Research Article

Symmetry Reduction, Exact Solutions, and Conservation Laws of the ZK-BBM Equation

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Employing the classical Lie method, we obtain the symmetries of the ZK-BBM equation. Applying the given Lie symmetry, we obtain the similarity reduction, group invariant solution, and new exact solutions. We also obtain the conservation laws of ZK-BBM equation of the corresponding Lie symmetry.

1. Introduction

Symmetry is one of the most important concepts in the area of partial differential equations, especially in integrable systems, which exist infinitely in many symmetries. These symmetry group techniques provide one method for obtaining exact and special solutions of a given partial differential equation. To find the Lie point symmetry of a nonlinear equation, some effective methods have been introduced, such as the classical Lie group approach, the nonclassical Lie group approach, the Clarkson-Kruskal (CK) direct symmetry method, and the compatibility method [1, 2].

The purpose of this paper is to apply the classical Lie method to the following Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation:

$$u_t + u_x + auu_x + b(u_{xxt} + u_{yyx}) = 0, \quad (1.1)$$

which was presented by Wazwaz [3, 4]. Wazwaz obtained some solitons solutions, periodic solutions, and complex solutions by using the sine-cosine method and the extended tanh

method. Abdou [5] used the extended F -expansion method to construct exact solutions. Song and Yang [6] employed the bifurcation method of dynamical systems to obtain traveling wave solutions.

This paper is arranged as follows. In Section 2, we get the symmetries and group invariant solutions of (1.1). In Section 3, the reductions of (1.1) are obtained, and in Section 4, we derive some new exact solutions of (1.1). Finally, the conservation laws of (1.1) will be presented in Section 5.

2. Symmetry of the ZK-BBM Equation

The main task of the classical Lie method is to seek some symmetries and look for exact solutions of a given partial differential equation. The vector field of (1.1) can be expressed as

$$V = \xi(x, y, t, u) \frac{\partial}{\partial x} + \zeta(x, y, t, u) \frac{\partial}{\partial y} + \tau(x, y, t, u) \frac{\partial}{\partial t} + \eta(x, y, t, u) \frac{\partial}{\partial u}, \quad (2.1)$$

and its third prolongation can be given as

$$\text{Pr}^{(3)}V = V + \phi^t \frac{\partial}{\partial u_t} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xxt} \frac{\partial}{\partial u_{xxt}} + \phi^{yyx} \frac{\partial}{\partial u_{yyx}}, \quad (2.2)$$

where $\phi^t, \phi^x, \phi^{xxt}, \phi^{yyx}$ are given explicitly in terms of ξ, ζ, τ, ϕ , and the derivatives of ϕ .

From $\text{Pr}^{(3)}VF|_{F=0} = 0$, it follows

$$\phi^t + \phi^x + au_x\phi + au\phi^x + b\phi^{xxt} + b\phi^{yyx} = 0. \quad (2.3)$$

Letting the coefficients of the polynomial be zero yields a set of differential equations of the functions ξ, ζ, τ , and ϕ . Solving these equations, one can arrive at

$$\xi = c_3, \quad \zeta = \frac{1}{2}c_1y + c_4, \quad \tau = c_1t + c_2, \quad \phi = -c_1 \frac{au + 1}{a}, \quad (2.4)$$

where c_1, c_2, c_3 , and c_4 are arbitrary constants.

The corresponding symmetries are

$$\sigma = c_3u_x + \left(\frac{1}{2}c_1y + c_4\right)u_y + (c_1t + c_2)u_t - c_1 \frac{au + 1}{a}u. \quad (2.5)$$

The vector field can be written as

$$v = c_3 \frac{\partial}{\partial x} + \left(\frac{1}{2}c_1y + c_4\right) \frac{\partial}{\partial y} + (c_1t + c_2) \frac{\partial}{\partial t} - c_1 \frac{au + 1}{a} \frac{\partial}{\partial u}. \quad (2.6)$$

In order to get some exact solutions from the known ones of (1.1), we will find the corresponding Lie symmetry groups. To this end, solving the following initial problems:

$$\begin{aligned}\frac{d}{d\epsilon}(\bar{x}, \bar{y}, \bar{t}, \bar{u}) &= (x, y, t, u), \\ (\bar{x}, \bar{y}, \bar{t}, \bar{u})|_{\epsilon=0} &= (x, y, t, u),\end{aligned}\tag{2.7}$$

where ϵ is a parameter.

From (2.7) we can obtain the Lie symmetry group $g : (\bar{x}, \bar{y}, \bar{t}, \bar{u}) \rightarrow (x, y, t, u)$. According to different ξ , ζ , τ , and ϕ in σ , we have the following group by solving (2.7):

$$\begin{aligned}g_1 : (\bar{x}, \bar{y}, \bar{t}, \bar{u}) &\longrightarrow (x + \epsilon, y, t, u), \\ g_2 : (\bar{x}, \bar{y}, \bar{t}, \bar{u}) &\longrightarrow (x, y + \epsilon, t, u), \\ g_3 : (\bar{x}, \bar{y}, \bar{t}, \bar{u}) &\longrightarrow (x, y, t + \epsilon, u), \\ g_4 : (\bar{x}, \bar{y}, \bar{t}, \bar{u}) &\longrightarrow \left(x, y + e^{(1/2)\epsilon}, t + e^\epsilon, u + e^{-\epsilon} - \frac{1}{a}\right).\end{aligned}\tag{2.8}$$

We can obtain the corresponding new solutions by applying the above groups g_1 , g_2 , g_3 , and g_4 as follows:

$$\begin{aligned}u_1 &= f(x - \epsilon, y, t), \\ u_2 &= f(x, y - \epsilon, t), \\ u_3 &= f(x, y, t - \epsilon), \\ u_4 &= f\left(x, y - e^{-(1/2)\epsilon}, t - e^{-\epsilon}\right) - \frac{1}{a} + e^{-\epsilon}.\end{aligned}\tag{2.9}$$

For example, taking the following periodic wave solution [3] of (1.1)

$$u(x, y, t) = \frac{3(c-1)}{2a} \sec^2\left(\frac{1}{2\sqrt{b}}\xi\right),\tag{2.10}$$

where $b > 0$ and $\xi = x + y - ct$.

One can obtain new exact solution of (1.1) by applying u_4 as follows:

$$u(x, y, t) = \frac{3(c-1)}{2a} \sec^2\left(\frac{1}{2\sqrt{b}}\xi\right) - \frac{1}{a} + e^{-\epsilon},\tag{2.11}$$

where $b > 0$ and $\xi = x + y - e^{-(1/2)\epsilon} - c(t - e^{-\epsilon})$.

3. Reductions of ZK-BBM Equation

Now we will reduce (1.1) by means of the symmetries (2.4). Here we discuss the following cases.

Case 1. Letting $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$. Then

$$\sigma = \frac{1}{2}c_1 y u_y + c_1 t u_t - c_1 \frac{au + 1}{a} u. \quad (3.1)$$

Solving the differential equation $\sigma = 0$, one can get

$$\xi = x, \quad \eta = \frac{y^2}{t}, \quad u = \frac{1}{t} f(\xi, \eta) - \frac{1}{a}. \quad (3.2)$$

Substituting (3.2) into (1.1), we have

$$-f - \eta f_\eta + f f_\xi + b(-f_{\xi\xi} - \eta f_{\xi\xi\eta} + 2f_{\eta\xi} + 4\eta f_{\eta\eta\xi}) = 0. \quad (3.3)$$

Case 2. Letting $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$, $c_4 \neq 0$. Then

$$\sigma = c_3 u_x + c_4 u_y + c_2 u_t. \quad (3.4)$$

Solving the differential equation $\sigma = 0$, one can get

$$\xi = x - \frac{c_3}{c_2} t, \quad \eta = y - \frac{c_4}{c_2} t, \quad u = f(\xi, \eta). \quad (3.5)$$

Substituting (3.5) into (1.1), one can reduce (1.1) to the following equation:

$$\left(1 - \frac{c_3}{c_2}\right) f_\xi - \frac{c_4}{c_2} f_\eta + a f f_\xi + b \left(-\frac{c_3}{c_2} f_{\xi\xi\xi} - \frac{c_4}{c_2} f_{\xi\xi\eta} + f_{\eta\eta\xi} \right) = 0. \quad (3.6)$$

Case 3. Letting $c_1 = 0$, $c_2 \neq 0$, $c_3 = 0$, $c_4 \neq 0$. Then

$$\sigma = c_4 u_y + c_2 u_t. \quad (3.7)$$

Solving the differential equation $\sigma = 0$, one can get

$$\xi = x, \quad \eta = y - \frac{c_4}{c_2} t, \quad u = f(\xi, \eta). \quad (3.8)$$

Substituting (3.8) into (1.1), we have

$$f_\xi - \frac{c_4}{c_2} f_\eta + a f f_\xi + b \left(-\frac{c_4}{c_2} f_{\xi\xi\eta} + f_{\eta\eta\xi} \right) = 0. \quad (3.9)$$

4. Similarity Solutions of ZK-BBM Equation

By solving the reduced equations (3.3), (3.6), and (3.9), we can get some new explicit solutions of (1.1). Now we discuss some cases in the following.

Case 1. Assuming (3.3) has the following solution:

$$f = r(\eta)\xi + \omega(\eta), \quad (4.1)$$

where $r(\eta)$ and $\omega(\eta)$ are the functions to be determined. Substituting (4.1) into (3.3) yields

$$f = \frac{1}{1 + c_1\eta}\xi + \omega(\eta), \quad (4.2)$$

then the new exact solution of (1.1) is expressed as

$$u_1 = \frac{1}{a(t + d_1 y^2)} \left(x + 2bd_1 \ln(t + d_1 y^2) - 4bd_1 \ln y + d_2 \right) - \frac{8btd_1}{a(t + d_1 y^2)^2} - \frac{1}{a}, \quad (4.3)$$

where d_1 and d_2 are constants.

Case 2. To solve (3.6), we apply the G'/G -expansion method [7] and look for the travelling wave solution of (3.6) by setting

$$f = \sum_{i=1}^N \alpha_i \left(\frac{G'(\omega)}{G(\omega)} \right)^i + \alpha_0, \quad \alpha_m \neq 0, \quad (4.4)$$

where $\omega = k\xi + l\eta$, α_i are constants to be determined later. It can be seen $m = 2$ by balancing f_ξ and ff_ξ in (3.4). Suppose that the solutions of (3.4) are of the form

$$f = \alpha_2 \left(\frac{G'(\omega)}{G(\omega)} \right)^2 + \alpha_1 \left(\frac{G'(\omega)}{G(\omega)} \right) + \alpha_0, \quad \alpha_2 \neq 0 \quad (4.5)$$

with $G(\omega)$ satisfying the second-order linear ordinary differential equation

$$G''(\omega) + \lambda G'(\omega) + \mu G(\omega) = 0, \quad (4.6)$$

where $\omega = k\xi + l\eta$, $\alpha_0, \alpha_1, \alpha_2, \lambda$, and μ are constants to be determined later. Substituting (4.5) into (3.6) along with (4.6) and setting the coefficients of $(G'(\omega)/G(\omega))^i$, ($i = 0, \dots, 4$) to zero yields a system of equations with respect to $\alpha_0, \alpha_1, \alpha_2, h$, and l . Solving the corresponding algebraic equations and using (3.5), (4.5), and (4.6), we get three types of traveling wave solutions of (1.1) as follows.

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} u_2 = & \frac{3b}{ac_2} (\lambda^2 - 4\mu) \left(c_3k + c_4l - c_2 \frac{l^2}{k} \right) \frac{d_1 \sin h\omega + d_2 \cos h\omega}{d_1 \cos h\omega + d_2 \sin h\omega} \\ & + \frac{2b}{ac_2} (\lambda^2 - 4\mu) \left(-c_3k - c_4l + c_2 \frac{l^2}{k} \right) + \frac{1}{ac_2} \left(c_4 \frac{l}{k} - c_2 + c_3 \right). \end{aligned} \quad (4.7)$$

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} u_3 = & -\frac{3b}{ac_2} (4\mu - \lambda^2) \left(c_3k + c_4l - c_2 \frac{l^2}{k} \right) \frac{d_1 \sin h\omega + d_2 \cos h\omega}{d_1 \cos h\omega + d_2 \sin h\omega} \\ & + \frac{2b}{ac_2} (4\mu - \lambda^2) \left(-c_3k - c_4l + c_2 \frac{l^2}{k} \right) + \frac{1}{ac_2} \left(c_4 \frac{l}{k} - c_2 + c_3 \right). \end{aligned} \quad (4.8)$$

When $\lambda^2 - 4\mu = 0$,

$$u_4 = \frac{12b}{ac_2} \left(c_3k + c_4l - c_2 \frac{l^2}{k} \right) \frac{d_1 \sin h\omega + d_2 \cos h\omega}{d_1 \cos h\omega + d_2 \sin h\omega} + \frac{1}{ac_2} \left(c_4 \frac{l}{k} - c_2 + c_3 \right), \quad (4.9)$$

where $\omega = kx + ly - ((kc_3 + lc - 4)/c_2)t$.

Case 3. Let $f = f(\omega)$, where $\omega = k\xi + l\eta$. Equation (3.9) becomes the following ordinary partial differential equation:

$$\left(k - \frac{c_4}{c_2} l \right) f' + akf f' + b \left[-\frac{c_4}{c_2} k^2 l + kl^2 \right] f''' = 0. \quad (4.10)$$

Integrating twice with respect to ω in (4.10), we get

$$(f')^2 = A_1 f^3 + A_2 f^2 + A_3 f, \quad (4.11)$$

where $A_1 = c_2 a / 3bl(c_4k - c_2l)$, $A_2 = (c_2k - c_4l) / bkl(c_4k - c_2l)$, and A_3 is a constant.

Since solutions of (4.11) have been given [8], we have some similarity solutions of (1.1) as follows.

When $A_1 = 4m^2$, $A_2 = -4(m^2 + 1)$, $A_3 = 4$,

$$u_5 = sn^2 \left[kx + ly - \frac{c_4}{c_2} lt \right]. \quad (4.12)$$

When $A_1 = 4m^2$, $A_2 = -4(2m^2 - 1)$, $A_3 = 4(1 - m^2)$,

$$u_6 = cn^2 \left[kx + ly - \frac{c_4}{c_2} lt \right]. \quad (4.13)$$

When $A_1 = -4$, $A_2 = 4(2 - m^2)$, $A_3 = -4(1 - m^2)$,

$$u_7 = dn^2 \left[kx + ly - \frac{c_4}{c_2} lt \right]. \quad (4.14)$$

When $A_1 f < 0$, $A_2 > 0$, $A_3 = 0$,

$$u_8 = -\frac{B_2}{B_1} \operatorname{sech}^2 \left[\frac{\sqrt{B_2}}{2} \left(kx + ly - \frac{c_4}{c_2} lt \right) \right]. \quad (4.15)$$

When $A_1 f > 0$, $A_2 > 0$, $A_3 = 0$,

$$u_9 = \frac{B_2}{B_1} \operatorname{csch}^2 \left[\frac{\sqrt{B_2}}{2} \left(kx + ly - \frac{c_4}{c_2} lt \right) \right]. \quad (4.16)$$

When $A_2 < 0$, $A_3 = 0$,

$$u_{10} = \frac{B_2}{B_1} \sec^2 \left[\frac{\sqrt{B_2}}{2} \left(kx + ly - \frac{c_4}{c_2} lt \right) \right]. \quad (4.17)$$

5. Conservation Laws of ZK-BBM Equation

In this section we will study the conservation laws by using the adjoint equation and symmetries of ZK-BBM equation. For (1.1), the adjoint equation has the form

$$v_t + v_x + auv_x + bv_{xxt} + bv_{yyx} = 0, \quad (5.1)$$

and the Lagrangian is

$$L = v(u_t + u_x + auu_x + bu_{xx}v_t + bu_{yy}v_x). \quad (5.2)$$

Since every Lie point, Lie-backlund and nonlocal symmetry of (1.1), provides a conservation law for (1.1) and the adjoint equation [9], the elements of conservation vector (C_1, C_2, C_3) are defined by the following expression:

$$\begin{aligned} C^i = & \xi^i L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right] \\ & + D_j (W^\alpha) \left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right] + D_j D_k (W^\alpha) \frac{\partial L}{\partial u_{ijk}^\alpha} \quad (i = 1, 2, 3), \end{aligned} \quad (5.3)$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

The conserved vector corresponding to an operator

$$v = \xi^1(x, y, t, u) \frac{\partial}{\partial x} + \xi^2(x, y, t, u) \frac{\partial}{\partial y} + \xi^3(x, y, t, u) \frac{\partial}{\partial t} + \eta(x, y, t, u) \frac{\partial}{\partial u}. \quad (5.4)$$

The operator v yields the conservation law

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0, \quad (5.5)$$

where the conserved vector $C = (C^1, C^2, C^3)$ is given by (5.3) and has the components

$$\begin{aligned} C^1 &= w^1[v(1 + au) - D_x(bvv_t)] + ku_{yy}vw^2 + bvv_tD_x(w^1), \\ C^2 &= \xi^2 L - D_y(kvv_x)w^1 + bvv_x D_y(w^1), \\ C^3 &= \xi^3 L + w^1 + bu_{xx}vw^2. \end{aligned} \quad (5.6)$$

Thus, (5.6) define components of a nonlocal conservation law for the system of (1.1), (5.1) corresponding to any operator v admitted by (1.1).

Let us make more detailed calculations for the operator $v = (1/2)y\partial_y + t\partial_t - ((au + 1)/a)\partial_u$. For this operator, we have $w^1 = -(au + 1)/a - ((1/2)y)u_y - tv_t$, $w^2 = (1/2)v - ((1/2)y)v_y - tv_t$ and (5.3) written for the sixth-order Lagrangian equation (5.2) yields the conserved vector

$$\begin{aligned} C^1 &= -\left(u + \frac{1}{a} + \frac{1}{2}yu_y + tu_t\right)(v + auv - bv_xv_t - bv_{tx}) \\ &\quad + bu_{yy}v\left(\frac{1}{2}v - \frac{1}{2}yv_y - tv_t\right) - bv_t\left(u_x + \frac{1}{2}yu_{yx} + tu_{tx}\right), \\ C^2 &= \frac{1}{2}yv(u_t + u_x + auu_x + bu_{xx}v_t + bu_{yy}v_x) - b(v_yv_x + vv_{xy}) \\ &\quad \cdot \left(u + \frac{1}{a} + \frac{1}{2}yu_y + tu_t\right) - bv_v\left(u_y + \frac{1}{2}u_y + \frac{1}{2}yu_{yy} + tu_{ty}\right), \\ C^3 &= tv(u_t + u_x + auu_x + bu_{xx}v_t + bu_{yy}v_x) \\ &\quad - \left(u + \frac{1}{a} + \frac{1}{2}yu_y + tu_t\right) + bu_{xx}v\left(\frac{1}{2}v - \frac{1}{2}yv_y - tv_t\right). \end{aligned} \quad (5.7)$$

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