International Scholarly Research Network
ISRN Biomathematics
Volume 2012, Article ID 796783, 8 pages
doi:10.5402/2012/796783

Research Article

Effects of Additional Foods to Predators on Nutrient-Consumer-Predator Food Chain Model

Banshidhar Sahoo

Department of Mathematics, Daharpur A. P. K. B Vidyabhaban, Paschim Medinipur, West Bengal, India

Correspondence should be addressed to Banshidhar Sahoo, banshivu@gmail.com

Received 8 February 2012; Accepted 18 March 2012

Academic Editors: J. R. C. Piqueira and I. Rogozin

We have proposed a nutrient-consumer-predator model with additional food to predator, at variable nutrient enrichment levels. The boundedness property and the conditions for local stability of boundary and interior equilibrium points of the system are derived. Bifurcation analysis is done with respect to quality and quantity of additional food and consumer's death rate for the model. The system has stable as well as unstable dynamics depending on supply of additional food to predator. This model shows that supply of additional food plays an important role in the biological controllability of the system.

1. Introduction

The interactions between living and nonliving organisms have significant role in ecological modelling. In every ecosystem, there always have been material fluxes from the outside to the system as well as from the system to outside. In a hypothetical steady state, these nutrient inputs and outputs balance. Many mathematical models have included these interactions. Effects of nutrient enrichment on a food chain model have been investigated, both empirically as well as theoretically by many scientists [1–3]. These nutrients enrichment may reduce species diversity and ecosystem functioning [4]. Also, many researchers [5–7] have shown that nutrient enrichment can lead to a complex dynamics as well as extinctions of species. In the late 1970s, Pimm and Lawton [8] simulated a large number of food webs including omnivorous links, as nonlinear interactions. They discovered that these additional interactions in general stable internal equilibria to become statistically rare.

The role of additional food as a tool in biological control programs has become a topic of great attention for many scientists due to its ecofriendly nature. In recent years, many biologists, experimentalists, and theoreticians have concentrated on investigating the effects of providing additional food to predators in a predator-prey system [9–14]. Srinivasu et al. [12] have studied qualitative behavior of a predator-prey system in the presence of additional food to the predators, and they concluded that handling times for the available foods to the predator play a key role in determining the state of the ecosystem. In the controllability studies by Srinivasu et al. [12], it is observed that, for properly chosen quality and quantity of the additional food, the asymptotic state of a solution of the system can either be an equilibrium or a limit cycle. Sahoo and Poria [13] discussed the dynamic behaviour for seasonal effects on additional food in a predator-prey model. Very recently, Sahoo [14] discussed that existence of species in a system depends on interaction functions and supply of the quality of additional food. The decline of large predators at the top of the food chain has disrupted ecosystems all over the planet, according to a review of recent findings conducted by an international team of scientists and published in Science (Estes et al. [15]). They concluded that the loss of apex consumers may be the most pervasive influence on the natural world. Therefore, analysis of strategies related to effects of additional foods to predators is important in real world.

In this paper, we propose a model of nutrient-consumer-predator interaction (Figure 1) with additional food (characterized by predator's handling time) to predator, at variable nutrient enrichment levels. We have derived the existence and local stability conditions of boundary and interior equilibrium points of the system. We have analyzed

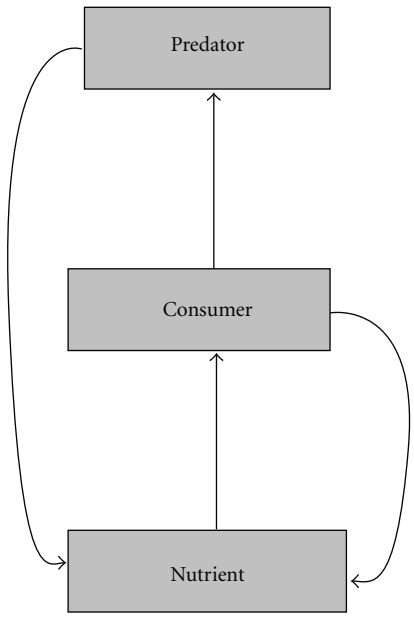

Figure 1: Diagrammatic representation of nutrient-consumer-predator interactions.

the behaviour of the proposed model through numerical simulations depending on some identified vital ecological parameters. We have done bifurcation analysis of our model with respect to quality and quantity of additional food and consumer's death rate, respectively, and finally conclusion is given.

2. Model Formulation

We formulate a nutrient-consumer-predator model as

$$\begin{aligned} \frac{dX}{dT} &= X(N^0 - aX) - A_1XY + w(E_1Y + E_2Z), \\ \frac{dY}{dT} &= A_2XY - A_3\frac{YZ}{B_1 + Y} - D_1Y, \\ \frac{dZ}{dT} &= A_4\frac{YZ}{B_1 + Y} - D_2Z. \end{aligned} \tag{1}$$

Here, X stands for the amount of nutrients present within the ecosystem, and Y and Z denote the number of the consumer species and predator, respectively. Here, T is time. Let N^0 be the constant rate of nutrient supply in the system; the constants A_1 and A_2 are conversion rates of nutrients supply to consumer; the constants A_3 and A_4 are conversion rates of consumer to predator for species Y and Z, respectively; D_1 and D_2 are constant death rates for species Y and Z respectively. The terms wE_1 and wE_2 are the nutrient regeneration rate from dead consumer and predator population. The constant B_1 is the half-saturation constant for Z.

If h_1 and e_1 are constants representing handling time of the predator Z per consumer item and ability of the predator to detect the consumer, then we have A_3 and B_1, representing the maximum predation rate and half-saturation values of the predator Z, to be $1/h_1$ and $1/e_1h_1$, respectively. If ϵ represents the efficiency with which the food consumed by the predator gets converted into predator biomass, then A_4, the maximum growth rate of the predator, is given by ϵ/h_1.

Now, we modify the model (1) by introducing "additional food" to predator population. We make the following assumptions:

(a) predator is provided with additional food of constant biomass A which is distributed uniformly in the habitat;

(b) the number of encounters per predator with the additional food is proportional to the density of the additional food;

(c) the proportionality constant characterizes the ability of the predator to identify the additional food.

Now, the modified model takes the following form:

$$\begin{aligned} \frac{dX}{dT} &= X(N^0 - aX) - A_1XY + w(E_1Y + E_2Z), \\ \frac{dY}{dT} &= A_2XY - A_3\frac{YZ}{B_1 + \alpha\mu A + Y} - D_1Y, \\ \frac{dZ}{dT} &= A_4\frac{(Y + \mu A)Z}{B_1 + \alpha\mu A + Y} - D_2Z. \end{aligned} \tag{2}$$

If h_2 represents the handling time of the predator Z per unit quantity of additional food, and e_2 represents the ability for the predator Z to detect the additional food, then we have $\mu = e_2/e_1$ and $\alpha = h_2/h_1$. The term μA represents effectual additional food level. The system has to be analyzed with the following conditions: $X(0) > 0$, $Y(0) > 0$, and $Z(0) > 0$.

To reduce the number of parameters and to determine which combinations of parameters control the behavior of the system, we nondimensionalize the system (2) with $N = X$, $C = Y/B_1$, $P = Z$, and $t = T$ and obtain the following system of equations:

$$\begin{aligned} \frac{dN}{dt} &= N(N^0 - aN) - \alpha_1NC + \omega(\gamma_1C + \gamma_2P), \\ \frac{dC}{dt} &= \alpha_2NC - \frac{\beta CP}{1 + \alpha\xi + C} - d_1C, \\ \frac{dP}{dt} &= \frac{\beta_1(C + \xi)P}{1 + \alpha\xi + C} - d_2P, \end{aligned} \tag{3}$$

where $\alpha_1 = A_1B_1$, $\alpha_2 = A_2B_1$, $\gamma_1 = E_1B_1$, $\gamma_2 = E_2$, $\beta = A_3$, $w = \omega$, $\beta_1 = A_4$, $\xi = \mu A/B_1$, $d_1 = D_1$, and $d_2 = D_2$. The system (3) has to be analyzed with the following initial conditions: $N(0) > 0$, $C(0) > 0$, and $P(0) > 0$.

The nutrient uptake rate per unit biomass of consumer per unit time is α_1. Nutrient involved in the system also undergos loss due to leaching at a rate a. Consumer growth rate per unit time is α_2. The terms $\omega\gamma_1$ and $\omega\gamma_2$ are the nutrient regeneration rate from dead consumer and predator population. It is assumed that input of external nutrient supply is dependent on the amount of nutrient present in the system.

Here, α represents the "quality" of the additional food (ratio between predator's handling time towards additional

food and consumer item), and ξ represents the "quantity" of the additional food for predator. The parameters α, ξ are the parameters which characterize the additional food. We do not make any distinction regarding the additional food like complementary, essential, or alternative. Here, we only assume that the predators are capable of reproducing by consuming the available food sources. Next, we shall analyze the dynamics of the model (3) theoretically and numerically.

3. Theoretical Study

In this section, positivity and boundedness for the system (3) are established. Since the state variables N, C, and P represent populations, positivity insures that they never become negative and population always survives. The boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources.

3.1. Positive Invariance. The system (3) can be put into the matrix form $F = F(\overline{X})$ with $\overline{X}(0) = \overline{X}_0 \in R^3_+$, where $\overline{X} = (N, C, P)^T \in R^3_+$. $F(\overline{X})$ is given by

$$F = F\left(\overline{X}\right) = \begin{pmatrix} N(N^0 - aN) - \alpha_1 NC + \omega(\gamma_1 C + \gamma_2 P) \\ \alpha_2 NC - \dfrac{\beta CP}{1+\alpha\xi + C} - d_1 C \\ \dfrac{\beta_1 (C+\xi)P}{1+\alpha\xi + C} - d_2 P \end{pmatrix}, \tag{4}$$

where $F : C_+ \to R^3$ and $F \in C^\infty(R^3)$.

It can be seen that whenever $\overline{X}(0) \in R^3_+$ such that, $X_i = 0$ then $F_i(\overline{X})|_{X_i=0} \geq 0$ (for $i = 1, 2, 3$). Now any solution of $F = F(\overline{X})$ with $\overline{X}_0 \in R^3_+$, say $\overline{X}(t) = \overline{X}(t, \overline{X}_0)$, is such that $\overline{X}(t) \in R^3_+$ for all $t > 0$ (Nagumo [16]).

3.2. Boundedness

Theorem 1. *All the solutions of the system* (3) *which start in* R^3_+ *are uniformly bounded.*

Proof. Let $(N(t), C(t), P(t))$ be any solution of the system (3) with positive initial conditions.

Let us consider that

$$\begin{aligned} w &= N + C + P, \\ \text{That is, } \frac{dw}{dt} &= \frac{dN}{dt} + \frac{dC}{dt} + \frac{dP}{dt}. \end{aligned} \tag{5}$$

Using (3), we have

$$\begin{aligned} \frac{dw}{dt} &= N(N^0 - aN) - \alpha_1 NC + \omega(\gamma_1 C + \gamma_2 P) + \alpha_2 NC \\ &\quad - \frac{\beta CP}{1+\alpha\xi + C} - d_1 C + \frac{\beta_1 (C+\xi)P}{1+\alpha\xi + C} - d_2 P. \end{aligned} \tag{6}$$

Therefore,

$$\begin{aligned} \frac{dw}{dt} &= NN^0 - aN^2 - (\alpha_1 - \alpha_2)NC + \omega(\gamma_1 C + \gamma_2 P) \\ &\quad - \frac{(\beta - \beta_1)CP}{1+\alpha\xi + C} + \frac{\beta_1 \xi P}{1+\alpha\xi + C} - d_1 C - d_2 P. \end{aligned} \tag{7}$$

Since $\alpha_1 \geq \alpha_2$ and $\beta \geq \beta_1$, we get the following expression:

$$\begin{aligned} \frac{dw}{dt} &\leq NN^0 + \omega(\gamma_1 C + \gamma_2 P) - d_1 C - d_2 P + \beta_1 \xi P, \\ \text{That is, } \frac{dw}{dt} &\leq 2NN^0 - NN^0 + \omega(\gamma_1 C + \gamma_2 P) - d_1 C \\ &\quad - d_2 P + \beta_1 \xi P, \\ \text{That is, } \frac{dw}{dt} &\leq (2NN^0 + \omega\gamma_1 C + \omega\gamma_2 P + \beta_1 \xi P) \\ &\quad - K(N + C + P), \end{aligned} \tag{8}$$

where $K = \min(N^0, d_1, d_2)$.

Hence,

$$\frac{dw}{dt} + Kw \leq \theta(2N^0 + \omega\gamma_1 + \omega\gamma_2 + \beta_1 \xi), \tag{9}$$

where $\theta = \max\{N(0), N^0/a,\ C(0), P(0)\}$.

Applying the theory of differential inequality, we obtain

$$\begin{aligned} 0 < w \leq{}& \frac{\theta(2N^0 + \omega\gamma_1 + \omega\gamma_2 + \beta_1 \xi)}{K}\left(1 - e^{-Kt}\right) \\ &+ w(N(0), C(0), P(0))e^{-Kt}. \end{aligned} \tag{10}$$

For $t \to \infty$, we have $0 < w \leq \theta(2N^0 + \omega\gamma_1 + \omega\gamma_2 + \beta_1 \xi)/K$.

Hence, all the solutions of (3) that initiate in R^3_+ are confined in the region

$$\begin{aligned} B = \Bigg\{ &(N, C, P) \in R^3_+ : \\ &0 < w \leq \frac{\theta(2N^0 + \omega\gamma_1 + \omega\gamma_2 + \beta_1 \xi)}{K} \Bigg\}. \end{aligned} \tag{11}$$

This proves the theorem. □

3.3. Existence and Local Stability of Boundary Equilibrium Points. The system (3) always has two boundary equilibrium points. $E_0(0, 0, 0)$ is the trivial equilibrium point. The axial equilibrium point is $E_1(N^0/a, 0, 0)$. The third boundary equilibrium point $E_2(\hat{N}, \hat{C}, 0)$ is the predator-free equilibrium point, where $\hat{N} = d_1/\alpha_2$ and $\hat{C} = d_1(N^0 - a(d_1/\alpha_2))/(\alpha_1 d_1 - \omega\gamma_1\alpha_2)$.

The predator-free equilibrium point E_2 exists if $d_1(N^0 - a(d_1/\alpha_2)) > 0$ and $(\alpha_1 d_1 - \omega\gamma_1\alpha_2) > 0$.

The Jacobian matrix J of the system (3) at any arbitrary point (N, C, P) is given by

$$J = \begin{pmatrix} F_{1N} & F_{1C} & F_{1P} \\ F_{2N} & F_{2C} & F_{2P} \\ F_{3N} & F_{3C} & F_{3P} \end{pmatrix}. \tag{12}$$

Theorem 2. *The trivial equilibrium point* E_0 *is always unstable. The axial equilibrium point* E_1 *is unstable if* $\alpha_2 N^0/a > d_1$ *and* $\beta_1\xi/(1+\alpha\xi) > d_2$. *The predator-free equilibrium point* E_2 *is locally stable if* $\beta_1(\hat{C}+\xi)/(1+\alpha\xi+\hat{C}) < d_2$, $N^0 - 2a\hat{N} - \alpha_1\hat{C} < 0$, *and* $\omega\gamma_1 < \alpha_1\hat{N}$.

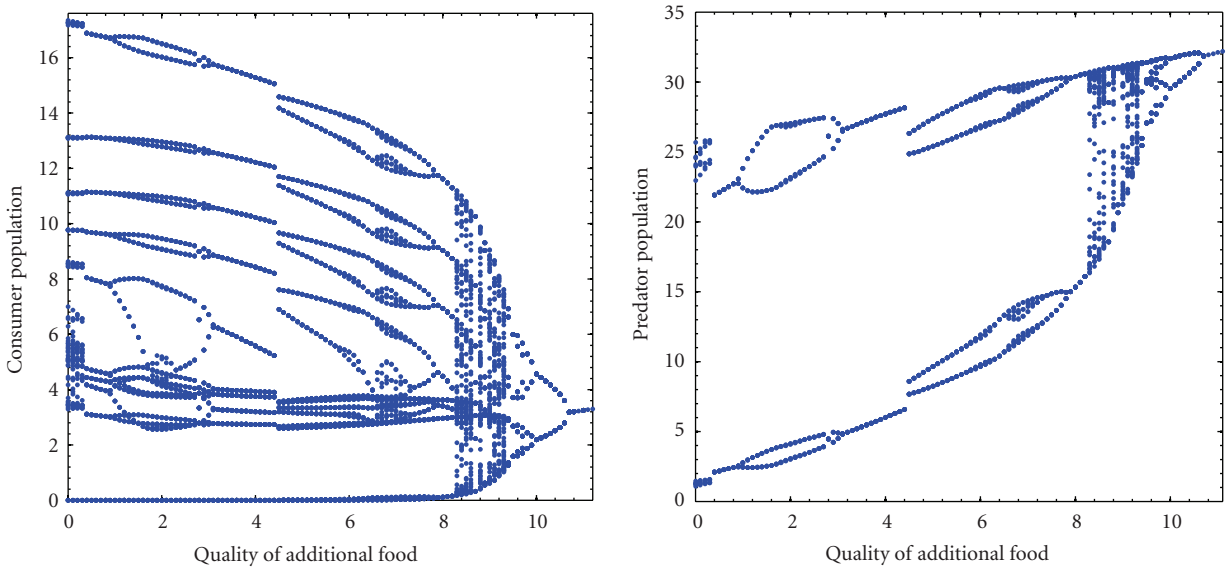

Figure 2: Bifurcation diagram of consumer and predator with respect to quality of additional food $\alpha \in [0, 11)$ keeping fixed $\xi = 0.2$ of the system (3) for $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_1 = 0.215$, $d_2 = 0.107$, and $\omega = 0.5$.

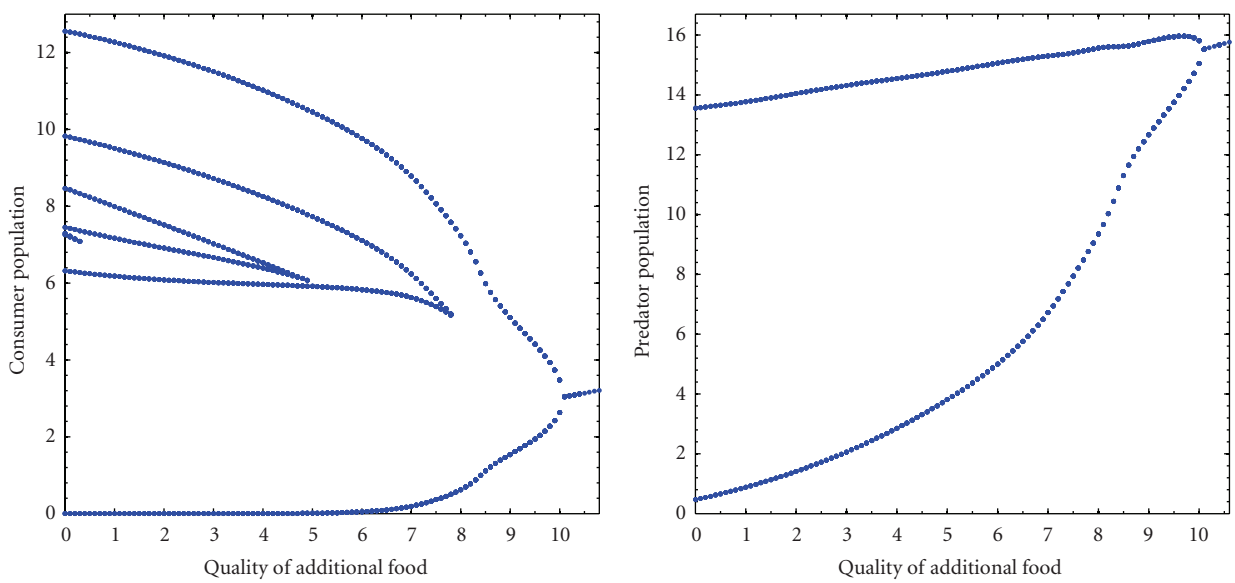

Figure 3: Bifurcation diagram of consumer and predator with respect to quality of additional food $\alpha \in [0.5, 7.5]$ taking $\xi = 0.2$, $d_1 = 0.115$, and the above set of other parameter values of the system (3).

Proof. The Jacobian matrix $J(E_0)$ at E_0 is given by

$$J(E_0) = \begin{pmatrix} N^0 & \omega\gamma_1 & \omega\gamma_2 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}, \tag{13}$$

which has one positive eigenvalue N^0 and two negativeeigen values $-d_1$ and $-d_2$, giving a point at the origin with nonempty stable manifolds and an unstable manifold. So, E_0 is always unstable.

The Jacobian matrix $J(E_1)$ at E_1 is given by

$$J(E_1) = \begin{pmatrix} -N^0 & \omega\gamma_1 - \dfrac{\alpha_1 N^0}{a} & \omega\gamma_2 \\ 0 & \dfrac{\alpha_2 N^0}{a} - d_1 & 0 \\ 0 & 0 & \dfrac{\beta_1 \xi}{1+\alpha\xi} - d_2 \end{pmatrix}. \tag{14}$$

From the Jacobian matrix $J(E_1)$, it is observed that it has one negative eigenvalue $(-N^0)$ and two positive eigenvalues if $\alpha_2 N^0/a > d_1$ and $\beta_1\xi/(1+\alpha\xi) > d_2$ and again has nonempty stable and unstable manifolds. Hence, the axial equilibrium point E_1 is unstable if $\alpha_2 N^0/a > d_1$ and $\beta_1\xi/(1+\alpha\xi) > d_2$.

The Jacobian matrix $J(E_2)$ at E_2 is given by

$$J(E_2) = \begin{pmatrix} N^0 - 2a\hat{N} - \alpha_1\hat{C} & \omega\gamma_1 - \alpha_1\hat{N} & \omega\gamma_2 \\ \alpha_2\hat{C} & 0 & \dfrac{-\beta_1\hat{C}}{1+\alpha\xi+\hat{C}} \\ 0 & 0 & \dfrac{\beta_1\left(\hat{C}+\xi\right)}{1+\alpha\xi+\hat{C}} - d_2 \end{pmatrix}. \tag{15}$$

The characteristic roots of the Jacobian matrix $J(E_2)$ are $(\beta_1(\hat{C}+\xi)/(1+\alpha\xi+\hat{C})) - d_2$ and roots of the equation

$$\lambda^2 - \left(N^0 - 2a\hat{N} - \alpha_1\hat{C}\right)\lambda - \alpha_2\hat{C}\left(\omega\gamma_1 - \alpha_1\hat{N}\right) = 0. \tag{16}$$

The predator-free equilibrium point E_2 is stable if $N^0 - 2a\hat{N} - \alpha_1\hat{C} < 0$ and $\omega\gamma_1 < \alpha_1\hat{N}$. Hence, the theorem is proved. □

3.4. Existence and Local Stability of Interior Equilibrium Point. The interior equilibrium point of the system (3) is given by $E^*(N^*, C^*, P^*)$, where $C^* = (d_2(1+\alpha\xi) - \beta_1\xi)/(\beta_1 - d_2)$, $P^* = ((1+\alpha\xi+C^*)/\beta_1)(\alpha_2N^* - d_1)$, and N^* is the positive root of the equation

$$PN^{*2} + QN^* + R = 0, \tag{17}$$

where $P = a$, $Q = \{(\alpha_1(d_2(1+\alpha\xi) - \beta_1\xi))/(\beta_1 - d_2) - N^0 - (\omega\gamma_2\alpha_2/\beta_1)(1+\alpha\xi+(d_2(1+\alpha\xi) - \beta_1\xi)/(\beta_1 - d_2))\}$, and $R = \omega d_1 - (\omega\gamma_1((d_2(1+\alpha\xi) - \beta_1\xi)/(\beta_1 - d_2)$.

The interior equilibrium point E^* exists if

$$d_2(1+\alpha\xi) > \beta_1\xi, \quad \beta_1 > d_2, \quad \alpha_2N^* > d_1, \quad Q^2 \geq 4PR. \tag{18}$$

Theorem 3. *The interior equilibrium point $E^*(N^*, C^*, P^*)$ for the system (3) is locally asymptotically stable if the following conditions hold: $\Omega_1 > 0$, $\Omega_3 > 0$, and $\Omega_1\Omega_2 - \Omega_3 > 0$, where*

$$\begin{aligned}
\Omega_1 &= -\left[N^0 - 2aN^* - \alpha_1C^* + \alpha_2N^* - \frac{\beta(1+\alpha\xi)(\alpha_2N^* - d_1)}{\beta_1(1+\alpha\xi+C^*)}\right], \\
\Omega_2 &= \left[\frac{(\alpha_2N^* - d_1)}{(1+\alpha\xi+C^*)} \cdot \frac{\beta(d_2(1+\alpha\xi) - \beta_1\xi)}{\beta_1} + (N^0 - 2aN^* - \alpha_1C^*) \cdot \left(\alpha_2N^* - \frac{\beta(1+\alpha\xi)(\alpha_2N^* - d_1)}{\beta_1(1+\alpha\xi+C^*)}\right) + (\alpha_1N^* - \omega\gamma_1)\alpha_2C^*\right], \\
\Omega_3 &= -\left[\frac{(\alpha_2N^* - d_1)(1+\alpha\xi-\xi)}{(1+\alpha\xi+C^*)^2} \cdot \left\{(N^0 - 2aN^* - \alpha_1C^*) \times \left(\frac{\beta(d_2(1+\alpha\xi) - \beta_1\xi)}{\beta_1(1+\alpha\xi-\xi)}\right) + \omega\gamma_2\alpha_2C^*\right\}\right].
\end{aligned} \tag{19}$$

Proof. The Jacobian matrix of the system (3) at the interior equilibrium point E^* is

$$J(E^*) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \tag{20}$$

where $A_{11} = N^0 - 2aN^* - \alpha_1C^*$, $A_{12} = \omega\gamma_1 - \alpha_1N^*$, $A_{13} = \omega\gamma_2$, $A_{21} = \alpha_2C^*$, $A_{22} = \alpha_2N^* - \beta(1+\alpha\xi)(\alpha_2N^* - d_1)/\beta_1(1+\alpha\xi+C^*)$, $A_{23} = -\beta(d_2(1+\alpha\xi) - \beta_1\xi)/\beta_1(1+\alpha\xi-\xi)$, $A_{31} = 0$, $A_{32} = (\alpha_2N^* - d_1)(1+\alpha\xi-\xi)/(1+\alpha\xi+C^*)$, $A_{33} = 0$. The characteristic equation of the Jacobian matrix E^* is given by

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0. \tag{21}$$

Using the Routh-Hurwitz criteria [17], we observe that the system (3) is stable around the positive equilibrium point E^* if the conditions $\Omega_1 > 0$, $\Omega_3 > 0$, and $\Omega_1\Omega_2 - \Omega_3 > 0$ hold. □

4. Numerical Study

For numerical simulation, we choose $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_1 = 0.215$, $d_2 = 0.107$, and $\omega = 0.5$ which remains the same for all numerical simulations. The remaining two parameters α (quality of additional food) and ξ (quantity of additional food) are varied to obtain different types of behaviours of the system.

4.1. Bifurcation Analysis with respect to the Quality of Additional Food α. We have done bifurcation analysis of the system (3) with respect to quality of additional food α within the range $0 \leq \alpha < 11$ taking $\xi = 0.2$ as fixed. From Figure 2, we observe that the system shows chaotic behaviour without any additional food. If we increase availability of the quality of additional food after $\alpha \geq 0.42$, the system shows periodic oscillations. The system again enters into chaotic region within $8.2 < \alpha < 9.4$, and after that, the system shows regular behaviour and finally settles down to steady state for $\alpha > 10.8$. However, from these bifurcation diagrams, we observe that increase of quality of additional food α up to a certain level reduces the prevalence of chaos, and the system enters into periodic region. Even, beyond a certain concentration level of food supply, the system will enter into a stable state. It shows that the consumer population has extinction risk for low quality of additional food. Figure 3 is the bifurcation diagram of the system with respect to quality of additional food α taking death rate $d_1 = 0.115$, $\gamma_1 = 0.1$ instead of $d_1 = 0.215$, $\gamma_1 = 0.2$. From Figure 3, we observe that the chaotic dynamics vanishes, and it shows oscillatory behaviour of the system (3) with suitable supply of additional food depending upon the death rate of the consumer.

4.2. Bifurcation Analysis with respect to the Quantity of Additional Food ξ. We have done bifurcation analysis with respect to quantity of additional food ξ within the range $0 \leq \xi \leq 2.6$ taking $\alpha = 2$. Figure 4 represents the bifurcation diagram of consumer and predator with respect to ξ for

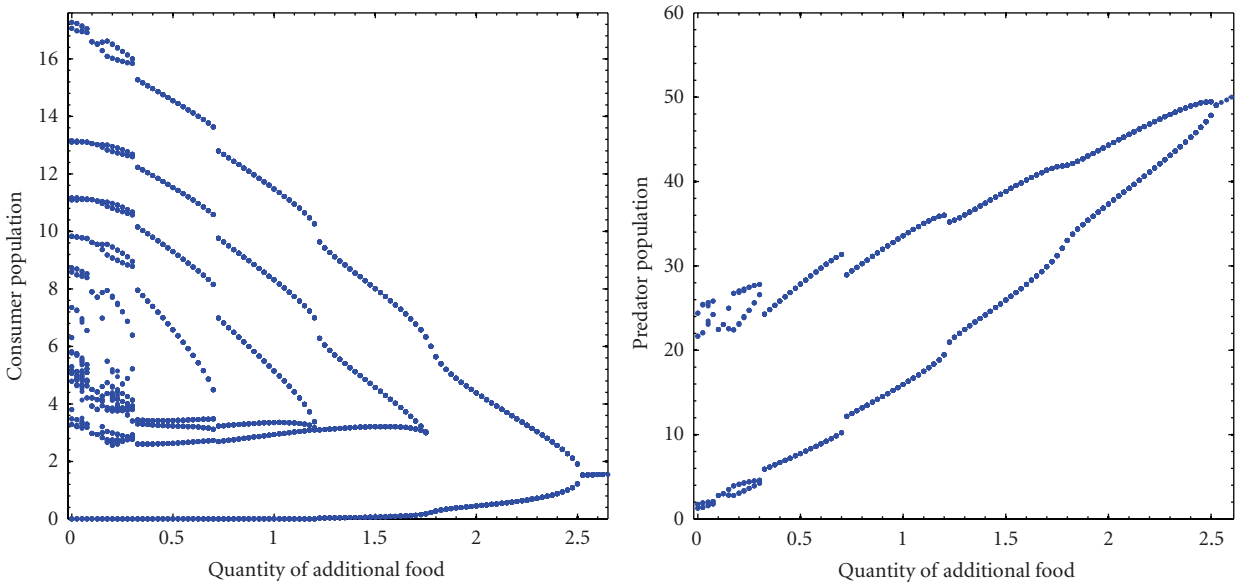

Figure 4: Bifurcation diagram of consumer and predator with respect to quantity of additional food $\xi \in [0, 2.6]$ keeping fixed $\alpha = 2$ of the system (3) for $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_1 = 0.215$, $d_2 = 0.107$, and $\omega = 0.5$.

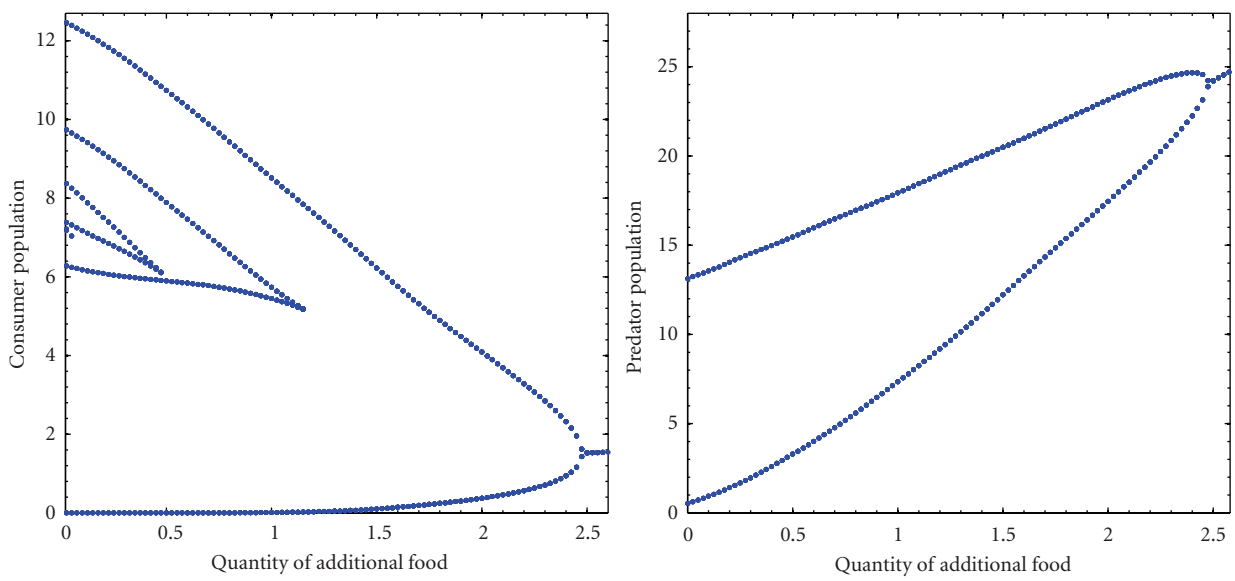

Figure 5: Bifurcation diagram of consumer and predator with respect to quantity of additional food $\xi \in [0.4, 7.5]$ taking $\alpha = 2$, $d_1 = 0.115$, and above set of other parameter values of the system (3).

fixed $\alpha = 2$. From Figure 4, we observe that in the absence of quantity of additional food ξ, that is, at $\xi = 0$, the system shows chaotic behaviour. After $\alpha > 0.3$, the system shows period 3, period 2, and limit cycle oscillation. With the increase of quantity of additional food ξ after certain level, system goes to steady state. If we take $\alpha = 2$, $d_1 = 0.115$, the chaos totally disappears from the system and shows periodic behaviour which is shown in Figure 5. From these diagrams, we conclude that the consumer population has extinction risk for small quantity of additional food, but it has stable behaviour for high quanity of additional food. The supply of additional food to predator decreases the predation pressure on consumer species, and as, a result, the consumer species survives.

4.3. Bifurcation Analysis with respect to Death Rate d_1 of Consumer. Figure 6 is the bifurcation diagram of the system (3) with respect to consumer's death rate d_1. The bifurcation diagram shows that some consumer species have high extinction risk in the system. Another consumers species survives due to presence of additional food to predator. On the other hand, predator populations have no extinction risk. They always survive in the system, but the growth rate of

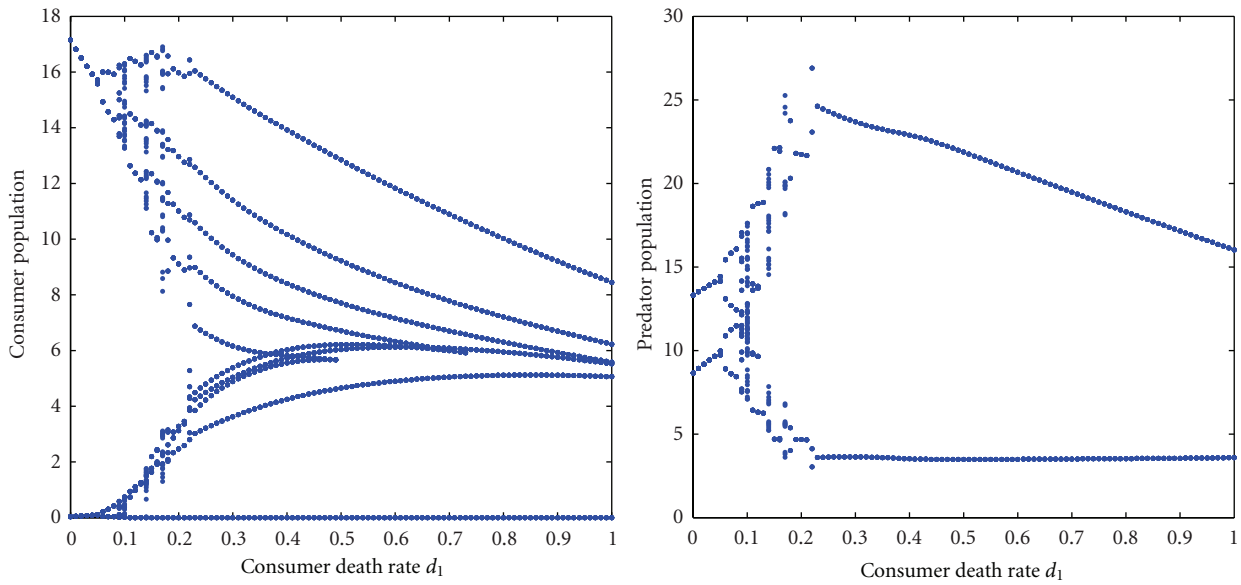

Figure 6: Bifurcation diagram of consumer and predator with respect to consumer's death rate $d_1 \in [0, 1]$ of the system (3) with additional food $\alpha = 2$ and $\xi = 0.2$ and for $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_2 = 0.107$, and $\omega = 0.5$.

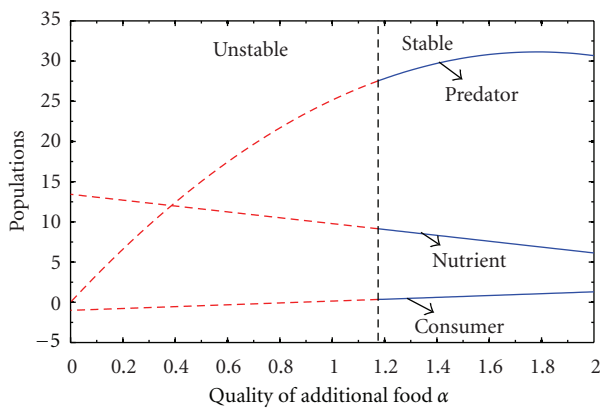

Figure 7: This figure illustrates that the smooth line indicates the stability and the dashed (-) line indicates the instability of the system (3) with respect to quality of additional food α for $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_1 = 0.215$, $d_2 = 0.107$, $\omega = 0.5$, and $\xi = 1$.

predator population decreases with the increase of death rate d_1 of consumer.

Figure 7 is the diagram of stable and unstable dynamics of the system (3) with respect to quality of additional food α for $N^0 = 2.5$, $a = 0.26$, $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.15$, $\beta = 0.4$, $\beta_1 = 0.2$, $d_1 = 0.215$, $d_2 = 0.107$, $\omega = 0.5$, and $\xi = 1$. The smooth lines indicate the stable dynamics, while the dashed lines indicate the unstable dynamics. From Figure 7, we can conclude that the system will have stable dynamic behaviour for proper choice of additional food.

5. Conclusions

In this paper, we have proposed a model of nutrient-consumer-predator interaction with additional food to predator. Here, we have derived boundedness criteria of our system. We have studied the existence and local stability conditions of boundary and interior equilibrium points of the system. We have done bifurcation analysis of our model with respect to quality of additional food α, quantity of additional food ξ, and death rate of consumer d_1 species, respectively. We observe that increasing quality and quantity of additional food supply, the system's chaos can be controlled.

Through the theoretical study and bifurcation analysis, we conclude that nutrient-consumer-predator system in the presence of additional food exhibits very rich dynamics. From the bifurcation diagrams, we observe that by varying quality and quantity of additional food we can control chaos of a food chain. Notice that the system becomes regular for some range of values of the death rate of consumer and regeneration rate of nutrient due to consumer's death. Therefore, the system's dynamics is sensitive to the death rate of consumer. An important observation the Figure 7 is that the system has stable and unstable dynamics with respect to quality of additional food. Therefore, the stability of a system highly depends on proper supply of additional food.

We observe that consumer species has extinction risk for low quality and small quantity of additional food, but consumer can survive only when we supply high quality and large quantity of additional food. This happened as predator is taking additional food, and the predation pressure on consumer is decreasing, and thus, consumer can servive and have a stable dynamic behaviour. Therefore for biological conservation, additional food may be very useful for servival

of consumer species in an ecosystem. So, we conclude that the proper choice of additional food to predator makes a food chain model more realistic, ecofriendly, and nonchaotic.

References

[1] P. A. Abrams and J. D. Roth, "The effects of enrichment of three-species food chains with nonlinear functional responses," *Ecology*, vol. 75, no. 4, pp. 1118–1130, 1994.

[2] M. Scheffer, S. H. Hosper, M. L. Meijer, and B. Moss, "Alternative equilibria in shallow lakes," *Trends in Ecology and Evolution*, vol. 8, pp. 352–356, 1993.

[3] D. L. DeAngelis, *Dynamics of Nutrient Cycling and Food Webs*, Chapman and Hall, London, UK, 1992.

[4] M. L. Rosenzweig, "Paradox of enrichment: destabilization of exploitation ecosystems in ecological time," *Science*, vol. 171, no. 3969, pp. 385–387, 1971.

[5] D. Ghosh and A. K. Sarkar, "Stability and oscillations in a resource-based model of two interacting species with nutrient cycling," *Ecological Modelling*, vol. 107, no. 1, pp. 25–33, 1998.

[6] G. Huxel and K. McCann, "Food web stability: the influence of trophic flows across habitats," *American Naturalist*, vol. 152, no. 3, pp. 460–469, 1998.

[7] R. Bhattacharyya, A. Bera, and B. Mukhopadhyay, "Role of different system delays on ecological food chain dynamics: mathematical modelling and analysis," *Journal of Systems Science and Complexity*, vol. 23, no. 4, pp. 727–737, 2010.

[8] S. L. Pimm and J. H. Lawton, "On feeding on more than one trophic level," *Nature*, vol. 275, no. 5680, pp. 542–544, 1978.

[9] J. D. Harwood and J. J. Obrycki, "The role of alternative prey in sustaining predator populations," in *Proceedings of the 2nd International Symposium on Biological Control of Arthropods*, M. S. Hoddle, Ed., vol. 2, pp. 453–462, 2005.

[10] M. W. Sabelis and P. C. J. van Rijn, "When does alternative food promote biological pest control?" in *Proceedings of the 2nd International Symposium on Biological Control of Arthropods*, M. S. Hoddle, Ed., vol. 2, pp. 428–437, 2005.

[11] M. van Baalen, V. Křivan, P. C. J. van Rijn, and M. W. Sabelis, "Alternative food, switching predators, and the persistence of predator-prey systems," *American Naturalist*, vol. 157, no. 5, pp. 512–524, 2001.

[12] P. D. N. Srinivasu, B. S. R. V. Prasad, and M. Venkatesulu, "Biological control through provision of additional food to predators: a theoretical study," *Theoretical Population Biology*, vol. 72, no. 1, pp. 111–120, 2007.

[13] B. Sahoo and S. Poria, "Dynamics of a predator-prey system with seasonal effects on additional food," *International Journal of Ecosystem*, vol. 1, no. 1, pp. 10–13, 2011.

[14] B. Sahoo, "A predator-prey model with general holling interactions in presence of additional food," *International Journal of Plant Research*, vol. 2, no. 1, pp. 47–50, 2012.

[15] J. A. Estes, J. Terborgh, J. S. Brashares et al., "Trophic downgrading of planet earth," *Science*, vol. 333, no. 6040, pp. 301–306, 2011.

[16] M. Nagumo, "Uber die Lage der Integralkurven gew onlicher Differentialgleichungen," *Journal of the Physical Society of Japan*, vol. 24, pp. 551–559, 1942.

[17] H. Freedman and P. Waltman, "Persistence in models of three interacting predator-prey populations," *Mathematical Biosciences*, vol. 68, no. 2, pp. 213–231, 1984.