Research Article

Study on Posbist Systems

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The probability theory, in general, with the help of the dichotomous state develops the theory of reliability. Recently, the fuzzy reliability has been developed based on the concept of possibility distribution and fuzzy-state assumption. In this paper, we derive the possibility distribution function and discuss the properties of a k -out-of- n ($1 \leq k \leq n$) system based on the assumption of the possibility theory and keeping the dichotomous state of the system unchanged when the lifetime distribution is either normal, Cauchy, or exponential. A few results contrary to the conventional reliability theory are obtained.

1. Introduction

It is well known that the conventional reliability theory is based on two fundamental assumptions (cf. Barlow and Proschan [1]): (a) probability assumption: the system (failure) behavior can be fully characterized in the context of probability measure; (b) binary state assumption: the meaning of system failure is defined precisely and thus at any time the system is in one of the two crisp states—fully functioning state and fully failed state. It is also called PROBIST reliability theory, since it is based on the probability and binary state assumption. Systems studied in the context of probist reliability theory are called probist systems. Although these two assumptions have been accepted in the past decades (since 1950s) and sound reasonable in extensive cases, it has been strongly argued that they are no longer the case in wide range of cases (cf. Zadeh [2, 3], Kai-Yuan et al. [4–6], and Cai [7]).

As a consequence, three forms of fuzzy reliability theory are developed and discussed in the literature (cf. Cai [7]): (i) the PROFUST reliability theory—based on the probability assumption and the fuzzy state assumption, (ii) POSBIST reliability theory—based on the possibility assumption and the binary state assumption, (iii) the POSFUST reliability theory—based on the possibility assumption and the fuzzy state assumption.

In this paper, we have dealt with the posbist reliability theory. To clarify the concept of fuzziness, Nahmias [8] proposed a theoretical framework based on fuzzy variable analogous to the random variable in the sample space model of the probability theory. Considering the lifetimes as fuzzy variables, Kai-Yuan et al. [5] derived the possibility distribution functions of a two-component series system and a two-component parallel system under the binary state assumption. Cross [9] considered the problem of evaluation of client-server networks as nested k -out-of- n systems ($1 \leq k \leq n$) with fuzzy reliability profiles. Huang et al. [10] provided a new technique, called fault tree analysis to characterize posbist system's failure. Bhattacharjee et al. [11] dealt with the possibility distribution function of a few posbist systems.

In this paper, some of the basic concepts regarding the posbist reliability theory have been described in Section 2. In Section 3, the possibility distribution function of k -out-of- n systems ($1 \leq k \leq n$) is stated. Section 4 deals with a few properties of posbist properties and compares them with the analogous results of the conventional reliability theory. A brief illustration of the component lifetimes having normal, Cauchy, or exponential fuzzy variables is given in Section 5. We conclude by some concluding remarks given in Section 6. Throughout the paper, we use $\mathbb{R}^+ = [0, \infty)$.

2. Basic Concepts of the Posbist Reliability Theory

The concept of the posbist reliability theory is introduced in Kai-Yuan et al. [5]. Since the binary state assumption is reserved, the failure of a system (or component) is defined precisely and the system lifetime, denoted by X , is calculated from the instant the system starts functioning to the instant the system fails. However, since the possibility assumption is taken, the instant of system failure occurrence is characterized in the context of possibility measure. The lifetimes of a posbist system and its components are taken as nonnegative real-valued fuzzy variables.

The definitions are borrowed from Kai-Yuan et al. [5] and Nahmias [8]. Nahmias [8] explores a possible axiomatic framework analogous to the sample space model of the probability theory, from which a rigorous theory of fuzziness can be constructed as given below.

Let Γ be an abstract space of generic elements $\gamma \in \Gamma$. The actual construction of Γ will depend upon the particular problem being modeled in much the same way as the construction of the sample space in probability depends upon the particular random experiment.

Definition 1. Suppose that $\mathcal{G}$ is the discrete topology on Γ (that is, the class of all subsets of Γ). A scale σ , defined on $\mathcal{G}$, satisfies the following axioms:

- (i) $\sigma(\phi) = 0$, $\sigma(\Gamma) = 1$;
- (ii) $\sigma(\cup A_\alpha) = \sup \sigma(A_\alpha)$ for any arbitrary collection of sets A_α in $\mathcal{G}$.

The triplet $(\Gamma, \mathcal{G}, \sigma)$ is called the *pattern space*.

Definition 2. A fuzzy variable X is a real-valued function defined on a pattern space $(\Gamma, \mathcal{G}, \sigma)$.

Definition 3. The possibility distribution function of a fuzzy variable X , denoted by $\mu_X(x)$, is a mapping from $\mathbb{R}$ to the unit interval $[0, 1]$ and is given by

$$\mu_X(x) = \sigma(\gamma : X(\gamma) = x) \quad \forall x \in \mathbb{R}. \quad (1)$$

Further, X is said to be normal (respectively, convex or strictly convex) if the fuzzy set $\{(x, \mu_X(x)) : x \in \mathbb{R}^+\}$ is normal (respectively, convex or strictly convex).

Definition 4. Given a pattern space $(\Gamma, \mathcal{G}, \sigma)$, the sets $A_1, \dots, A_n \subset \mathcal{G}$ are said to be mutually unrelated if $\sigma(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \min(\sigma(A_{i_1}), \sigma(A_{i_2}), \dots, \sigma(A_{i_k}))$, for $1 \leq i_1 \neq i_2 \neq \dots \neq i_k \leq n$ and $1 \leq k \leq n$.

Definition 5. Given a pattern space $(\Gamma, \mathcal{G}, \sigma)$, the fuzzy variables $X_1, \dots, X_n$ are said to be mutually unrelated if the sets $\{X_{i_1} = x_1\}, \dots, \{X_{i_k} = x_k\}$ are unrelated for $x_1, \dots, x_k \in \mathbb{R}$ whenever $1 \leq i_1 \neq i_2 \neq \dots \neq i_k \leq n$ and $1 \leq k \leq n$.

The lifetime of a posbist system (posbist component) is a nonnegative real-valued fuzzy variable.

Definition 6. Posbist reliability $R(t)$ of a system is the possibility that the system performs the functions properly during a predefined exposure period under a given environment; that is,

$$R(t) = \sigma(X > t) = \sup_{u > t} \mu_X(u), \quad t \in \mathbb{R}^+. \quad (2)$$

Next, we state a lemma which will be used in proving the upcoming theorems in the next section.

Lemma 7. For any sets $A_1, A_2, \dots, A_n$ with $A_i \subseteq A_{i+1} \subseteq \mathbb{R}$, $i = 1, 2, \dots, n-1$,

$$\max\left(\min_{1 \leq i \leq n} A_i\right) = \min A_1, \quad (3)$$

where minimum of a set is defined as the elementwise minimum; that is, for $A = \{a_1, \dots, a_n\}$, $\min A = \min_{1 \leq i \leq n} a_i$.

We now state a lemma from Kai-Yuan et al. [5] to be used in proving the subsequent theorems.

Lemma 8. Suppose X is the system lifetime defined on a possibility space $(\Gamma, \mathcal{G}, \sigma)$. Suppose that X is a strictly convex fuzzy variable with the possibility distribution function $\mu_X(x)$ which is continuous. Then there exists a unique point, say x_0 (finite or infinite), such that

$$\sup_{u \geq x} \mu_X(u) = \begin{cases} \mu_X(x_0) & \text{if } x \leq x_0, \\ \mu_X(x) & \text{if } x > x_0. \end{cases} \quad (4)$$

3. The Possibility Distribution Function of Posbist Systems

The following theorem gives the possibility distribution function of a k -out-of- n system in terms of those of the components, which generalize Kai-Yuan et al. [5] work giving the possibility distribution a parallel system with two components.

Theorem 9. For a k -out-of- n system ($1 \leq k < n$), let X_j denote the component lifetime of the j th component, $j = 1, 2, \dots, n$. Assume that X_j 's are normal unrelated and strictly convex fuzzy variables with the corresponding continuous possibility distribution function $\mu_{X_j}(x)$, $j = 1, 2, \dots, n$. Let X denote the system lifetime. Then there exists a set of unique real numbers $\{a_1, a_2, \dots, a_n\}$, $a_i \in \mathbb{R}^+$ for $i = 1, 2, \dots, n$ such that the possibility distribution function of X , denoted by $\mu_X(x)$, with

$$a_1 \leq a_2 \leq \dots \leq a_i < x \leq a_{i+1} \leq a_{i+2} \leq \dots \leq a_n \quad (5)$$

is given by

$$\mu_X(x) = \begin{cases} \min\{\mu_{X_j}(x), j = i+1, \dots, n\} & \text{if } n-i \geq k \\ \max\left[\min\{\mu_{X_j}(x), j = i+1, \dots, n, \mu_{l_1}(x), \right. \\ \quad \left. \mu_{l_2}(x), \dots, \mu_{l_{k-n+i}}(x), \right. \\ \quad \left. 1 \leq l_1 < l_2 < \dots < l_{k-n+i} \leq i\} \right] & \text{if } n-i < k. \end{cases} \quad (6)$$

Proof. We consider the following:

$$\begin{aligned}
 \mu_X(x) &= \sigma(X = x) \\
 &= \sigma(\text{each of at least } k \text{ of } X_1, X_2, \dots, X_n \text{ equals } x) \\
 &= \sigma\left(\bigcup_{s=0}^{n-k} A_{k+s}\right) \\
 &= \max_{0 \leq s \leq n-k} (\sigma(A_{k+s})),
 \end{aligned} \tag{7}$$

where

$$\begin{aligned}
 A_{k+s} &= \bigcup_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \{X_{i_1} = x, X_{i_2} = x, \dots, X_{i_{k+s}} = x, \\
 &\quad \langle \langle X_m \leq x : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle\}
 \end{aligned} \tag{8}$$

with $0 \leq s \leq n-k$. Here, the suffixes of A denote the number of X_i 's equal to x out of n number of X_i 's and the rest are less than x . Note that in (8),

$$\langle \langle X_m \leq x : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle \tag{9}$$

represents a collection of $X_m \leq x$ with $1 \leq m \leq n$, and $m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s}$.

Without loss of generality, we assume that $a_1 \leq a_2 \leq \dots \leq a_n$. It is to be noted that, for $p = 1, 2, \dots, n$,

$$\sup_{u \leq x} \mu_{X_p}(u) = \begin{cases} \mu_{X_p}(x) & \text{if } x \leq a_p \\ 1 & \text{if } x > a_p. \end{cases} \tag{10}$$

Thus, for $0 \leq s \leq n-k$,

$$\begin{aligned}
 &\sigma(A_{k+s}) \\
 &= \sigma \left[\bigcup_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \{X_{i_1} = x, X_{i_2} = x, \dots, X_{i_{k+s}} = x, \right. \\
 &\quad \left. \langle \langle X_m \leq x : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle\} \right] \\
 &= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} [\sigma(X_{i_1} = x, X_{i_2} = x, \dots, X_{i_{k+s}} = x, \langle \langle X_m \leq x : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle)] \\
 &= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} [\min\{\sigma(X_{i_1} = x), \sigma(X_{i_2} = x), \dots, \sigma(X_{i_{k+s}} = x), \\
 &\quad \langle \langle \sigma(X_m \leq x) : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle \gg \}] \\
 &= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left[\min \left\{ \sigma(X_{i_1} = x), \sigma(X_{i_2} = x), \dots, \sigma(X_{i_{k+s}} = x), \right. \right. \\
 &\quad \left. \left. \left\langle \left\langle \sup_{u \leq x} \sigma(X_m = u) : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \right\rangle \right\rangle \right\} \right] \\
 &= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left[\min \left\{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s}}}(x), \right. \right. \\
 &\quad \left. \left. \left\langle \left\langle \sup_{u \leq x} \mu_{X_m}(u) : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \right\rangle \right\rangle \right\} \right].
 \end{aligned} \tag{11}$$

Further in (11), we find that

$$\begin{aligned}
 \sigma(A_{k+s}) &= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left[\min \left\{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s}}}(x), \right. \right. \\
 &\quad \left. \left. \left\langle \left\langle \sup_{u \leq x} \mu_{X_m}(u) : 1 \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \right\rangle \right\rangle \right\} \right],
 \end{aligned}$$

$$\begin{aligned}
& \left\langle \left\langle \sup_{u \leq x} \mu_{X_m}(u) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \right\rangle \right\rangle \Bigg] \\
&= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left[\min \{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s}}}(x), 1, \right. \\
&\quad \left. \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle \} \right] \\
&= \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left[\min \{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s}}}(x), \right. \\
&\quad \left. \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle \} \right] \\
&= \max_{0 \leq s \leq n-k} \left\{ \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+s} \leq n} \left\{ \min \{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s}}}(x), \right. \right. \\
&\quad \left. \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+s} \rangle \rangle \} \right\} \Bigg\} \\
&= \max_{0 \leq s \leq n-k} \left\{ \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_k \leq n} \left\{ \min \{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_k}}(x), \right. \right. \\
&\quad \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_k \rangle \rangle \} \Bigg\}, \\
&\quad \max_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+1} \leq n} \left\{ \min \{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+1}}}(x), \right. \\
&\quad \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+1} \rangle \rangle \} \Bigg\}, \\
&\quad \dots, \min \{ \mu_{X_1}(x), \mu_{X_2}(x), \dots, \mu_{X_n}(x) \} \Bigg\} \\
&= \max_{0 \leq s \leq n-k} \left\{ \min_{1 \leq i_1 \neq i_2 \neq \dots \neq i_k \leq n} \left\{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_k}}(x), \right. \right. \\
&\quad \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_k \rangle \rangle \Bigg\}, \\
&\quad \min_{1 \leq i_1 \neq i_2 \neq \dots \neq i_{k+1} \leq n} \left\{ \mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+1}}}(x), \right. \\
&\quad \langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+1} \rangle \rangle \Bigg\}, \\
&\quad \dots, \min \{ \mu_{X_1}(x), \mu_{X_2}(x), \dots, \mu_{X_n}(x) \} \Bigg\}.
\end{aligned} \tag{12}$$

(12) is obtained by using (10) and the fact that $a_1 \leq a_2 \leq \dots \leq a_i < x \leq a_{i+1} \leq a_{i+2} \leq \dots \leq a_n$, whereas (12) follows by using the fact that $\mu_X(x) \leq 1$ for all $x \in \mathbb{R}^+$. Now two cases may arise.

Case I ($n - i \geq k$). Let $n - i = k + s_1$, for some integer $s_1 \geq 0$. We have to obtain the maximum of the minimum of the sets enlisted in (12). By Lemma 7, it follows that the maximum of the minimum of these sets will be the one which is the smallest subset of all the sets. The smallest subset will be the one for which $\{i_1, i_2, \dots, i_{k+s_1}\}$ is a permutation of $\{i+1, i+2, \dots, n\}$; that is,

$$\{\mu_{X_{i_1}}(x), \mu_{X_{i_2}}(x), \dots, \mu_{X_{i_{k+s_1}}}(x)\}$$

$$= \{\mu_{X_{i+1}}(x), \mu_{X_{i+2}}(x), \dots, \mu_{X_n}(x)\}.$$

(13)

Then $\{\langle \langle \mu_{X_m}(x) : (i+1) \leq m \leq n, m \neq i_1 \neq i_2 \neq \dots \neq i_{k+1} \rangle \rangle\}$ is a void set. In that case, the set $\{\mu_{X_{i+1}}(x), \mu_{X_{i+2}}(x), \dots, \mu_{X_n}(x)\}$ becomes the smallest subset in (12), thereby using Lemma 7 in (12); we conclude that

$$\mu_X(x) = \min \{ \mu_{X_j}(x), j = i+1, \dots, n \}. \tag{14}$$

Case II ($n - i < k$). Let $r = k - n + i$. Then the smallest subset will be the one for which $k - r$ of $i_1, i_2, \dots, i_k$ is a

permutation of $\{i+1, i+2, \dots, n\}$, and the rest r values come from $\{1, 2, \dots, i\}$. Since here we have to consider all possible combinations in which r of $i_1, i_2, \dots, i_k$ takes values from $\{1, 2, \dots, i\}$, it follows from (12) that

$$\mu_X(x) = \max \left[\min \left\{ \mu_{X_j}(x), j = i+1, \dots, n, \mu_{l_1}(x), \mu_{l_2}(x), \dots, \mu_{l_{k-n+i}}(x), \right. \right. \\ \left. \left. 1 \leq l_1 \neq l_2 \neq \dots \neq l_{k-n+i} \leq i \right\} \right]. \quad (15)$$

This completes the proof. $\square$

Interestingly, the possibility distribution function as given in Theorem 9 for a k -out-of- n system is true only when $k < n$. For $k = n$, the theorem is stated below, which generalize Kai-Yuan et al. [5] work giving the possibility distribution a series system with two components. The proof is omitted for brevity.

Theorem 10. For a series system, let X_j denote the lifetime of the j th component. Assume that X_j 's are normal unrelated and strictly convex fuzzy variables with the corresponding continuous possibility distribution function $\mu_{X_j}(x)$, $j = 1, 2, \dots, n$. Let X denote the system lifetime. Then there exists a set of unique real numbers $\{a_1, a_2, \dots, a_n\}$, $a_i \in \mathbb{R}^+$ for $i = 1, 2, \dots, n$, such that the possibility distribution function of X , denoted by $\mu_X(x)$, with $a_1 \leq a_2 \leq \dots \leq a_i < x \leq a_{i+1} \leq a_{i+2} \leq \dots \leq a_n$ is given by

$$\mu_X(x) = \begin{cases} \min \{ \mu_{X_j}(x) : j = i, i-1, \dots, 1 \} & \text{if } 1 \leq i \leq n, \\ \max \{ \mu_{X_j}(x) : j = 1, 2, \dots, n \} & \text{if } x < a_1 \leq a_2 \leq \dots \leq a_n. \end{cases} \quad (16)$$

4. Properties of the Posbist Systems

In this section, we study a few properties of the posbist systems and compare them with the analogous results of the conventional reliability theory. For the sake of brevity, we omit the proofs of the subsequent theorems.

Theorem 11. Assume that X_j is a normal unrelated and strictly convex fuzzy variable denoting the lifetime of the j th component with corresponding continuous possibility distribution function $\mu_{X_j}(x)$, $j = 1, 2, \dots, n$. Let the lifetime values of a k -out-of- n system and a $(k+1)$ -out-of- n system formed by the components X_j , $j = 1, 2, \dots, n$, have the possibility distribution functions $\mu_{k:n}(x)$ and $\mu_{k+1:n}(x)$, respectively. Then, for $a_1 \leq a_2 \leq \dots \leq a_i < x \leq a_{i+1} \leq a_{i+2} \leq \dots \leq a_n$, one has

$$\mu_{k:n}(x) \geq \mu_{k+1:n}(x), \quad (17)$$

where $\{a_1, a_2, \dots, a_n\}$, $a_i \in \mathbb{R}^+$, $i = 1, 2, \dots, n$, is a unique set of real numbers.

Corollary 12. Under the conditions stated in Theorem 11, it follows that

$$R_{k:n}(x) \geq R_{k+1:n}(x) \quad \text{for } x \in \mathbb{R}^+, \quad (18)$$

where $R_{k:n}(x)$ and $R_{k+1:n}(x)$ denote the posbist reliability of k -out-of- n and $(k+1)$ -out-of- n system, respectively.

Below we generalize Theorems 4.6 and 4.8 of Kai-Yuan et al. [5] where the results are proved for series and parallel systems, respectively.

Theorem 13. The reliability of a k -out-of- n system with an arbitrary number of unrelated identical components coincides with the reliability of a single component.

Proof. Let us denote the lifetime of a k -out-of- n system by $X_{k:n}$ having its components with lifetime represented by X_i for $i \in \{1, 2, \dots, n\}$. Since the components are identical, we have $\mu_{X_i}(x) = \mu_{X_j}(x)$ for $1 \leq i \neq j \leq n$. It follows from Theorem 9 and Definition 6 that

$$\mu_{X_{k:n}}(x) = \mu_{X_i}(x), \quad \text{for } i \in \{1, 2, \dots, n\}. \quad (19)$$

Thus, the system reliability $R_{k:n}(x)$ of a k -out-of- n system is given by

$$\begin{aligned} R_{k:n}(x) &= \sigma(X_{k:n} > x) \\ &= \sup_{u > x} \mu_{X_{k:n}}(u) = \sup_{u > x} \mu_{X_i}(u) \\ &= R_i(x). \end{aligned} \quad (20)$$

This completes the proof. $\square$

Remark 14. The reliability of a k -out-of- n system consisting of arbitrary number of unrelated identical components does not change due to the addition of an identical component as contrary to the conventional reliability theory.

Similarly, we give the results for n -out-of- n system (series system with n components) using Theorem 10.

Theorem 15. The reliability of an n -out-of- n system of unrelated identical components coincides with the reliability of a single component.

Remark 16. The reliability of an n -out-of- n system of unrelated identical components does not change due to the addition of an identical component as contrary to the conventional reliability theory.

In conventional reliability, the reliability of a parallel system increases monotonically with the number of components contained in it, whereas the reliability of a series system decreases monotonically with the number of components contained in it. But these may not be true in case of the posbist reliability.

5. Illustration with a Few Examples

The change with time in the possibility distribution function and reliability function has been studied in this section when the component lifetimes are normal or Cauchy (cf. Cai [7]) as follows.

FIGURE 1: Component lifetimes having the Cauchy possibility distribution function of k -out-of- n systems with $n = 10$.

(I) A fuzzy variable X is said to have the Cauchy possibility distribution function if

$$\mu_X(x) = \frac{1}{1 + \alpha(x - a)^\beta}, \quad (21)$$

for $\alpha > 0$ and $\beta > 0$, is an even integer. Then the corresponding reliability function becomes

$$R(t) = \begin{cases} 1 & \text{if } t \leq a, \\ \frac{1}{1 + \alpha(t - a)^\beta} & \text{if } t > a. \end{cases} \quad (22)$$

Figure 1 shows the graph of the membership function for a k -out-of-10 system for different values of k when the component lifetimes are Cauchy fuzzy variables.

(II) A fuzzy variable X is said to have normal possibility distribution if its possibility distribution function is given by

$$\mu_X(x) = \exp[-k(x - a)^2], \quad k > 0. \quad (23)$$

Thus, the reliability function becomes

$$R(t) = \begin{cases} 1 & \text{if } t \leq a, \\ \exp[-k(t - a)^2] & \text{if } t > a. \end{cases} \quad (24)$$

The membership function of a k -out-of-10 system when the component lifetimes are normal fuzzy variables is plotted in Figure 2 for different values of k .

6. Concluding Remarks

The work done in this paper on various posbist systems can be concluded in the following points.

FIGURE 2: Component lifetimes having normal possibility distribution function of k -out-of- n systems with $n = 10$.

- (i) The possibility distribution functions of 1-out-of-2 and 2-out-of-2 systems (given in Kai-Yuan et al. [5]) can be obtained (as special cases of the present work) from the possibility distribution functions of k -out-of- n and n -out-of- n systems by using $k = 1$ and $n = 2$.
- (ii) In the posbist reliability theory, the possibility distribution function and hence the reliability function of a series system, that is, n -out-of- n system, cannot be obtained from the possibility distribution function of a k -out-of- n system by using $k = n$, whereas in the conventional reliability theory, the reliability function of an n -out-of- n system can be obtained as a special case from the reliability function of a k -out-of- n system.
- (iii) From Figures 1 and 2, it is clear that the possibility distribution and hence the reliability function of a k -out-of- n system are greater than that of $(k + 1)$ -out-of- n system. This matches with the result of the conventional reliability theory.
- (iv) It is also evident that the differences in the values of the possibility distribution function of a k -out-of- n system with different values of k become appreciable for large values of x .

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