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Research Article

Relations between Stochastic and Partial Differential Equations in Hilbert Spaces

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The aim of the paper is to introduce a generalization of the Feynman-Kac theorem in Hilbert spaces. Connection between solutions to the abstract stochastic differential equation $dX(t) = AX(t)dt + BdW(t)$ and solutions to the deterministic partial differential (with derivatives in Hilbert spaces) equation for the probability characteristic $\mathbb{E}^{t,x}h(X(T))$ is proved. Interpretation of objects in the equations is given.

1. Introduction

The Feynman-Kac theorem in the numerical (vector) case relates solutions of the Cauchy problem for stochastic equations with a Brownian motion $W(t)$, $t \geq 0$:

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t), \quad t \in [0, T], X(0) = y, \tag{1.1}$$

with solutions of the Cauchy problem for deterministic partial differential equations:

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0, \quad g(T, x) = h(x), \tag{1.2}$$

for the probability characteristic $g(t, x) = \mathbb{E}^{t,x}h(X(T))$ with an arbitrary Borel function h. Here $\mathbb{E}^{t,x}$ means the mathematical expectation of a solution to (1.1) with initial value $X(t) = x$, $0 \leq t \leq T$.

Study of the relationship between the problems (1.1) and (1.2) was initially caused by the needs of physics. For example, the process $X(t)$ describes the random motion of particles in a liquid or gas, and $g(t,x)$ is a probability characteristic such as temperature, determined by the Kolmogorov equation. In the last years, the importance of the relationship between stochastic (1.1) and deterministic (1.2) problems has become more acute with the development of numerical methods and applications in financial mathematics. Here $X(t)$, for example, stock price at time t, then $g(t,x)$ is the value of stock options, determined by the famous Black-Scholes equation [1, 2]. Moreover, there exist recent applications of infinite-dimensional stochastic equations in financial mathematics [3]. For example, consider function $P(t,T)$ that is the price at time $t \le T$ of the coupon bond with maturity date T. Let $P(t,T)$ be parametrized as $P(t,t) = 1$ for all t and $f(t,T)$, $t \le T$ be the forward curve; that is, $P(t,T) = \exp(-\int_t^T f(t,s)ds)$. Then the Musiela reparametrization $r(t,z) := f(t,z+t)$, $z \ge 0$, in the special case of zero HJM shift, satisfies the following equation in a Hilbert space H of functions acting from $\mathbb{R}_+$ to $\mathbb{R}$:

$$dr(t) = Ar(t)dt + \sigma(t,r(t))dW(t), \quad r(0) = y, \tag{1.3}$$

where A is the generator of the right-shifts semigroup in H, W is a U-valued Wiener (in particular Q-Wiener) process, and σ is a random mapping from Hilbert space U to H. Here the value of bond options may be calculated, at least numerically, via $g(t,x)$ defined for $X(t) = r(t)$. Thereby it requires an infinite-dimensional analogue of the connection between problems (1.1) and (1.2), which the present paper is devoted to.

Generalization of the Feynman-Kac theorem to the infinite-dimensional case raises many questions related with the very formulation of the problem in infinite-dimensional spaces, the definition of relevant objects and a rigorous rationale for the relationship between mentioned problems.

The paper considers the stochastic Cauchy problem in Hilbert spaces, which is the infinite-dimensional generalization of the problem (1.1):

$$dX(t) = AX(t)dt + BdW(t), \quad t \in [0,T], X(0) = y. \tag{1.4}$$

We prove the infinite-dimensional case of the Feynman-Kac theorem in a standard basic conditions such as A is the generator of a C_0-semigroup in a Hilbert space H, B is a bounded operator from a Hilbert space U to H, and W is a U-valued Q-Wiener process. This is done for the clarity of proof despite the fact that the conditions are likely to be weakened, and the statement is true in more general assumptions. For example, the main result can be applied for (1.3) related to the bond price in more general case $B = B(t,r(t))$. The reduction of constraints to the Feynman-Kac theorem is also the subject of further research.

For the problem (1.4) we associate a problem for a deterministic partial differential equation which is a generalization of the problem (1.2) in the case of Hilbert spaces. The paper gives the rigorous interpretation of the objects included in the stochastic and deterministic equations and proves the connection between their solutions. The proof of the relation is obtained as a generalization of the finite-dimensional Feynman-Kac theorem [2]. In contrast to the well-known works as [4], the proof is performed without using semigroup technique for some operator family $\{P_t, t \ge 0\}$ defined as $P_t h(x) := g(t,x)$ that gives a rigorous justification of the study only in certain cases. Particular attention is paid to the subtle issue of transition from zero expectation for a function of g to equality for g itself.

2. Definitions and Auxiliary Statements

We start with interpretation for the objects of the stochastic problem (1.4). Let the operator A be the generator of the C_0-semigroup in Hilbert space H. This ensures uniform well-posedness of the Cauchy problem for the corresponding homogeneous equation $X'(t) = AX(t)$, as well as existence and uniqueness of a weak solution to the stochastic problem (1.4) with Q-Wiener process W, where Q is a trace class operator [4, 5].

Let a U-valued stochastic process W be Q-Wiener, that is, satisfy the following conditions:

(i) $W(0) = 0$,

(ii) W has continuous trajectories,

(iii) W has independent increments,

(iv) $W(t) - W(s)$ is normally distributed with $\mathbb{E}[W(t) - W(s)] = 0$ and $\operatorname{Var}[W(t) - W(s)] = (t-s)Q$, $t \geq s \geq 0$.

The Q-Wiener process can be expanded into the convergent series

$$W(t) = \sum_i \sqrt{\lambda_i}\beta_i(t)e_i, \tag{2.1}$$

where β_i are independent Brownian motions, $\{\lambda_i\}$ is the sequence of eigenvalues of Q, and $\{e_i\}$ is the complete orthonormal system in U corresponding to the $\{\lambda_i\}$: $\lambda_i e_i = Qe_i$.

Define the function $g(t,x) := \mathbb{E}^{t,x}h(X(T))$ that transforms $[0,T] \times H$ into $\mathbb{R}$. The h is a measurable function from H to $\mathbb{R}$. We show that g satisfies the following infinite-dimensional deterministic problem:

$$\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x)Ax + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^* \frac{\partial^2 g}{\partial x^2}(t,x)\left(BQ^{1/2}\right)\right] = 0, \quad g(T,x) = h(x), \tag{2.2}$$

corresponding to the stochastic one (1.4).

At the beginning we make the sense to the terms of (2.2). The derivatives $\partial g/\partial x$ and $\partial^2 g/\partial x^2$ are understood in the sense of Frechet, that means $\partial g/\partial x : [0,T] \times H \to H^*$ and $\partial^2 g/\partial x^2 : [0,T] \times H \to \mathcal{L}(H,H^*)$. More precisely

$$\begin{gathered} \frac{\partial g}{\partial x}(t,x)(\cdot) : H \to \mathbb{R}, \quad \frac{\partial^2 g}{\partial x^2}(t,x)(\cdot) : H \to H^*, \quad \text{for any fixed } t \in [0,T], x \in H, \\ BQ^{1/2} : U \to H, \quad \left(BQ^{1/2}\right)^* : H^* \to U^*. \end{gathered} \tag{2.3}$$

The term $\operatorname{Tr}[(BQ^{1/2})^*(\partial^2 g/\partial x^2)(BQ^{1/2})]$ requires special attention. Expression Tr is usually defined as the trace of an operator acting in the same Hilbert space. The operator under the trace sign in (2.2) maps Hilbert space U to its adjoint U^*.

Using the traditional definition of the trace, we can make sense to its expression, using the Riesz theorem on the isomorphism U and U^*, that is, identifying U^* with U. Note that the isomorphism allows us to consider operators $BQ^{1/2}$, $(BQ^{1/2})^*$, and $\partial^2 g/\partial x^2$

as mappings from U to H, from H to U, and H to H, respectively. Then operator $(BQ^{1/2})^*(\partial^2 g/\partial x^2)(BQ^{1/2})$ transfers the Hilbert space U to U, and trace of this operator can be understood in the usual sense.

We give more rigorous interpretation to the concept of the trace for an operator acting from a Hilbert space U to U^*: for the purpose we write an arbitrary operator $R : U \to U^*$ in the form

$$Rz = \sum_{k=1}^{\infty} a_k(z)\ b_k, \quad z \in U, a_k, b_k \in U^*. \tag{2.4}$$

Then $\operatorname{Tr} R$ can be understood as $\operatorname{Tr} R = \sum_{k=1}^{\infty} Re_j(e_j)$, $e_j \in U$, where $\{e_j\}$ is the basis in U. Considering the proposed interpretation of the trace, the expression $\operatorname{Tr}[(BQ^{1/2})^*(\partial^2 g/\partial x^2)(BQ^{1/2})]$ with $R = (BQ^{1/2})^*(\partial^2 g/\partial x^2)(BQ^{1/2})$ has a clear and definite sense.

Now we prove necessary properties of the process X that is a solution of (1.4), and function $g(t,x)$ that determines the relationship between the solutions of problems (1.4) and (2.2). We obtain the required properties for the case of more general processes, the diffusion processes, to which the solution of (1.4) is a special case.

An H-valued Ito process $\{X(t), t \geq 0\}$ is called diffusion if it can be written in the form

$$dX(t) = \delta(X(t))dt + \sigma(X(t))dW(t). \tag{2.5}$$

In the paper we consider the diffusion processes for which the existence and uniqueness of solution to the stochastic Cauchy problem for (2.5) are fulfilled. For example, such additional condition is guaranteed by the estimate to the coefficients δ and σ: $\|\delta(z_1) - \delta(z_2)\| + \|\sigma(z_1) - \sigma(z_2)\| \leq c\|z_1 - z_2\|$, z_1, $z_2 \in H$, $c \in \mathbb{R}$ (see Theorem 2.1 [6], ch. VII). Note that in the particular case of the problem (1.4) the unique solution of (2.5) can be written as sum of the term depending on the initial value and the stochastic convolution term (see, e.g., [4, 5]):

$$X(t) = S(t)y + \int_0^t S(t-\tau)BdW(\tau), \tag{2.6}$$

where the family $\{S(t)\}$ is the C_0-semigroup with the generator A.

To prove the relationship under the study it is important to establish the Markov property for the solution of the Cauchy problem (1.4). The following statement is a generalization of the finite-dimensional case result (Theorem 7.1.2 [7]) to the case of Hilbert spaces.

Proposition 2.1. *Let $h(z)$, $z \in H$ be Borel-measurable and let $X(t), t \geq 0$ be a diffusion Ito process. Then X satisfies the Markov property with respect to a σ-algebra $\mathfrak{F}_t$ defined by Q-Wiener process W:*

$$\mathbb{E}[h(X(t+s)) \mid \mathfrak{F}_t] = \mathbb{E}^{0,X(t)}[h(X(s))]. \tag{2.7}$$

Proof. Let $X^{t,x}$, $t \in [0,T]$, $x \in H$ be the solution of (2.5) with the condition $X(t) = x$. By the uniqueness of a solution to the Cauchy problem for (2.5) we have $X(r) = X^{t,X(t)}(r)$, $r \geq t$, almost surely. Define the map $F(x,t,r,\omega) := X^{t,x}(r)$. Then to prove the property (2.7) is sufficient to obtain the equality

$$\mathbb{E}[h(F(X(t),t,t+s,\omega)) \mid \mathfrak{F}_t] = \mathbb{E}[h(F(X(t),0,s,\omega))]. \tag{2.8}$$

Introduce the function $v^{t,x}(t+s,\omega) = h(F(x,t,t+s,\omega))$. It is measurable as superposition of measurable functions. Fix some $m \in \mathbb{N}$ and consider some partition of the segment $[t,T]$: $t_k = t + k(T-t)/m$, $k = 0,\dots,m$. Let

$$v_m^{t,X(t)}(\tau,\omega) = \sum_{k=0}^{m} v^{t,X(t)}(t_{k+1},\omega)\chi_{\tau\in[t_k,t_{k+1})}, \tag{2.9}$$

where $\chi_{\tau\in[t_k,t_{k+1})}$ is a characteristic function of the semiopen interval $[t_k,t_{k+1})$. Then we have

$$\begin{aligned}
\mathbb{E}\left[v_m^{t,X(t)}(\tau,\omega) \mid \mathfrak{F}_t\right] &= \mathbb{E}\left[\sum_{k=0}^{m} v^{t,X(t)}(t_{k+1},\omega)\chi_{\tau\in[t_k,t_{k+1})} \mid \mathfrak{F}_t\right] = \sum_{k=0}^{m}\chi_{\tau\in[t_k,t_{k+1})}\mathbb{E}\left[v^{t,X(t)}(t_{k+1},\omega) \mid \mathfrak{F}_t\right] \\
&= \sum_{k=0}^{m}\chi_{\tau\in[t_k,t_{k+1})}\mathbb{E}\left[v^{t,X(t)}(t_{k+1},\omega)\right] = \mathbb{E}\left[\sum_{k=0}^{m}\chi_{\tau\in[t_k,t_{k+1})}v^{t,X(t)}(t_{k+1},\omega)\right] \\
&= \mathbb{E}\left[v_m^{t,X(t)}(\tau,\omega)\right].
\end{aligned} \tag{2.10}$$

The first and the last equalities follow from definition of the $v_m^{t,X(t)}$. The second one is a consequence of the fact that the characteristic functions of intervals do not depend on the variable ω. Since $t_{k+1} > t$, the third equality holds because $X(t_{k+1})$ is independent of the filter $\mathfrak{F}_t$ for all $k \in [0,m]$.

Now note that $v_m^{t,X(t)}(\tau,\omega) \to_{m\to\infty} v^{t,X(t)}(\tau,\omega)$ in $L_2(\Omega)$. So, let m tends to infinity in the established relation (2.10); then we have

$$\mathbb{E}\left[v^{t,X(t)}(\tau,\omega) \mid \mathfrak{F}_t\right] = \mathbb{E}\left[v^{t,X(t)}(\tau,\omega)\right]. \tag{2.11}$$

Thus, as $\tau = t+s$ we conclude that

$$\mathbb{E}[h(F(X(t),t,t+s,\omega)) \mid \mathfrak{F}_t] = \mathbb{E}[h(F(X(t),t,t+s,\omega))]. \tag{2.12}$$

Using the diffusion property of Ito process $X(t)$ we obtain

$$\mathbb{E}[h(F(z,t,t+s,\omega))]_{z=X(t)} = \mathbb{E}[h(F(z,0,s,\omega))]_{z=X(t)}. \tag{2.13}$$

The last two equalities imply the desired relation (2.7). □

Note that if the Ito process $X(t)$ is a solution to (1.4), it is diffusive and by the statement established previously has the Markov property.

Equation (2.13) obtained in the proof of Proposition 2.1 can be written in the ensuing form.

Corollary 2.2. *By the homogeneity in time of diffusion processes the following relation is fulfilled:*

$$\mathbb{E}^{0,X(t)}[h(X(s))] = \mathbb{E}^{t,X(t)}[h(X(t+s))]. \tag{2.14}$$

As a consequence of Proposition 2.1 and Corollary 2.2 we obtain the following.

Corollary 2.3. *Markov property can be written as follows:*

$$\mathbb{E}[h(X(t+s)) \mid \mathfrak{F}_t] = \mathbb{E}^{t,X(t)}[h(X(t+s))]. \tag{2.15}$$

The following statement generalizes Theorem 5.50 [8].

Proposition 2.4. *Suppose a process $X(t)$ satisfies the conditions of Proposition 2.1. Then the process $g(t,X(t)) := \mathbb{E}^{t,x}h(X(T))|_{x=X(t)}$ is martingale, that is:*

$$\mathbb{E}\big[g(t,X(t)) \mid \mathfrak{F}(s)\big] = g(s,X(s)), \quad 0 \le s \le t \le T. \tag{2.16}$$

Proof. According to the Proposition 2.1, $X(t)$ has the Markov property. Therefore

$$\mathbb{E}[h(X(T)) \mid \mathfrak{F}(t)] = \mathbb{E}^{t,X(t)}h(X(T)) = g(t,X(t)), \tag{2.17}$$

and we obtain the following equalities:

$$\begin{aligned}\mathbb{E}\big[g(t,X(t)) \mid \mathfrak{F}(s)\big] &= \mathbb{E}[\mathbb{E}[h(X(T)) \mid \mathfrak{F}(t)] \mid \mathfrak{F}(s)] = \mathbb{E}[h(X(T)) \mid \mathfrak{F}(s)] \\ &= \mathbb{E}^{s,X(s)}h(X(T)) = g(s,X(s)).\end{aligned} \tag{2.18}$$

The first equality implies the obtained representation for the process $g(t,X(t))$ via the conditional expectation. The second equality follows from the properties of conditional expectation. The third one is the direct consequence of the Markov property for $g(t,X(t))$. The last equality follows from the definition of the process $g(t,X(t))$ and completes the proof. □

Now we can proceed to prove the connection between the problems (1.4) and (2.2).

3. Proof of the Main Result

Theorem 3.1. *Consider the stochastic differential Cauchy problem* (1.4). *Fix some $T > 0$ and suppose that $\mathbb{E}^{t,x}|h(X(T))| < \infty$ for all pairs t and x. Then $g(t,x) = \mathbb{E}^{t,x}h(X(T)) : [0,T] \times H \to \mathbb{R}$ is the solution of infinite-dimensional (backward) Kolmogorov problem* (2.2).

Proof. Applying the Ito formula in Hilbert spaces [4] to g as a function from the solution of the problem (1.4) we obtain

$$dg(t,X(t)) = \left(\frac{\partial g}{\partial t}(t,X(t)) + \frac{\partial g}{\partial x}(t,X(t))AX(t) + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(t,X(t))\left(BQ^{1/2}\right)\right]\right)dt + \frac{\partial g}{\partial x}(t,X(t))BdW(t). \tag{3.1}$$

This equality is written in the form of differentials (increments). In the integral form it can be written as follows:

$$\begin{aligned} g(t,X(t)) = g(0,y) &+ \int_0^t \frac{\partial g}{\partial x}(s,X(s))BdW(s) \\ &+ \int_0^t \left(\frac{\partial g}{\partial s}(s,X(s)) + \frac{\partial g}{\partial x}(s,X(s))AX(s) \right. \\ &\left. + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(s,X(s))\left(BQ^{1/2}\right)\right]\right)ds. \end{aligned} \tag{3.2}$$

Apply the expectation to both sides of the equation. From the definition of Ito integral (via the approximation in the mean square by step processes) and the properties of the Q-Wiener process, we obtain $\mathbb{E}[\int_0^t (\partial g/\partial x)(s,X(s))BdW(s)] = 0$. Further, since the process $g(t,X(t))$ is martingale, we have

$$\mathbb{E}[g(t,X(t))] = \mathbb{E}[g(t,X(t)) \mid \mathfrak{F}_0] = g(0,y). \tag{3.3}$$

Hence, using the theorem of Tonelli-Fubini in Hilbert spaces and equalities mentioned previously we conclude

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_0^t \left(\frac{\partial g}{\partial s}(s,X(s)) + \frac{\partial g}{\partial x}(s,X(s))AX(s) + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(s,X(s))\left(BQ^{1/2}\right)\right]\right)ds\right] \\ &= \int_0^t \mathbb{E}\left(\frac{\partial g}{\partial s}(s,X(s)) + \frac{\partial g}{\partial x}(s,X(s))AX(s) + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(s,X(s))\left(BQ^{1/2}\right)\right]\right)ds. \end{aligned} \tag{3.4}$$

The last equality is true for all $t \in [0,T]$. Therefore,

$$\mathbb{E}\left[\frac{\partial g}{\partial t}(t,X(t)) + \frac{\partial g}{\partial x}(t,X(t))AX(t) + \frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(t,X(t))\left(BQ^{1/2}\right)\right]\right] = 0. \tag{3.5}$$

Rewrite this equality at the origin point $(0,y)$:

$$\mathbb{E}\left[\frac{\partial g}{\partial t}(0,y)+\frac{\partial g}{\partial x}(0,y)Ay+\frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)\right]\right]=0, \tag{3.6}$$

that is

$$\mathbb{E}\left[\frac{\partial g}{\partial t}(0,y)\right]+\mathbb{E}\left[\frac{\partial g}{\partial x}(0,y)Ay\right]+\frac{1}{2}\mathbb{E}\left[\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)\right]\right]=0. \tag{3.7}$$

Note that Ay does not depend on ω; thus $\mathbb{E}[(\partial g/\partial x)(0,y)Ay]=\mathbb{E}[(\partial g/\partial x)(0,y)]Ay$. Using the Lebesgue dominated convergence theorem, the fact that the mappings $\partial/\partial t$, $\partial/\partial x$ and $\partial^2/\partial x^2$ are independent of the variable ω, and that the expectation $\mathbb{E}$ is an integral of the variable ω, we conclude that all these operators commute with the operator $\mathbb{E}$. Furthermore, according to the interpretation of trace given in Section 2, it also commutes with the operator $\mathbb{E}$ by the following arguments:

$$\begin{aligned}\mathbb{E}\left[\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)\right]\right]&=\mathbb{E}\left[\sum_{k=1}^{\infty}\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)e_j(e_j)\right]\\&=\sum_{k=1}^{\infty}\mathbb{E}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)e_j(e_j)\right]=\operatorname{Tr}\mathbb{E}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)e_j(e_j)\right].\end{aligned} \tag{3.8}$$

Note

$$\mathbb{E}g(0,y)=\mathbb{E}\left[\mathbb{E}^{0,y}[h(X(T))]\right]=\mathbb{E}[\mathbb{E}[h(X(T))]]=\mathbb{E}[h(X(T))]=g(0,y). \tag{3.9}$$

Hence, we obtain

$$\frac{\partial g}{\partial t}(0,y)+\frac{\partial g}{\partial x}(0,y)Ay+\frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(0,y)\left(BQ^{1/2}\right)\right]=0. \tag{3.10}$$

Until now we have considered the Cauchy problem (1.4) where $t\in[0,T]$. Consider this problem for the same equation with the initial condition at the moment $\tau\in[0,T]$:

$$dX(t)=AX(t)dt+BdW(t),\quad t\in[\tau,T],X(\tau)=x. \tag{3.11}$$

Then by the arguments similar to that conducted previously the equality

$$\frac{\partial g}{\partial t}(\tau,x)+\frac{\partial g}{\partial x}(\tau,x)Ax+\frac{1}{2}\operatorname{Tr}\left[\left(BQ^{1/2}\right)^{*}\frac{\partial^2 g}{\partial x^2}(\tau,x)\left(BQ^{1/2}\right)\right]=0 \tag{3.12}$$

holds. Varying $\tau \in [0, T]$ we obtain (3.12) for $x = X(\tau)$. It remains to note

$$g(T, x) = \mathbb{E}^{T,x} h(X(T)) = h(X(T))|_{x=X(T)} = h(x). \tag{3.13}$$

That completes the proof. □

In conclusion we note that the Feynman-Kac theorem in the numerical case establishes the interrelation between the stochastic and the deterministic problems on both sides [1, 2]. In numerical methods this relationship is indeed important to both sides: numerical methods obtained for stochastic equations are used for solving differential equations in partial derivatives, and basic methods for partial differential equations allow to obtain the characteristics of solutions to stochastic problems (see, e.g., [9]). Therefore, in addition to the previous result, it is important to establish the connection in the opposite direction and to construct the methods of solution for infinite-dimensional problem. Those are the subjects of the future studies related to the generalization of the Feynman-Kac theorem for the case of Hilbert spaces.

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