International Scholarly Research Network
ISRN Discrete Mathematics
Volume 2012, Article ID 850538, 7 pages
doi:10.5402/2012/850538

Research Article
Splitting Lemma for 2-Connected Graphs

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Received 1 October 2012; Accepted 17 October 2012

Academic Editors: S. Bozapalidis, E. Kranakis, and W. Wang

Using a splitting operation and a splitting lemma for connected graphs, Fleischner characterized connected Eulerian graphs. In this paper, we obtain a splitting lemma for 2-connected graphs and characterize 2-connected Eulerian graphs. As a consequence, we characterize connected graphic Eulerian matroids.

1. Introduction

Fleischner [1] introduced a splitting operation to characterize Eulerian graphs as follows. Let G be a connected graph and $v \in V(G)$ with $d(v) \geq 3$. If $x = vv_1$ and $y = vv_2$ are two edges incident with v, then splitting away the pair $\{x, y\}$ of edges from the vertex v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2 (see Figure 1).

The following splitting lemma established by Fleischner [1] has been widely recognized as a useful tool in the graph theory.

Splitting Lemma 1.1 (see [1, page III-29]). *Let G be a connected bridgeless graph. Suppose $v \in V(G)$ such that $d(v) \geq 3$ and x, y, z are the edges incident with v. Form the graphs $G_{x,y}$ and $G_{x,z}$ by splitting away the pairs $\{x, y\}$ and $\{x, z\}$, respectively, and assume x and z belong to different blocks if v is a cut vertex of G. Then either $G_{x,y}$ or $G_{x,z}$ is connected and bridgeless.*

This lemma is used to obtain the following characterization of Eulerian graphs.

Theorem 1.2 (see [1, page V-6]). *A graph G has an Eulerian trail T if and only if G can be transformed into a cycle C through repeated applications of the splitting procedure on vertices of a degree exceeding* 2. *Moreover, the number of Eulerian trails of G equals the number of different labeled cycles into which G can be transformed this way.*

Thus a connected graph G is Eulerian if and only if there exists a sequence $G = G_0 < G_1 < \cdots < G_n$ of connected graphs such that G_n is a cycle and G_{i+1} is obtained from G_i by applying splitting operation once.

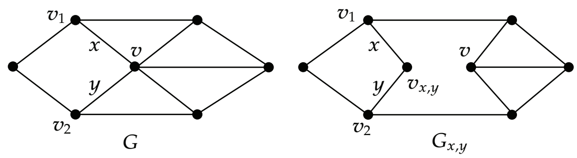

Figure 1

The splitting operation may not preserve 2-connectedness of the graph. Consider the graph G of Figure 2. It is 2-connected but the graph $G_{x,y}$ is not 2-connected for any two adjacent edges x and y.

We obtain the splitting lemma for 2-connected graphs as follows.

Theorem 1.3. *Let G be a 2-connected graph and let v be a vertex of G with $d(v) \geq 4$. Then either $G_{x,y}$ is 2-connected for some pair $\{x, y\}$ of edges incident with v or for any pair $\{x_1, y_1\}$ of edges incident with v; there is another pair $\{x_2, y_2\}$ of adjacent edges of G such that $(G_{x_1,y_1})_{x_2,y_2}$ is 2-connected.*

The next theorem is a consequence of the above result.

Theorem 1.4. *Let G be a 2-connected graph. Then G is Eulerian if and only if there exists a sequence of 2-connected graphs $G = G_0 < G_1 < G_2 < \cdots < G_n$ such that G_n is a cycle and G_{i+1} is obtained from G_i by applying splitting operation once or twice for $i = 0, 1, 2, \ldots, n-1$.*

A matroid M is *Eulerian* if its ground set can be partitioned into disjoint circuits, and it is *connected* if any pair of its elements is contained in a circuit. It is clear that an Eulerian matroid may not be connected. A matroid is *graphic* if it is isomorphic to the cycle matroid of a graph. For matroid concepts and terminology, we refer to Oxley [2]. Raghunathan et al. [3] generalized the splitting operation of graphs to binary matroids and characterized Eulerian matroids in terms of this operation. We characterize connected Eulerian graphic matroids.

In Section 2, we prove Theorems 1.3 and 1.4. The matroid extension is considered in Section 3.

2. Eulerian 2-Connected Graphs

A block of a connected graph G is a *pendant block* if it contains exactly one cut vertex of G. For an edge $e \in E(G)$, we denote the set of end vertices of e by $V(e)$. For a vertex v of G, let $E_G(v)$ denote the set of edges of G which are incident with v, that is, $E_G(v) = \{e \in E(G) \mid v \in V(e)\}$. Raghunathan et al. [3] characterized the circuits of the graph $G_{x,y}$ in terms of circuits of G as follows.

Lemma 2.1 (see [3]). *Let G be a graph and let $\{x, y\}$ be a pair of adjacent edges of G. Then a subset C of edges of the graph $G_{x,y}$ is a circuit in $G_{x,y}$ if and only if C satisfies one of the following conditions:*

(i) *C is a circuit in G containing x and y;*

(ii) *C is a circuit in G containing neither x nor y;*

(iii) *$C = C_1 \cup C_2$, where C_1 and C_2 are edge disjoint circuits of G with $x \in C_1$, $y \in C_2$, and $C_1 \cup C_2$ does not contain a circuit in G satisfying either (i) or (ii) above.*

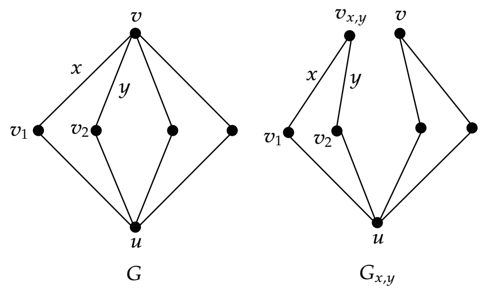

Figure 2

Lemma 2.2. *Let G be a 2-connected graph and v be a vertex of G with $d(v) \geq 3$ and $\{x, y\} \subseteq E_G(v)$ such that the graph $G_{x,y}$ is not 2-connected. Then $G_{x,y}$ is connected and has exactly two pendant blocks. Further, one pendant block contains $\{x, y\}$, and the other pendant block contains $E_G(v) - \{x, y\}$.*

Proof. The proof is straightforward (see Figure 3). □

Lemma 2.3. *Let G be a 2-connected graph and let v be a vertex of G with $d(v) \geq 4$ such that $G_{x,y}$ is not 2-connected for all $\{x, y\} \subseteq E_G(v)$. Then, for a given $\{x, y\} \subseteq E_G(v)$, the graph $G_{x,y}$ is connected and has one cut vertex and two blocks.*

Proof. Let $\{x, y\}$ be a pair of edges incident with v. By Lemma 2.2, $G_{x,y}$ is connected and has exactly two pendant blocks, say B_1 and B_2. We may assume that B_1 contains $\{x, y\}$ and B_2 contains $E_G(v) - \{x, y\}$. As $d(v) \geq 4$, we can choose two edges z, w from $E_G(v) - \{x, y\}$. Let P_1 and P_2 be paths in $G_{x,y}$ from v to $v_{x,y}$ with $\{x, z\} \subset E(P_1)$, $\{y, w\} \cap E(P_1) = \phi$, $\{y, w\} \subset E(P_2)$ and $\{x, z\} \cap E(P_2) = \phi$. Each of P_1 and P_2 corresponds to a cycle in G. By Lemma 2.1, these cycles are preserved in the graph $G_{y,w}$. Therefore P_i is contained in a block of $G_{y,w}$ for $i = 1, 2$. By Lemma 2.2, $G_{y,w}$ has two pendant blocks one containing edges y, w and the other containing $E_G(v) - \{y, w\}$. Hence $P_1$and P_2 belong to different pendant blocks of $G_{y,w}$. Hence P_1 and P_2 share at most one vertex of $G_{y,w}$. However, P_1 and P_2 share all cut vertices of $G_{x,y}$. This implies that $G_{x,y}$ has exactly one cut vertex. Therefore, by Lemma 2.2, $G_{x,y}$ is connected and has exactly two blocks. □

Lemma 2.4. *Let G and v be as stated in Lemma 2.3. Then there exists a vertex u in G such that u is the cut vertex of $G_{x,y}$ for all $\{x, y\} \subseteq E_G(v)$.*

Proof. Let $\{x, y\} \subseteq E_G(v)$. By Lemma 2.3, $G_{x,y}$ is connected and has one cut vertex, say u. Let $\{z, w\} \subseteq E_G(v)$. Then, by Lemma 2.3, $G_{z,w}$ is also connected and has two blocks and one cut vertex. It suffices to prove that u is the cut vertex of $G_{z,w}$. If $\{x, y\} = \{z, w\}$, then there is nothing to prove. Suppose $\{x, y\} \neq \{z, w\}$. We may assume that $x \notin \{z, w\}$ and $y \neq w$. By Lemma 2.3, $G_{x,w}$ is connected and has two blocks, say B_1 and B_2. By Lemma 2.2, we may assume that B_1 contains $\{x, w\}$ and B_2 contains $E_G(v) - \{x, w\}$. Let e be an edge of G incident with v such that $e \notin \{x, y, w\}$. Let P_1 and P_2 be paths in $G_{x,y}$ from the vertex $v_{x,y}$ to v such that $\{x, w\} \subset E(P_1)$, $\{y, e\} \subset E(P_2)$, and $\{x, w\} \cap E(P_2) = \phi$, $\{y, e\} \cap E(P_1) = \phi$. Then each of P_1 and P_2 contains all cut vertices of $G_{x,y}$. Therefore u is a common vertex of P_1 and P_2. Further, each of P_1 and P_2 corresponds to a cycle in G. By Lemma 2.1, these cycles are preserved in the graph

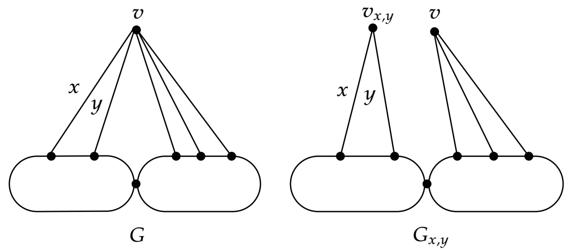

Figure 3

$G_{x,w}$ and hence are contained in blocks of $G_{x,w}$. Therefore B_i contains P_i for $i = 1, 2$. Thus u is a common vertex of B_1 and B_2. This implies that u is a cut vertex of $G_{x,w}$. By Lemma 2.3, u is the only cut vertex of $G_{x,w}$. Let Q_1 and Q_2 be paths in $G_{x,w}$ from v to $v_{x,w}$ such that $\{z, w\} \subseteq E(Q_1)$ and $x \in E(Q_2) - E(Q_1)$. Then Q_i contains the cut vertex u and, further, it corresponds to a cycle in G for $i = 1, 2$. By Lemma 2.1, Q_i corresponds to a cycle of the graph $G_{z,w}$ and hence is contained in a block of $G_{z,w}$ for $i = 1, 2$. These cycles are contained in different blocks of $G_{z,w}$. By Lemma 2.2, one block of $G_{z,w}$ contains the edges z, w and the other block contains the remaining edges of G that are incident with v. Hence Q_1 and Q_2 belong to different blocks of $G_{z,w}$. As u is a common vertex of Q_1 and Q_2, it is a cut vertex of $G_{z,w}$. Thus u is the cut vertex of $G_{z,w}$. □

Lemma 2.5. *Let G be a 2-connected graph and v be a vertex of G with the set of neighbours $\{v_1, v_2, \ldots, v_k\}$, where $k \geq 4$. Suppose $G_{x,y}$ is not 2-connected for all $\{x, y\} \subseteq E_G(v)$. Then there exists a vertex u in G such that $V(P) \cap V(Q) = \{u\}$ for any uv_i-path P and uv_j-path Q in $G - v$ with $i \neq j$.*

Proof. By Lemma 2.4, there exists a vertex u in G such that it is the cut vertex of $G_{x,y}$ for all $\{x, y\} \subseteq E_G(v)$. Let P be a uv_i-path and Q be a uv_j-path in $G - v$ with $i \neq j$. We prove that $V(P) \cap V(Q) = \{u\}$. If P or Q is a trivial graph, then there is nothing to prove. Assume that $|E(P)| \geq 1$ and $|E(Q)| \geq 1$. Without loss of generality, we may assume that $v_i = v_1$ and $v_j = v_2$. Let $e_1 = vv_1$ and $e_2 = vv_2$. Then G_{e_1,e_2} is connected and has two blocks, say B_1 and B_2 (see Figure 4). By Lemma 2.2, we may assume that B_1 contains $\{e_1, e_2\}$ and B_2 contains $E_G(v) - \{e_1, e_2\}$. Since u is the cut vertex of G_{e_1,e_2}, the paths P, Q are contained in B_1. Let $e_3 = vv_3$ and $e_4 = vv_4$. Let P_1 be a uv-path in B_2 containing the edge e_3 but avoiding e_4. Let P_2 be an uv-path in B_2 containing e_4 and avoiding e_3. Then $V(P_k) \cap V(P) = V(P_k) \cap V(Q) = \{u\}$ for $k = 1, 2$. Let $C_1 = P_1 \cup P \cup e_1$ and $C_2 = P_2 \cup Q \cup e_2$. Then each of C_1 and C_2 corresponds to a cycle in G. Further, C_1 contains e_1, e_3 and C_2 contains e_2, e_4. Therefore, by Lemma 2.1, C_1 and C_2 correspond to cycles in G_{e_1,e_3}. By Lemmas 2.2 and 2.3, G_{e_1,e_3} has exactly two blocks one of them contains C_1 and the other contain C_2. Hence C_1 and C_2 can share at most one vertex. This implies that P and Q can share at most one vertex. Thus $V(P) \cap V(Q) = \{u\}$. □

Proof of Theorem 1.3. Let G be a 2-connected graph and let v be a vertex of G with $d(v) \geq 4$. Suppose $G_{x,y}$ is not 2-connected for every pair $\{x, y\}$ of edges incident with v. Let $\{v_1, v_2, \ldots, v_r\}$ be the set of neighbours of v. Let x_1 and y_1 be any two edges of G incident with v. We may assume that $x_1 = vv_1$ and $y_1 = vv_2$. By Lemma 2.5, there exists a vertex u in G such that $V(P_i) \cap V(P_j) = \{u\}$ for any i, j with $i \neq j$, where P_i is a uv_i-path and P_j is a uv_j-path in $G - v$ (see Figure 5). It is easy to see that $d(u) \geq d(v) \geq 4$. If $u = v_i$ for some i,

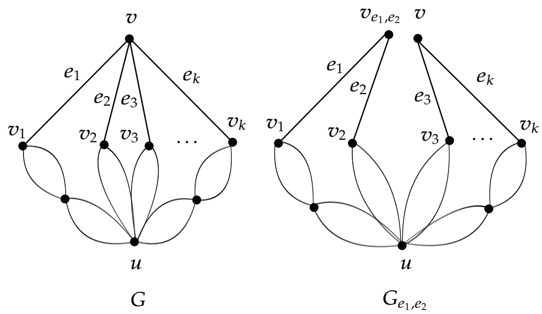

Figure 4

then P_i is the trivial graph containing only the vertex u. If $v_2 = u$, then set $x_2 = y_1 = uv$. If $v_3 = u$, then set $y_2 = vu$. In other cases, P_2 and P_3 are nontrivial graphs and hence we can take $x_2 = uu_2 \in E(P_2)$ and $y_2 = uu_3 \in E(P_3)$. It is easy to see that $(G_{x_1,y_1})_{x_2,y_2}$ is 2-connected. □

Now, we prove Theorem 1.4. Let $d_H(v)$ denotes the degree of a vertex v in a graph H.

Proof of Theorem 1.4. Let $G = G_0$ be an Eulerian 2-connected graph. Suppose G is not a cycle. Then G has a vertex v of a degree of at least 4. By Theorem 1.3, we get a pair $\{x_1, y_1\}$ of edges incident with v such that either G_{x_1,y_1} is 2-connected or $(G_{x_1,y_1})_{x_2,y_2}$ is 2-connected for some pair $\{x_2, y_2\}$ of edges of G having a common vertex other than v. Denote this new 2-connected graph by G_1. If $G_1 = G_{x_1,y_1}$, then $d_{G_1}(v) = d_G(v) - 2$ and $d_{G_1}(w) = d_G(w)$ for any $w \in V(G) - \{v\}$. If $G_1 = (G_{x_1,y_1})_{x_2,y_2}$, then $d_{G_1}(v) = d_G(v) - 2$, $d_{G_1}(u) = d_G(u) - 2$, and $d_{G_1}(w) = d_G(w)$ for any $w \in V(G) - \{u, v\}$, where u is the common vertex of x_2 and y_2 other than v. Further, the new vertices of G_1 that are created in the splitting procedure have degree two. Obviously, G_1 is Eulerian. If G_1 is not a cycle, then we obtain a 2-connected Eulerian graph G_2 from G_1 by applying splitting operation once (or twice) which results in reducing the degree of a vertex (or two vertices) of G_1 by 2. By repeating the same procedure and through a sequence of once or twice splitting operations performed in such a way that at each step the resulting graph is still 2-connected one finally arrives at a cycle which corresponds to an Eulerian trail of G. The converse is obvious. □

3. Eulerian 2-Connected Matroids

In this section, we extend Theorem 1.4 to connected Eulerian matroids. Raghunathan et al. [3] generalized the splitting operation of graphs to binary matroids and characterized Eulerian matroids in terms of this operation. In this section, we characterize connected Eulerian graphic matroids.

Definition 3.1 (see [3]). Let $M = M[A]$ be a binary matroid and suppose $x, y \in E(M)$. Let $A_{x,y}$ be the matrix obtained from A by adjoining the row that is zero everywhere except for the entries of 1 in the columns labeled by x and y. The splitting matroid $M_{x,y}$ is defined to be the vector matroid of the matrix $A_{x,y}$. The transition from M to $M_{x,y}$ is called a splitting operation. The splitting operation for binary matroids is also studied in [3–6].

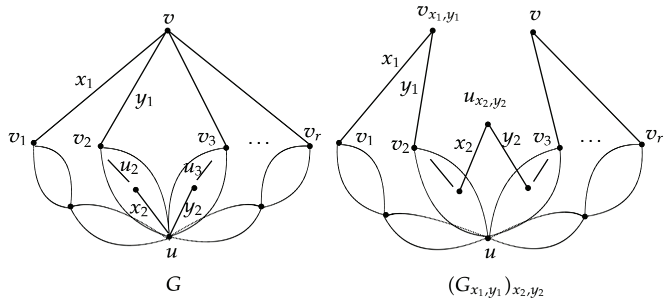

Figure 5

We need the following three results.

Lemma 3.2 (see [3]). *If $M(G)$ denotes the circuit matroid of a graph G, then for a pair $\{x, y\}$ of adjacent edges in a graph G, $M(G_{x,y}) = (M(G))_{x,y}$.*

Lemma 3.3 (see [3]). *Let M be a binary matroid and $x, y \in E(M)$. Then M is Eulerian if and only if $M_{x,y}$ is Eulerian.*

Theorem 3.4 (see [2, page 127]). *Let G be a loopless graph without isolated vertices. If G has at least three vertices, then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.*

We obtain the following characterization of connected Eulerian graphic matroids.

Theorem 3.5. *Let M be a connected graphic matroid. Then M is Eulerian if and only if it can be transformed into a circuit C through a sequence $M = M_0 < M_1 < \cdots < M_n = C$ of connected graphic matroids such that M_{i+1} is obtained from M_i by applying splitting operation once or twice.*

Proof. Let M be a connected graphic matroid. Then M is isomorphic to a cycle matroid $M(G)$ of some graph G. In view of Theorem 3.4, we may assume that G is 2-connected. Suppose M is Eulerian. Then the graph G is Eulerian. By Theorem 1.4, there is a sequence of 2-connected graphs $G = G_0 < G_1 < G_2 < \cdots < G_n$ such that G_n is a cycle, and G_{i+1} is obtained from G_i by applying splitting operation once or twice for $i = 0, 1, 2, \ldots, n-1$. Let $M_i = M(G_i)$ for $i = 0, 1, \ldots, n$. By Theorem 3.4, each M_i is connected. It follows from Lemma 3.2 that if G_{i+1} is obtained from G_i by applying splitting operation once or twice then M_{i+1} is obtained from M_i by applying splitting operation once or twice, respectively. Further, by Lemma 3.3, M_i is Eulerian for $i = 1, 2, \ldots, n$.

Conversely, suppose there exists a sequence of connected graphic matroids $M = M_0 < M_1 < \cdots < M_n = C$, where M_{i+1} is obtained from M_i by applying splitting operation once or twice for $i = 0, 1, 2, \ldots, n-1$. Since $M_n = C$ is Eulerian, by Lemma 3.3, M_{n-1} is Eulerian. By repeated applications of Lemma 3.3, we see that M_i is Eulerian for each i. Thus $M = M_0$ is Eulerian. □

Acknowledgment

The author would like to thank the referees for their valuable suggestions. The paper is supported by University of Pune under BCUD project scheme.

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