Research Article

# Oscillating Flows of Fractionalized Second Grade Fluid 

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#### Abstract

New exact solutions for the motion of a fractionalized (this word is suitable when fractional derivative is used in constitutive or governing equations) second grade fluid due to longitudinal and torsional oscillations of an infinite circular cylinder are determined by means of Laplace and finite Hankel transforms. These solutions are presented in series form in term of generalized $G_{a, b, c}(\cdot, t)$ functions and satisfy all imposed initial and boundary conditions. In special cases, solutions for ordinary second grade and Newtonian fluids are obtained. Furthermore, other equivalent forms of solutions for ordinary second grade and Newtonian fluids are presented and written as sum of steady-state and transient solutions. The solutions for Newtonian fluid coincide with the well-known classical solutions. Finally, by means of graphical illustrations, the influence of pertinent parameters on fluid motion as well as comparison among different models is discussed.


## 1. Introduction

In recent years, the non-Newtonian fluids have received considerable attention by scientist and engineers. Such interest is inspired by practical applications of non-Newtonian fluids in industry and engineering applications. The shear stress and shear rate in non-Newtonian fluids are connected by a relation in a nonlinear manner. Because of diverse fluids characteristics in nature, all the non-Newtonian fluids cannot be described by a single constitutive relation [1-6]. Thus, among the several existing non-Newtonian fluid models, there is one which is most famous model called second grade fluid [7]. Although the constitutive equation of second grade fluid is simpler than that for the rate type fluids (those fluids which encounter viscoelastic and memory effects), it has been shown by Walters [8] that, for many
types of problems in which the flow is slow enough in the viscoelastic sense, the results given using Oldroyd fluid will be substantially similar to those obtained for second grade fluid. Therefore, if we discuss the result in this manner, it is reasonable to use the second grade fluid to carry out the calculations as compared to other non-Newtonian fluids. This fact seems to be true, not only for exact analytic solutions but even for numerical solutions. The second grade fluid is the simplest subclass of non-Newtonian fluids for which one can reasonably hope to obtain exact analytic solutions. Some recent attempts regarding exact analytic solutions for the flow of a second grade fluid are present in [9-16].

Linear viscoelasticity is certainly the field of most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory. During the twentieth century, a number of authors have (implicitly or explicitly) used the fractional calculus as an empirical method of describing the properties of viscoelastic materials [17]. A motivation for using fractional order operators in viscoelasticity is that a whole spectrum of viscoelastic mechanisms can be included in a single internal variable [18]. The stress relaxation spectrum for the fractional order model is continuous with the relaxation constant as the most probable relaxation time, while the order of the operator plays the role of a distribution parameter. Note that the spectrum is discrete for the classical model that is based on integer order derivatives. By a suitable choice of material parameters for the classical viscoelastic model, it is observed both numerically and analytically that the classical model with a large number of internal variables (each representing a specific viscoelastic mechanism) converges to the fractional model with a single internal variable [18, 19]. In other cases, it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [20]. The use of fractional derivatives within the context of viscoelasticity was firstly proposed by Germant [21]. Later, Bagley and Torvik [22] demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives, and Makris et al. [23] achieved a very good fit of the experimental data when the fractional derivative Maxwell model has been used instead of the Maxwell model for the silicon gel fluid. Some important recent attempts of fractional derivative approach to non-Newtonian fluids to obtain exact analytic solutions are listed here [24-30].

The oscillating flow of the viscoelastic fluid in cylindrical pipes has been applied in many fields, such as industries of petroleum, chemistry, and bioengineering. In the field of bioengineering, this type of investigation is of particular interest since blood in veins is forced by a periodic pressure gradient. In the petroleum and chemical industries, there are also many problems which involve the dynamic response of the fluid to the frequency of the periodic pressure gradient. An excellent collection of papers on oscillating flow can be found in the paper by Yin and Zhu [31]. We also include some important studies of nonNewtonian fluids, where oscillating boundary value problems are used in cylindrical region [32-40]. Consequently, for completeness and motivated by the above remarks, we solve our problem for fractionalized second grade fluid. The aim of this paper is to find some new and closed-form exact solutions for the oscillating flows of fractionalized second grade fluid. More precisely, our objective is to find the velocity field and the shear stresses corresponding to the motion of a fractionalized second grade fluid through a cylinder due to longitudinal and torsional oscillations of an infinite circular cylinder. The general solutions are obtained using the discrete Laplace and finite Hankel transforms. They are presented in series form in term of the $G_{a, b, c}(\cdot, t)$ functions in simpler forms as comparison to known results from literature. The solutions for similar motion of ordinary second grade and Newtonian fluids are obtained as spacial cases from general solutions. Equivalent forms of the solutions for ordinary second
grade and Newtonian fluids are also constructed and presented as a sum between steadystate and transient solutions. The equivalent forms of general solutions for Newtonian fluid coincide with the well known classical solutions from the literature. Finally, the influence of material and fractional parameters on the motion of fractionalized second grade fluid is underlined by graphical illustrations. The difference among fractionalized, ordinary second grade and Newtonian fluid models is also spotlighted.

## 2. Governing Equations for Fractionalized Second Grade Fluid

The Cauchy stress $T$ in an incompressible homogeneous fluid of second grade is related to the fluid motion in the following manner:

$$
\begin{equation*}
T=-p I+S, \quad S=\mu A_{1}+\alpha_{1} A_{2}+\alpha_{2} A_{1}^{2} \tag{2.1}
\end{equation*}
$$

where $-p I$ is the indeterminate part of the stress due to the constraint of incompressibility, $S$ is the extra-stress tensor, $\mu$ is the dynamic viscosity, $\alpha_{1}$ and $\alpha_{2}$ are the normal stress moduli, and $A_{1}$ and $A_{2}$ are the kinematic tensors defined through

$$
\begin{equation*}
A_{1}=(\nabla V)+(\nabla V)^{T}, \quad A_{2}=\frac{d A_{1}}{d t}+A_{1}(\nabla V)+(\nabla V)^{T} A_{1} \tag{2.2}
\end{equation*}
$$

In the above equations, $V$ is the velocity field, $\nabla$ is the gradient operator, and $d / d t$ denotes the material time derivative. Since the fluid is incompressible, it can undergo only isochoric motion, and the equations of motion are

$$
\begin{equation*}
\nabla \cdot V=0, \quad \nabla \cdot T=\rho \frac{d V}{d t}+\rho b \tag{2.3}
\end{equation*}
$$

where $\rho$ is the constant density of the fluid and $b$ is the body force. If the model (2.1) is required to be compatible with thermodynamics in the sense that all motions satisfy the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material moduli must meet the following restrictions [41]:

$$
\begin{equation*}
\mu \geq 0, \quad \alpha_{1} \geq 0, \quad \alpha_{1}+\alpha_{2}=0 \tag{2.4}
\end{equation*}
$$

The sign of the material moduli $\alpha_{1}$ and $\alpha_{2}$ has been the subject of much controversy. A comprehensive discussion on the restrictions given in (2.4) as well as a critical review on the fluids of differential type can be found in the extensive work by Dunn and Rajagopal [42].

For the problem under consideration, we shall assume a velocity field and an extrastress of the form

$$
\begin{equation*}
V=V(r, t)=w(r, t) e_{\theta}+v(r, t) e_{z} \tag{2.5}
\end{equation*}
$$

where $e_{\theta}$ and $e_{z}$ are unit vectors in the $\theta$ and $z$-directions of the cylindrical coordinate system $r, \theta$ and $z$. For such flows the constraint, of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t=0$, then

$$
\begin{equation*}
V(r, 0)=0 \tag{2.6}
\end{equation*}
$$

and (2.1) implies $S_{r r}=0$ and the meaningful equations

$$
\begin{equation*}
\tau_{1}(r, t)=\left(\mu+\alpha_{1} \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right) w(r, t), \quad \tau_{2}(r, t)=\left(\mu+\alpha_{1} \frac{\partial}{\partial t}\right) \frac{\partial v(r, t)}{\partial r}, \tag{2.7}
\end{equation*}
$$

where $\tau_{1}=S_{r \theta}$ and $\tau_{2}=S_{r z}$ are the shear stresses that are different of zero.
The equation of motion $(2.3)_{2}$, in the absence of a pressure gradient in the axial direction and neglecting body forces, leads to the relevant equations ( $\partial_{\theta} p=0$ due to the rotational symmetry)

$$
\begin{equation*}
\rho \frac{\partial w(r, t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right) \tau_{1}(r, t), \quad \rho \frac{\partial v(r, t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \tau_{2}(r, t) \tag{2.8}
\end{equation*}
$$

Eliminating $\tau_{1}$ and $\tau_{2}$ between (2.7) and (2.8), we attain the governing equations

$$
\begin{align*}
& \frac{\partial w(r, t)}{\partial t}=\left(v+\alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}-\frac{1}{r^{2}}\right) w(r, t), \quad r \in(0, R), t>0 \\
& \frac{\partial v(r, t)}{\partial t}=\left(v+\alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}\right) v(r, t), \quad r \in(0, R), t>0 \tag{2.9}
\end{align*}
$$

where $v=\mu / \rho$ is the kinematic viscosity and $\alpha=\alpha_{1} / \rho$ is the material parameter of the fluid. The governing equations corresponding to an incompressible fractionalized second grade fluid, performing the same motion, are

$$
\begin{gather*}
\frac{\partial w(r, t)}{\partial t}=\left(v+\alpha D_{t}^{\beta}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}-\frac{1}{r^{2}}\right) w(r, t), \quad r \in(0, R), t>0  \tag{2.10}\\
\frac{\partial v(r, t)}{\partial t}=\left(v+\alpha D_{t}^{\beta}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}\right) v(r, t), \quad r \in(0, R), t>0 \tag{2.11}
\end{gather*}
$$

$$
\begin{gather*}
\tau_{1}(r, t)=\left(\mu+\alpha_{1} D_{t}^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right) w(r, t)  \tag{2.12}\\
\tau_{2}(r, t)=\left(\mu+\alpha_{1} D_{t}^{\beta}\right) \frac{\partial v(r, t)}{\partial r} \tag{2.13}
\end{gather*}
$$

where $0<\beta<1$ is the fractional parameter. Of course, the new material constant $\alpha_{1}$, although for simplicity we keep the same notation, tends to the original $\alpha_{1}$ as $\beta \rightarrow 1$. The fractional differential operator so-called Caputo fractional operator $D_{t}^{\beta}$ defined by [43, 44]

$$
D_{t}^{\beta} f(t)= \begin{cases}\frac{1}{\Gamma(1-\beta)} \int_{0}^{t} \frac{f^{\prime}(\tau)}{(t-\tau)^{\beta}} d \tau, & 0 \leq \beta<1  \tag{2.14}\\ \frac{d f(t)}{d t}, & \beta=1\end{cases}
$$

and $\Gamma(\cdot)$ is the Gamma function.

## 3. Oscillating Flows of Fractionalized Second Grade Fluids

Let us consider an incompressible fractionalized second grade fluid at rest, in an infinitely long cylinder of radius $R$ as shown in Figure 1. At time $t=0^{+}$, the cylinder starts to oscillate according to

$$
\begin{equation*}
\mathbf{V}(R, t)=\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right] e_{\theta}+\left[V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)\right] e_{z} \tag{3.1}
\end{equation*}
$$

where $\omega_{1}$ and $\omega_{2}$ are the frequencies of the velocity of the cylinder and $V_{1}, V_{2}, W_{1}$, and $W_{2}$ are constant amplitudes. Owing to the shear, the fluid in cylinder is gradually moved, its velocity being of the form (2.5). The governing equations are given by (2.10)-(2.13) while the associated initial and boundary conditions are

$$
\begin{equation*}
w(r, 0)=v(r, 0)=0, \quad r \in(0, R) \tag{3.2}
\end{equation*}
$$

respectively, and

$$
\begin{equation*}
w(R, t)=W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right), \quad v(R, t)=V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right), \quad t \geq 0 \tag{3.3}
\end{equation*}
$$

where $H(t)$ is the Heaviside function [45]. In the following, the system of fractional partial differential equations (2.10)-(2.13), with appropriate initial and boundary conditions, will be solved by means of Laplace and finite Hankel transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [24-30].


Figure 1: Geometry of the problem for oscillating flows of fractionalized second grade fluid through a cylinder.

### 3.1. Calculation of the Velocity Field

Applying the Laplace transform to (2.10) and (2.11) and having in mind the initial and boundary conditions (3.2) and (3.4), we find that

$$
\begin{gather*}
q \bar{w}(r, q)=\left(v+\alpha q^{\beta}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}-\frac{1}{r^{2}}\right) \bar{w}(r, q), \quad r \in(0, R),  \tag{3.4}\\
q \bar{v}(r, q)=\left(v+\alpha q^{\beta}\right)\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}\right) \bar{v}(r, q), \quad r \in(0, R), \tag{3.5}
\end{gather*}
$$

where the image functions $\bar{w}(r, q)$ and $\bar{v}(r, q)$ of $w(r, t)$ and $v(r, t)$ have to satisfy the conditions

$$
\begin{equation*}
\bar{w}(R, q)=\frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}}, \quad \bar{v}(R, q)=\frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}} . \tag{3.6}
\end{equation*}
$$

Multiplying now both sides of (3.4) and (3.5) by $r J_{1}\left(r r_{m}\right)$ and $r J_{0}\left(r r_{n}\right)$, respectively, integrating them with respect to $r$ from 0 to $R$ and taking into account the conditions (3.6) and the known relations $[46,47]$

$$
\begin{gather*}
\int_{0}^{R} r\left(\frac{\partial^{2} \bar{w}}{\partial r^{2}}+\frac{1}{r} \frac{\partial \bar{w}}{\partial r}-\frac{\bar{w}}{r^{2}}\right) J_{1}\left(r r_{m}\right) d r=R r_{m} J_{2}\left(R r_{m}\right) \bar{w}(R, t)-r_{m}^{2} \bar{w}_{H}\left(r_{m}, t\right), \\
\int_{0}^{R} r\left(\frac{\partial^{2} \bar{v}}{\partial r^{2}}+\frac{1}{r} \frac{\partial \bar{v}}{\partial r}\right) J_{0}\left(r r_{n}\right) d r=R r_{n} J_{1}\left(R r_{n}\right) \bar{v}(R, t)-r_{n}^{2} \bar{v}_{H}\left(r_{n}, t\right), \tag{3.7}
\end{gather*}
$$

we find that

$$
\begin{align*}
\bar{w}_{H}\left(r_{m}, q\right) & =R r_{m} J_{2}\left(R r_{m}\right) \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}} \frac{v+\alpha q^{\beta}}{q+\alpha r_{m}^{2} q^{\beta}+v r_{m}^{2}} \\
\bar{v}_{H}\left(r_{n}, q\right) & =R r_{n} J_{1}\left(R r_{m}\right) \frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}} \frac{v+\alpha q^{\beta}}{q+\alpha r_{n}^{2} q^{\beta}+v r_{n}^{2}}, \tag{3.8}
\end{align*}
$$

where $[46,47]$

$$
\begin{equation*}
\bar{w}_{H}\left(r_{m}, q\right)=\int_{0}^{R} r \bar{w}(r, q) J_{1}\left(r r_{m}\right) d r, \quad \bar{v}_{H}\left(r_{n}, q\right)=\int_{0}^{R} r \bar{v}(r, q) J_{0}\left(r r_{n}\right) d r, \quad m, n=1,2,3, \ldots \tag{3.9}
\end{equation*}
$$

are the Hankel transforms of $\bar{w}(r, q)$ and $\bar{v}(r, q)$, while $r_{m}$ and $r_{n}$ are the positive roots of the transcendental equations $J_{1}(R r)=0$ and $J_{0}(R r)=0$, respectively. In order to determine $\bar{w}(r, q)$ and $\bar{v}(r, q)$, we must apply the inverse Hankel transforms. However, for a more suitable presentation of final results, we firstly rewrite in (3.8), in the equivalent forms:

$$
\begin{align*}
\bar{w}_{H}\left(r_{m}, q\right) & =\frac{R J_{2}\left(R r_{m}\right)}{r_{m}} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}}-\frac{R J_{2}\left(R r_{m}\right)}{r_{n}} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}} \frac{q}{q+\alpha r_{m}^{2} q^{\beta}+v r_{m}^{2}}  \tag{3.10}\\
\bar{v}_{H}\left(r_{n}, q\right) & =\frac{R J_{1}\left(R r_{n}\right)}{r_{n}} \frac{V_{1} q+V_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}}-\frac{R J_{1}\left(R r_{n}\right)}{r_{n}} \frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{1}^{2}} \frac{q}{q+\alpha r_{n}^{2} q^{\beta}+v r_{n}^{2}}
\end{align*}
$$

and apply the inverse Hankel transform formulae [46, 47]

$$
\begin{equation*}
\bar{w}(r, q)=\frac{2}{R^{2}} \sum_{m=1}^{\infty} \bar{w}_{H}\left(r_{m}, q\right) \frac{J_{1}\left(r r_{m}\right)}{J_{2}^{2}\left(R r_{m}\right)}, \quad \bar{v}(r, q)=\frac{2}{R^{2}} \sum_{n=1}^{\infty} \bar{v}_{H}\left(r_{n}, q\right) \frac{J_{0}\left(r r_{n}\right)}{J_{1}^{2}\left(R r_{n}\right)} \tag{3.11}
\end{equation*}
$$

Taking into account the following results [47]:

$$
\begin{equation*}
\int_{0}^{R} r^{2} J_{1}\left(r r_{m}\right) d r=\frac{R^{2}}{r_{m}} J_{2}\left(R r_{m}\right), \quad \int_{0}^{R} r J_{0}\left(r r_{n}\right) d r=\frac{R}{r_{n}} J_{1}\left(R r_{n}\right), \tag{3.12}
\end{equation*}
$$

we find that

$$
\begin{align*}
& \bar{w}(r, q)=\frac{R}{r} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}}-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}} \frac{q}{q+\alpha r_{m}^{2} q^{\beta}+v r_{m}^{2}} \\
& \bar{v}(r, q)=\frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}}-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{2}\left(R r_{n}\right)} \frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}} \frac{q}{q+\alpha r_{n}^{2} q^{\beta}+v r_{n}^{2}} \tag{3.13}
\end{align*}
$$

Finally, in order to obtain $w(r, t)=\mathcal{L}^{-1}\{\bar{w}(r, q)\}$ and $v(r, t)=\mathcal{L}^{-1}\{\bar{v}(r, q)\}$ and to avoid the lengthy calculations of residues and contour integrals, we will apply the discrete inverse Laplace transform method [24-30]. For this, we firstly write (3.13) in series form

$$
\begin{align*}
\bar{w}(r, q)= & \frac{R}{r} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}}-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} \\
& \times\left[W_{1} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i}}{i!} \frac{1}{q^{(i-k) \beta+k+2 j+1}}+W_{2} \omega_{1} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i}}{i!} \frac{1}{q^{(i-k) \beta+k+2 j+2}}\right],  \tag{3.14}\\
\bar{v}(r, q)= & \frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}}-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k} \\
& \times\left[V_{1} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i}}{i!} \frac{1}{q^{(i-k) \beta+k+2 j+1}}+V_{2} \omega_{2} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i}}{i!} \frac{1}{q^{(i-k) \beta+k+2 j+2}}\right],
\end{align*}
$$

where we used the fact that

$$
\begin{equation*}
\binom{k}{i}=\frac{(-1)^{i}(-k)_{i}}{i!} \tag{3.15}
\end{equation*}
$$

and $(-k)_{i}$ is the Pochhammer symbol

$$
(k)_{i}= \begin{cases}1, & i=0  \tag{3.16}\\ k(k+1) \cdots(k+i-1), & i \in N\end{cases}
$$

In particular $(0)_{0}=1,(k)_{0}=1$ and $(0)_{i}=0$, for $i \in N$. Applying the discrete inverse Laplace transform, we get

$$
\begin{aligned}
w(r, q)= & \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} \\
& \times\left[W_{1} H(t) \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i} t^{(i-k) \beta+k+2 j}}{i!\Gamma((i-k) \beta+k+2 j+1)}+W_{2} \omega_{1} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i} t^{(i-k) \beta+k+2 j+1}}{i!\Gamma((i-k) \beta+k+2 j+2)}\right]
\end{aligned}
$$

$$
\begin{align*}
v(r, q)= & V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k} \\
& \times\left[V_{1} H(t) \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i} t^{k((i-k) \beta+k+2 j)}}{i!\Gamma((i-k) \beta+k+2 j+1)}+V_{2} \omega_{2} \sum_{i=0}^{\infty} \frac{(-k)_{i}(-v / \alpha)^{i} t^{(i-k) \beta+k+2 j+1}}{i!\Gamma((i-k) \beta+k+2 j+2)}\right] \tag{3.17}
\end{align*}
$$

In terms of the generalized $G_{a, b, c}(\cdot, t)$ functions [48], we rewrite the above equations in simple forms:

$$
\begin{align*}
w(r, q)= & \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} \\
& \times\left[W_{1} H(t) G_{\beta,-k-2 j-1,-k}\left(-\frac{v}{\alpha}, t\right)+W_{2} \omega_{1} G_{\beta,-k-2 j-2,-k}\left(-\frac{v}{\alpha}, t\right)\right]  \tag{3.18}\\
v(r, q)= & V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k}  \tag{3.19}\\
& \times\left[V_{1} H(t) G_{\beta,-k-2 j-1,-k}\left(-\frac{v}{\alpha}, t\right)+V_{2} \omega_{2} G_{\beta,-k-2 j-2,-k}\left(-\frac{v}{\alpha}, t\right)\right]
\end{align*}
$$

where the generalized $G_{a, b, c}(\cdot, t)$ function is defined by [48]

$$
\begin{equation*}
G_{a, b, c}(d, t)=\sum_{j=0}^{\infty} \frac{(c)_{j} d^{j}}{j!} \frac{t^{(c+j) a-b-1}}{\Gamma[(c+j) a-b]}, \quad \operatorname{Re}(a c-b)>0, \quad \operatorname{Re}(q)>0, \quad\left|\frac{d}{q^{a}}\right|<1 \tag{3.20}
\end{equation*}
$$

### 3.2. Calculation of the Shear Stress

Applying the Laplace transform to (2.12) and (2.13), we find that

$$
\begin{align*}
& \overline{\tau_{1}}(r, q)=\left(\mu+\alpha_{1} q^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right) \bar{w}(r, q)  \tag{3.21}\\
& \overline{\tau_{2}}(r, q)=\left(\mu+\alpha_{1} q^{\beta}\right) \frac{\partial \bar{v}(r, q)}{\partial r}
\end{align*}
$$

where

$$
\begin{align*}
\frac{\partial \bar{w}(r, q)}{\partial r}-\frac{1}{r} \bar{w}(r, q) & =\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{W_{1} q+W_{2} \omega_{1}}{q^{2}+\omega_{1}^{2}} \frac{q}{q+\alpha r_{m}^{2} q^{\beta}+v r_{m}^{2}} \\
\frac{\partial \bar{v}(r, q)}{\partial r} & =\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{J_{1}\left(R r_{m}\right)} \frac{V_{1} q+V_{2} \omega_{2}}{q^{2}+\omega_{2}^{2}} \frac{q}{q+\alpha r_{n}^{2} q^{\beta}+v r_{n}^{2}} \tag{3.22}
\end{align*}
$$

have been obtained form (3.13), using the identities

$$
\begin{equation*}
r r_{m} J_{1}^{\prime}\left(r r_{m}\right)-J_{1}\left(r r_{m}\right)=-r r_{m} J_{2}\left(r r_{m}\right), \quad J_{0}^{\prime}\left(r r_{n}\right)=-r_{n} J_{1}\left(r r_{n}\right) \tag{3.23}
\end{equation*}
$$



$$
000 t=2.1 \mathrm{~s} \quad \Leftrightarrow \theta t=2.3 \mathrm{~s}
$$

(a)

$\theta=2.1 \mathrm{~s} \quad \Leftrightarrow t=2.3 \mathrm{~s}$
$\Delta \Delta t=2.2 \mathrm{~s}$
(b)

Figure 2: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=33, \alpha=0.5, \beta=0.5$, and different values of $t$.

Substituting (3.22) into (3.21), respectively, and applying again the discrete inverse Laplace transform method, we find that the shear stresses $\tau_{1}(r, t)$ and $\tau_{2}(r, t)$ have the following forms:

$$
\begin{align*}
\tau_{1}(r, t)=\frac{2 \alpha \rho}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} & {\left[W_{1} H(t) G_{\beta,-k-2 j-1,-k-1}\left(-\frac{v}{\alpha}, t\right)\right.} \\
& \left.+W_{2} \omega_{1} G_{\beta,-k-2 j-2,-k-1}\left(-\frac{v}{\alpha}, t\right)\right],  \tag{3.24}\\
\tau_{2}(r, t)=\frac{2 \alpha \rho}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k}[ & V_{1} H(t) G_{\beta,-k-2 j-1,-k-1}\left(-\frac{v}{\alpha^{\prime}}, t\right) \\
+ & \left.V_{2} \omega_{2} G_{\beta,-k-2 j-2,-k-1}\left(-\frac{v}{\alpha}, t\right)\right] .
\end{align*}
$$

## 4. Limiting Cases

### 4.1. Ordinary Second Grade Fluid $(\beta \rightarrow 1)$

Making $\beta \rightarrow 1$ into (3.18), (3.19), (3.24), we obtain the solutions

$$
\begin{aligned}
w_{\mathrm{OSG}}(r, q)= & \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} \\
& \times\left[W_{1} H(t) G_{1,-k-2 j-1,-k}\left(-\frac{v}{\alpha}, t\right)+W_{2} \omega_{1} G_{1,-k-2 j-2,-k}\left(-\frac{v}{\alpha}, t\right)\right], \\
v_{\mathrm{OSG}}(r, q)= & V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k} \\
& \times\left[V_{1} H(t) G_{1,-k-2 j-1,-k}\left(-\frac{v}{\alpha}, t\right)+V_{2} \omega_{2} G_{1,-k-2 j-2,-k}\left(-\frac{v}{\alpha}, t\right)\right],
\end{aligned}
$$

$$
\begin{align*}
\tau_{1 \mathrm{OSG}}(r, t)= & \frac{2 \alpha \rho}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{m}^{2}\right)^{k} \\
& \times\left[W_{1} H(t) G_{1,-k-2 j-1,-k-1}\left(-\frac{v}{\alpha}, t\right)+W_{2} \omega_{1} G_{1,-k-2 j-2,-k-1}\left(-\frac{v}{\alpha}, t\right)\right], \\
\tau_{2 \mathrm{OSG}}(r, t)= & \frac{2 \alpha \rho}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j} \sum_{k=0}^{\infty}\left(-\alpha r_{n}^{2}\right)^{k} \\
& \times\left[V_{1} H(t) G_{1,-k-2 j-1,-k-1}\left(-\frac{v}{\alpha}, t\right)+V_{2} \omega_{2} G_{1,-k-2 j-2,-k-1}\left(-\frac{v}{\alpha}, t\right)\right], \tag{4.1}
\end{align*}
$$

corresponding to an ordinary second grade fluid, performing the same motion. Other equivalent forms of solutions for ordinary second grade fluids can be directly obtained from (3.10) by substituting $\beta=1$, and performing the inverse Laplace transform. The expressions for velocity field are given by

$$
\begin{equation*}
w_{\mathrm{OSG}}(r, t)=w_{\mathrm{OSS}}(r, t)+w_{\mathrm{OST}}(r, t), \quad v_{S}(r, t)=v_{\mathrm{OSS}}(r, t)+v_{\mathrm{OST}}(r, t) \tag{4.2}
\end{equation*}
$$

where

$$
\begin{align*}
& w_{\mathrm{OSS}}= \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2 \omega_{1}}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \\
& \times\left[\frac{W_{1} H(t)\left[\omega_{1}\left(1+\alpha r_{m}^{2}\right) \cos \left(\omega_{1} t\right)-v r_{m}^{2} \sin \left(\omega_{1} t\right)\right]}{v^{2} r_{m}^{4}+\omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)^{2}}\right. \\
&\left.+\frac{W_{2}\left[v r_{m}^{2} \cos \left(\omega_{1} t\right)+\omega_{1}\left(1+\alpha r_{m}^{2}\right) \sin \left(\omega_{1} t\right)\right]}{v^{2} r_{m}^{4}+\omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)^{2}}\right], \\
& w_{\text {OST }}=-\frac{2 v}{R} \sum_{m=1}^{\infty} \frac{r_{m} J_{1}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{W_{1} H(t) v r_{m}^{2}-W_{2} \omega_{1}\left(1+\alpha r_{m}^{2}\right)}{\left(1+\alpha r_{m}^{2}\right)\left[v^{2} r_{m}^{4}+\omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)^{2}\right]} \exp \left(-\frac{v r_{m}^{2} t}{1+\alpha r_{m}^{2}}\right),  \tag{4.3}\\
& v_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2 \omega_{2}}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \\
& \times\left[\frac{V_{1} H(t)\left[\omega_{2}\left(1+\alpha r_{n}^{2}\right) \cos \left(\omega_{2} t\right)-v r_{n}^{2} \sin \left(\omega_{2} t\right)\right]}{v^{2} r_{n}^{4}+\omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)^{2}}\right. \\
&\left.\quad+\frac{V_{2}\left[v r_{n}^{2} \cos \left(\omega_{2} t\right)+\omega_{2}\left(1+\alpha r_{n}^{2}\right) \sin \left(\omega_{2} t\right)\right]}{v^{2} r_{n}^{4}+\omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)^{2}}\right], \\
& w_{\text {OST }}=-\frac{2 v}{R} \sum_{n=1}^{\infty} \frac{r_{n} J_{0}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} \frac{V_{1} H(t) v r_{n}^{2}-V_{2} \omega_{2}\left(1+\alpha r_{n}^{2}\right)}{\left(1+\alpha r_{n}^{2}\right)\left[v^{2} r_{n}^{4}+\omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)^{2}\right]} \exp \left(-\frac{v r_{n}^{2} t}{1+\alpha r_{n}^{2}}\right),
\end{align*}
$$



$$
w_{1}=0.9 \mathrm{~s} \quad w_{1}=1.1 \mathrm{~s}
$$

(a)

$w_{2}=0.9 \mathrm{~s} \quad w_{2}=1.1 \mathrm{~s}$
$\leftrightarrow w_{2}=1 \mathrm{~s} \quad--\cdot w_{2}=1.2 \mathrm{~s}$
(b)

Figure 3: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, v=0.5566, \mu=33, \alpha=0.5, \beta=0.5$, and different values of $w_{1}$ and $w_{2}$, respectively.


Figure 4: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=33, \beta=0.5, t=2.5 \mathrm{~s}$, and different values of $\alpha$.
are the steady-state and transient solutions. Introducing (4.2) into (2.7), we find that

$$
\begin{equation*}
\tau_{1 \mathrm{OSG}}(r, t)=\tau_{1 \mathrm{OSS}}(r, t)+\tau_{1 \mathrm{OST}}(r, t), \quad \tau_{2 \mathrm{OSG}}(r, t)=\tau_{2 \mathrm{OSS}}(r, t)+\tau_{2 \mathrm{OST}}(r, t), \tag{4.4}
\end{equation*}
$$




$$
\begin{array}{ll}
\leftrightarrow \beta=0.1 & \leftrightarrow \beta=0.5 \\
\leftrightarrow \beta=0.3 & -\beta=0.8
\end{array}
$$

(a)

$$
\begin{array}{ll}
\leftrightarrow \beta=0.1 & \leftrightarrow \beta=0.5 \\
\leftrightarrow \beta=0.3 & -\beta=0.8
\end{array}
$$

(b)

Figure 5: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, \nu=0.5566, \mu=33, \alpha=0.5, t=2.5 \mathrm{~s}$, and different values of $\beta$.
where the steady-state and transient components are given by

$$
\begin{gather*}
\begin{aligned}
& \tau_{1 \mathrm{OSS}}= \frac{2 \rho \omega_{1}}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{1}{v^{2} r_{m}^{4}+\omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)^{2}} \\
& \times\left[W_{1} H(t)\left\{v \omega_{1} \cos \left(\omega_{1} t\right)-\left[v^{2} r_{m}^{2}+\alpha \omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)\right] \sin \left(\omega_{1} t\right)\right\}\right. \\
&\left.+W_{2}\left\{\left[v^{2} r_{m}^{2}+\alpha \omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)\right] \cos \left(\omega_{1} t\right)+v \omega_{1} \sin \left(\omega_{1} t\right)\right\}\right], \\
& \tau_{1 \mathrm{OST}}=\frac{2 \rho v^{2}}{R} \sum_{m=1}^{\infty} \frac{r_{m}^{2} J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{W_{1} H(t) v r_{m}^{2}-W_{2} \omega_{1}\left(1+\alpha r_{m}^{2}\right)}{\left(1+\alpha r_{m}^{2}\right)^{2}\left[v^{2} r_{m}^{4}+\omega_{1}^{2}\left(1+\alpha r_{m}^{2}\right)^{2}\right]} \exp \left(-\frac{v r_{m}^{2} t}{1+\alpha r_{m}^{2}}\right), \\
& \tau_{2 \mathrm{OSS}}= \frac{2 \rho \omega_{2}}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{m}\right)} \frac{1}{v^{2} r_{n}^{4}+\omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)^{2}} \\
& \times\left[V_{1} H(t)\left\{v \omega_{2} \cos \left(\omega_{2} t\right)-\left[v^{2} r_{n}^{2}+\alpha \omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)\right] \sin \left(\omega_{2} t\right)\right\}\right. \\
&\left.+V_{2}\left\{\left[v^{2} r_{n}^{2}+\alpha \omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)\right] \cos \left(\omega_{2} t\right)+v \omega_{2} \sin \left(\omega_{2} t\right)\right\}\right], \\
& \tau_{2 \mathrm{OST}}=\frac{2 \rho v^{2}}{R} \sum_{m=1}^{\infty} \frac{r_{n}^{2} J_{1}\left(r r_{m}\right)}{J_{1}\left(R r_{n}\right)} \frac{V_{1} H(t) v r_{n}^{2}-V_{2} \omega_{2}\left(1+\alpha r_{n}^{2}\right)}{\left(1+\alpha r_{n}^{2}\right)^{2}\left[v^{2} r_{n}^{4}+\omega_{2}^{2}\left(1+\alpha r_{n}^{2}\right)^{2}\right]} \exp \left(-\frac{v r_{n}^{2} t}{1+\alpha r_{n}^{2}}\right)
\end{aligned}
\end{gather*}
$$



Figure 6: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, \rho=59.289, \alpha=0.1, \beta=0.8, t=2.5 \mathrm{~s}$, and different values of $\mathcal{v}$.

In practice, the steady-state solutions for unsteady motions of Newtonian or nonNewtonian fluids are important for those who need to eliminate transients from their rheological measurements. Consequently, an important problem regarding the technical relevance of these solutions is to find the approximate time after which the fluid is moving according to the steady-state. More exactly, in practice it is necessary to know the required time to reach the steady-state.

### 4.2. Newtonian Fluids ( $\alpha_{1} \rightarrow 0$ )

Making the limit $\alpha_{1}$ and then $\alpha \rightarrow 0$ into (3.13), (3.22), and proceeding as in the last section, the solutions for a Newtonian fluid

$$
\begin{aligned}
w_{N}(r, q)= & \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \\
& \times\left[W_{1} H(t) G_{1,-2 j, 1}\left(-v r_{m}^{2}, t\right)+W_{2} \omega_{2} G_{1,-2 j-1,1}\left(-v r_{m}^{2}, t\right)\right], \\
v_{N}(r, q)= & V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j} \\
& \times\left[V_{1} H(t) G_{1,-2 j, 1}\left(-v r_{n}^{2}, t\right)+V_{2} \omega_{2} G_{1,-2 j-1,1}\left(-v r_{n}^{2}, t\right)\right], \\
\tau_{1 N}(r, t)= & \frac{2 \mu}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \\
& \times \sum_{j=0}^{\infty}\left(-\omega_{1}^{2}\right)^{j}\left[W_{1} H(t) G_{1,-2 j, 1}\left(-v r_{m}^{2}, t\right)+W_{2} \omega_{1} G_{1,-2 j-1,1}\left(-v r_{m}^{2}, t\right)\right],
\end{aligned}
$$

$$
\begin{align*}
\tau_{2 N}(r, t)= & \frac{2 \mu}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} \\
& \times \sum_{j=0}^{\infty}\left(-\omega_{2}^{2}\right)^{j}\left[V_{1} H(t) G_{1,-2 j, 1}\left(-v r_{n}^{2}, t\right)+V_{2} \omega_{2} G_{1,-2 j-1,1}\left(-v r_{n}^{2}, t\right)\right] \tag{4.9}
\end{align*}
$$

are obtained. Similarly by making $\alpha_{1}$ and then $\alpha \rightarrow 0$ into (4.2)-(4.8), the corresponding solutions

$$
\begin{gather*}
w_{N}(r, t)=w_{N S}(r, t)+w_{N T}(r, t), \quad v_{N}(r, t)=v_{N S}(r, t)+v_{N T}(r, t),  \tag{4.10}\\
\tau_{1 N}(r, t)=\tau_{1 N S}(r, t)+\tau_{1 N T}(r, t), \quad \tau_{2 N}(r, t)=\tau_{2 N S}(r, t)+\tau_{2 N T}(r, t), \tag{4.11}
\end{gather*}
$$

where

$$
\begin{align*}
w_{N S}= & \frac{r}{R}\left[W_{1} H(t) \cos \left(\omega_{1} t\right)+W_{2} \sin \left(\omega_{1} t\right)\right]-\frac{2 \omega_{1}}{R} \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} \\
& \times\left[\frac{W_{1} H(t)\left[\omega_{1} \cos \left(\omega_{1} t\right)-v r_{m}^{2} \sin \left(\omega_{1} t\right)\right]+W_{2}\left[v r_{m}^{2} \cos \left(\omega_{1} t\right)+\omega_{1} \sin \left(\omega_{1} t\right)\right]}{v^{2} r_{m}^{4}+\omega_{1}^{2}}\right]  \tag{4.12}\\
w_{N T}= & -\frac{2 v}{R} \sum_{m=1}^{\infty} \frac{r_{m} J_{1}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{W_{1} H(t) v r_{m}^{2}-W_{2} \omega_{1}}{v^{2} r_{m}^{4}+\omega_{1}^{2}} e^{-v r_{m}^{2} t}  \tag{4.13}\\
v_{N S}= & V_{1} H(t) \cos \left(\omega_{2} t\right)+V_{2} \sin \left(\omega_{2} t\right)-\frac{2 \omega_{2}}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} \\
& \times\left[\frac{V_{1} H(t)\left[\omega_{2} \cos \left(\omega_{2} t\right)-v r_{n}^{2} \sin \left(\omega_{2} t\right)\right]+V_{2}\left[v r_{n}^{2} \cos \left(\omega_{2} t\right)+\omega_{2} \sin \left(\omega_{2} t\right)\right]}{v^{2} r_{n}^{4}+\omega_{2}^{2}}\right]  \tag{4.14}\\
v_{N T}= & -\frac{2 v}{R} \sum_{n=1}^{\infty} \frac{r_{n} J_{0}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} \frac{V_{1} H(t) v r_{n}^{2}-V_{2} \omega_{2}}{v^{2} r_{n}^{4}+\omega_{2}^{2}} e^{-v r_{m}^{2} t}  \tag{4.15}\\
\tau_{1 N S}= & \frac{2 \rho \omega_{1}}{R} \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{1}{v^{2} r_{m}^{4}+\omega_{1}^{2}} \\
& \times\left[W_{1} H(t)\left\{v \omega_{1} \cos \left(\omega_{1} t\right)-v^{2} r_{m}^{2} \sin \left(\omega_{1} t\right)\right\}+W_{2}\left\{v^{2} r_{m}^{2} \cos \left(\omega_{1} t\right)+v \omega_{1} \sin \left(\omega_{1} t\right)\right\}\right]  \tag{4.16}\\
\tau_{1 N T}= & \frac{2 \rho v^{2}}{R} \sum_{m=1}^{\infty} \frac{r_{m}^{2} J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} \frac{W_{1} H(t) v r_{m}^{2}-W_{2} \omega_{1}}{v^{2} r_{m}^{4}+\omega_{1}^{2}} e^{-v r_{m}^{2} t}, \tag{4.17}
\end{align*}
$$

$$
\begin{align*}
\tau_{2 N S}= & \frac{2 \rho \omega_{2}}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{m}\right)} \frac{1}{v^{2} r_{n}^{4}+\omega_{2}^{2}} \\
& \times\left[V_{1} H(t)\left\{v \omega_{2} \cos \left(\omega_{2} t\right)-v^{2} r_{n}^{2} \sin \left(\omega_{2} t\right)\right\}+V_{2}\left\{v^{2} r_{n}^{2} \cos \left(\omega_{2} t\right)+v \omega_{2} \sin \left(\omega_{2} t\right)\right\}\right]  \tag{4.18}\\
\tau_{2 N T}= & \frac{2 \rho v^{2}}{R} \sum_{m=1}^{\infty} \frac{r_{n}^{2} J_{1}\left(r r_{m}\right)}{J_{1}\left(R r_{n}\right)} \frac{V_{1} H(t) v r_{n}^{2}-V_{2} \omega_{2}}{v^{2} r_{n}^{4}+\omega_{2}^{2}} e^{-v r_{n}^{2} t} \tag{4.19}
\end{align*}
$$

for Newtonian fluids are obtained. Substituting $W_{1}=R \Omega, W_{2}=0, V_{1}=U, V_{2}=0, \omega_{1}=0$, and $\omega_{2}=0$ in (4.9) and using the definition of generalized $G_{a, b, c}(\cdot, t)$ functions, the solutions

$$
\begin{align*}
& w_{N}(r, t)=r \Omega-2 \Omega \sum_{m=1}^{\infty} \frac{J_{1}\left(r r_{m}\right)}{r_{m} J_{2}\left(R r_{m}\right)} e^{-v r_{m}^{2} t} \\
& v_{N}(r, t)=U-\frac{2 U}{R} \sum_{n=1}^{\infty} \frac{J_{0}\left(r r_{n}\right)}{r_{n} J_{1}\left(R r_{n}\right)} e^{-v r_{n}^{2} t}  \tag{4.20}\\
& \tau_{1 N}(r, t)=2 \mu \Omega \sum_{m=1}^{\infty} \frac{J_{2}\left(r r_{m}\right)}{J_{2}\left(R r_{m}\right)} e^{-v r_{m}^{2} t} \\
& \tau_{2 N}(r, t)=\frac{2 \mu U}{R} \sum_{n=1}^{\infty} \frac{J_{1}\left(r r_{n}\right)}{J_{1}\left(R r_{n}\right)} e^{-v r_{n}^{2} t}
\end{align*}
$$

obtained in [49, equations (36)-(39)] by a different technique are recovered. Of course the above expressions can also be obtained form (4.10)-(4.19).

## 5. Numerical Results and Conclusions

The velocity fields and the adequate shear stresses corresponding to the unsteady motions of an incompressible fractionalized second grade fluid due to longitudinal and torsional oscillations of an infinite circular cylinder have been determined by means of the Laplace and finite Hankel transforms. The general solutions are written in series form in term of generalized $G_{a, b, c}(\cdot, t)$ functions and satisfy all imposed initial and boundary conditions. The solutions for ordinary second grade and Newtonian fluids performing the same motion are obtain as special cases of general solutions. Furthermore, another equivalent solutions for ordinary second grade and Newtonian fluids are presented, in terms of steady-state and transient solutions. They describe the motion of the fluid sometime after its initiation. After that time, when the transients disappear, they tend to the steady-state solutions, which are periodic in time and independent of the initial conditions. It is also shown that for $W_{1}=R \Omega$, $W_{2}=0, V_{1}=U, V_{2}=0, \omega_{1}=0$ and $\omega_{2}=0$, (4.9) reduce to the well-known classical solutions [49, equations (36)-(39)]. The similar solutions corresponding to the sine and cosine oscillations of the boundary are immediately obtained by making $V_{2}=W_{2}=0$, respectively, $V_{1}=W_{1}=0$ into general solutions.

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity components $w(r, t)$ and $v(r, t)$ are depicted against $r$ for different values of $t$ and the pertinent parameters. Figure 2 contains the diagrams of the velocity components for four different times $t$; it is obvious to see the impact of rigid boundary of


Figure 7: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ given by (3.18) and (3.19), for $R=3, W_{1}=$ $W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, \nu=0.5566, \mu=33, \alpha=0.5, \beta=0.5$, and different values of $r$.


Figure 8: Profiles of the velocity components $w(r, t)$ given by (3.18), for $R=3, w_{1}=1, v=0.5566, \mu=$ $33, \alpha=0.5, \beta=0.5, t=2.5 \mathrm{~s}$, and different values of $W_{1}$ and $W_{2}$.
the cylinder on the motion of the fluid. Further, the amplitude of oscillations for the two components of the velocity decreases with increasing values of $t$. However, this conclusion cannot be generalized. The influence of the frequency of oscillations $\omega_{1}$ and $\omega_{2}$, on fluid motion is shown in Figure 3. The amplitudes of both components of velocity are decreasing functions of frequency of oscillations $\omega_{1}$ and $\omega_{2}$ respectively. The effect of material parameter


Figure 9: Profiles of the velocity components $v(r, t)$ given by (3.19), for $R=3, w_{2}=1, v=0.5566, \mu=$ 33, $\alpha=0.5, \beta=0.5, t=2.5 \mathrm{~s}$, and different values of $V_{1}$ and $V_{2}$.
$\alpha$ on fluid motion is discussed in Figure 4. The influence of material parameter $\alpha$ is quite similar to that of Figures 2 and 3. Nowadays fractional derivative approach in viscoelastic fluid plays an important role to describe the behavior of complex fluid. Therefore, it is important to see the effect of fractional parameter on oscillating fluid. Figure 5 depict the influence of fractional parameter $\beta$ on fluid motion. It is again clear that the amplitude of fluid oscillations decreases with respect to fractional parameter $\beta$. The viscosity is an important property of the fluid. It is observed that the amplitude of oscillations is an increasing function of kinematic viscosity $v$ in this geometry as shown in Figure 6. The influence of radius $r$ against time $t$ is shown in Figure 7. The oscillating behavior of fluid motion clearly results from these figures. As expected, the amplitude of oscillations increases with the increasing values of $r$. The influence of $W_{1}$ and $W_{2}$ on rotational component $w(r, t)$ and the effect of $V_{1}$ and $V_{2}$ on longitudinal component $v(r, t)$ are presented in Figures 8 and 9. The influence of $W_{1}$ and $W_{2}$ on $w(r, t)$ is quite opposite. For instance, $w(r, t)$ is a decreasing function with respect to $W_{1}$ and an increasing one of $W_{2}$. The longitudinal component $v(r, t)$ is an increasing function of $V_{1}$ near the center of cylinder and of $V_{2}$ on the whole flow domain.

Finally, for comparison, the diagrams of $w(r, t)$ and $v(r, t)$ corresponding to the three models, fractionalized second grade $(\beta=0.3$ and $\beta=0.6)$, ordinary second grade $(\beta=1)$, and Newtonian fluids ( $\alpha=0$ and $\beta=1$ ) are presented in Figures 10-13. It is clearly seen from Figure 10 that the fractionalized second grade fluid is the swiftest and the Newtonian one is the slowest. However, the behavior of these models is quite opposite at time $t=4 \mathrm{~s}$ as shown by Figure 11. The large time effect on oscillating fluid is shown in Figures 12 and 13. It is observed that for large time the non-Newtonian effects can be neglected for rotational component of velocity $w(r, t)$. This seems to be not true for the longitudinal component $v(r, t)$ of the velocity. The units of the material constants in Figures $2-13$ are SI units and the roots $r_{m}$ and $r_{n}$ have been approximated by $(4 m+1) \pi / 4 R$ and $(4 n-1) \pi / 4 R$, respectively.


Figure 10: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ for fractionalized second grade, ordinary second grade, and Newtonian fluids, for $R=3, W_{1}=W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=$ 33, $\alpha=0.9, \beta=0.3$ and 0.6 and different values of $t=1 \mathrm{~s}$.


Figure 11: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ for fractionalized second grade, ordinary second grade, and Newtonian fluids, for $R=3, W_{1}=W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=$ 33, $\alpha=0.9, \beta=0.3$ and 0.6 and different values of $t=4 \mathrm{~s}$.


Figure 12: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ for fractionalized second grade, ordinary second grade, and Newtonian fluids, for $R=3, W_{1}=W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=$ $33, \alpha=0.9, \beta=0.3$ and 0.6 and for large time $t$.


Figure 13: Profiles of the velocity components $w(r, t)$ and $v(r, t)$ for fractionalized second grade, ordinary second grade, and Newtonian fluids, for $R=3, W_{1}=W_{2}=V_{1}=V_{2}=1, w_{1}=w_{2}=1, v=0.5566, \mu=$ 33, $\alpha=0.9, \beta=0.3$ and 0.6 and for large time $t$.

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