi\right)\\ &=\left(\Delta^{m}\left(\phi\times\Lambda^{2\beta}\phi\right),\Delta^{m}\Lambda^{2\beta}\phi\right)\\ &=\left(\Delta^{m}\left(\phi\times\Lambda^{2\beta}\phi\right),\Delta^{m}\Lambda^{2\beta}\phi\right)\\ &=\left(\Delta^{m}\phi\times\Lambda^{2\beta}\phi+\sum_{\alpha=1}^{2m-1}C_{\alpha}D^{2m-\alpha}\phi\times\Lambda^{2\beta}D^{\alpha}\phi,\Delta^{m}\Lambda^{2\beta}\phi\right)\\ &=\left(\Lambda^{\beta}\left(\Delta^{m}\phi\times\Lambda^{2\beta}\phi+\sum_{\alpha=1}^{2m-1}C_{\alpha}D^{2m-\alpha}\phi\times\Lambda^{2\beta}D^{\alpha}\phi\right),\Delta^{m}\Lambda^{\beta}\phi\right)\\ &\leq C\left\|\Lambda^{2\beta}\phi\right\|_{L^{\infty}}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}^{2}+C\left\|\Delta^{m}\phi\right\|_{L^{p}}\left\|\Lambda^{3\beta}\phi\right\|_{L^{q}}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}\\ &+\sum_{\alpha=1}^{2m-2}C_{\alpha}\left(\left\|D^{2m-\alpha}\phi\right\|_{L^{p_{\alpha}}}\left\|\Lambda^{3\beta}D^{\alpha}\phi\right\|_{L^{q_{\alpha}}}+\left\|D^{2m-\alpha}\Lambda^{\beta}\phi\right\|_{L^{p_{\alpha}}}\left\|\Lambda^{2\beta}D^{\alpha}\phi\right\|_{L^{q_{\alpha}}}\right)\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}\\ &\leq C\left\|\Lambda^{2\beta}\phi\right\|_{L^{\infty}}\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}^{2}\\ &\leq C\left(1+\left\|\Lambda^{2\beta}\phi\right\|_{\dot{B}^{0}_{\infty,\infty}}\log\left(e+\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}\right)\right)\left\|\Delta^{m}\Lambda^{\beta}\phi\right\|_{L^{2}}^{2}, \end{split} \tag{3.2}$$

which yields

$$\left\| \Lambda^{\beta} \phi(t) \right\|_{H^{2m}} \le C. \tag{3.3}$$

Here we have used the following interesting Gagliardo-Nirenberg inequalities:

$$\|\Delta^{m}\phi\|_{L^{p}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{1-\theta_{0}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{\theta_{0}} \quad \text{with } p = \frac{2m-\beta}{m-\beta}, \ \theta_{0} = \frac{2m-2\beta}{2m-\beta},$$

$$\|\Lambda^{3\beta}\phi\|_{L^{q}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{\theta_{0}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{1-\theta_{0}} \quad \text{with } q = \frac{2(2m-\beta)}{\beta},$$

$$\|D^{2m-\alpha}\phi\|_{L^{p_{\alpha}}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{1-\theta_{\alpha}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{\theta_{\alpha}} \quad \text{with } \theta_{\alpha} = \frac{2m-\alpha-2\beta}{2m-\beta}, \ p_{\alpha} = \frac{4m-2\beta}{2m-\alpha-2\beta},$$

$$\|\Lambda^{3\beta}D^{\alpha}\phi\|_{L^{q_{\alpha}}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{\theta_{\alpha}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{1-\theta_{\alpha}} \quad \text{with } q_{\alpha} = \frac{4m-2\beta}{\alpha+\beta},$$

$$\|D^{2m-\alpha}\Lambda^{\beta}\phi\|_{L^{p_{\alpha}}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{1-\tilde{\theta}_{\alpha}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{\tilde{\theta}_{\alpha}} \quad \text{with } \tilde{\theta}_{\alpha} = \frac{2m-\alpha-\beta}{2m-\beta}, \ \tilde{p}_{\alpha} = \frac{4m-2\beta}{2m-\alpha-\beta},$$

$$\|\Lambda^{2\beta}D^{\alpha}\phi\|_{L^{\tilde{q}_{\alpha}}} \leq C \|\Lambda^{2\beta}\phi\|_{L^{\infty}}^{\tilde{\theta}_{\alpha}} \|\Delta^{m}\Lambda^{\beta}\phi\|_{L^{2}}^{1-\tilde{\theta}_{\alpha}} \quad \text{with } \tilde{q}_{\alpha} = \frac{4m-2\beta}{\alpha}.$$

$$(3.4)$$

This completes the proof.

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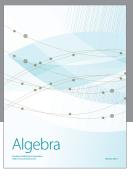
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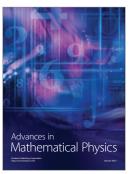


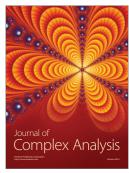




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