Research Article

Regularity Criterion for the 3D Nematic Liquid Crystal Flows

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We study the hydrodynamic theory of liquid crystals. We prove a logarithmically improved regularity criterion for two simplified Ericksen-Leslie systems.

1. Introduction

The hydrodynamic theory of liquid crystals was established by Ericksen and Leslie [1–4]. However, since the equations are too complicated, we consider the first simplified Ericksen-Leslie system:

$$u_t + u \cdot \nabla u + \nabla \pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \quad (1.1)$$

$$d_t + u \cdot \nabla d = \Delta d - f(d), \quad (1.2)$$

$$\operatorname{div} u = 0, \quad (1.3)$$

$$(u, d)(x, 0) = (u_0, d_0)(x) \quad \text{in } \mathbb{R}^3, \quad (1.4)$$

which include the velocity vector $u := (u_1, u_2, u_3)^t$, the scalar pressure π being and the direction vector $d := (d_1, d_2, d_3)^t$. $f(d) := 1/\eta(|d|^2 - 1)d$ with η , a positive constant. $(\nabla d \odot \nabla d)_{i,j} := \sum_k \partial_i d_k \partial_j d_k$, and hence $\nabla \cdot (\nabla d \odot \nabla d) = \sum_k \Delta d_k \nabla d_k + (1/2)\nabla|\nabla d|^2$.

Lin-Liu [5] proved that the system (1.1)–(1.4) has a unique smooth solution globally in 2 space dimensions and locally in 3 dimensions. They also proved the global existence of weak solutions. However, the regularity of solutions to the system is still open. Fan-Guo [6] and Fan-Ozawa [7] showed the following regularity criteria:

$$\begin{aligned} u &\in L^r(0, T; L^s(\mathbb{R}^3)) \quad \text{for } \frac{2}{r} + \frac{3}{s} = 1, \quad 3 < s \leq \infty, \\ \nabla u &\in L^r(0, T; L^s(\mathbb{R}^3)) \quad \text{for } \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leq \infty, \\ u &\in L^2(0, T; \dot{B}_{\infty, \infty}^0), \\ \omega := \operatorname{curl} u &\in L^1(0, T; \dot{B}_{\infty, \infty}^0), \end{aligned} \tag{1.5}$$

where $\dot{B}_{\infty, \infty}^0$ denotes the homogeneous Besov space.

The first aim of this paper is to prove a new regularity criterion as follows.

Theorem 1.1. *Let $u_0 \in H^3, d_0 \in H^4$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. Let (u, d) be a smooth solution to the problem (1.1)–(1.4) on $[0, T)$. If u satisfies*

$$\int_0^T \frac{\|u(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}}^{2/(1-s)}}{1 + \log(e + \|u(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}})} dt < \infty \tag{1.6}$$

for some s with $0 < s < 1$, then the solution (u, d) can be extended beyond $T > 0$.

When the penalization parameter $\eta \rightarrow 0$, (1.1)–(1.4) reduce to

$$u_t + u \cdot \nabla u + \nabla \pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \tag{1.7}$$

$$\operatorname{div} u = 0, \tag{1.8}$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad |d| = 1, \tag{1.9}$$

$$(u, d)(x, 0) = (u_0, d_0)(x) \quad \text{in } \mathbb{R}^3, \quad |d_0| = 1. \tag{1.10}$$

When $u = 0$, then (1.9) is the well-known harmonic heat flow equation onto a sphere. Fan-Gao-Guo [8] proved the following blow-up criteria:

$$\begin{aligned} u, \nabla d &\in L^2(0, T; \dot{B}_{\infty, \infty}^0), \\ \omega, \Delta d &\in L^1(0, T; \dot{B}_{\infty, \infty}^0). \end{aligned} \tag{1.11}$$

We will prove the following theorem:

Theorem 1.2. *Let $u_0, \nabla d_0 \in H^3(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0, |d_0| = 1$ in $\mathbb{R}^3$. Let (u, d) be a smooth solution to the problem (1.7)–(1.10) on $[0, T)$. If the following condition is satisfied:*

$$\int_0^T \frac{\|u(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}}^{2/(1-s)} + \|\nabla d(\cdot, t)\|_{\dot{B}_{\infty, \infty}^0}^2}{1 + \log(e + \|u(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}} + \|\nabla d(\cdot, t)\|_{\dot{B}_{\infty, \infty}^0})} dt < \infty, \quad (1.12)$$

for some s with $0 < s < 1$, then the solution (u, d) can be extended beyond $T > 0$.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is well-known that there are $T_0 > 0$ and a unique smooth solution (u, d) to the problem (1.1)–(1.4) in $(0, T_0]$, we only need to show a priori estimates.

Testing (1.1) by u and using (1.3), we see that

$$\frac{1}{2} \frac{d}{dt} \int u^2 dx + \int |\nabla u|^2 dx = - \int (u \cdot \nabla) d \cdot \Delta d dx. \quad (2.1)$$

Testing (1.2) by $\Delta d - f(d)$ and using (1.3), we find that

$$\frac{d}{dt} \int \frac{1}{2} |\nabla d|^2 + \frac{1}{4\eta} (|d|^2 - 1)^2 dx + \int |\Delta d - f(d)|^2 dx = \int (u \cdot \nabla) d \cdot \Delta d dx. \quad (2.2)$$

Summing up (2.1) and (2.2), we infer that

$$\begin{aligned} & \int \frac{1}{2} u^2 + \frac{1}{2} |\nabla d|^2 + \frac{1}{4\eta} (|d|^2 - 1)^2 dx + \int_0^T \int |\nabla u|^2 + |\Delta d - f(d)|^2 dx \\ & \leq \int \frac{1}{2} u_0^2 + \frac{1}{2} |\nabla d_0|^2 + \frac{1}{4\eta} (|d_0|^2 - 1)^2 dx. \end{aligned} \quad (2.3)$$

Testing (1.2) by d and using (1.3), we deduce that

$$\frac{1}{2} \frac{d}{dt} \int d^2 dx + \int |\nabla d|^2 dx + \frac{1}{\eta} \int |d|^4 dx = \frac{1}{\eta} \int d^2 dx, \quad (2.4)$$

which yields

$$\|d\|_{L^\infty(0, T; L^2)} + \|d\|_{L^2(0, T; H^1)} \leq C. \quad (2.5)$$

Next, we prove the following estimate:

$$\|d\|_{L^\infty(0, T; L^\infty)} \leq \max(\|d_0\|_{L^\infty}, 1). \quad (2.6)$$

Without loss of generality, we assume that $1 \leq \|d_0\|_{L^\infty}$. Multiplying (1.2) by d , we get

$$\phi_t + u \cdot \nabla \phi - \Delta \phi + 2|d|^2 \phi = -\frac{2}{\eta} \left(\|d_0\|_{L^\infty}^2 - 1 \right) |d|^2 - 2|\nabla d|^2 \leq 0 \quad (2.7)$$

with $\phi = |d|^2 - \|d_0\|_{L^\infty}^2$ and $\phi|_{t=0} = |d_0|^2 - \|d_0\|_{L^\infty}^2 \leq 0$. Then (2.6) follows immediately from $\phi \leq 0$ by the maximum principle.

Testing (1.1) by $-\Delta u$ and using (1.3), we see that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \int |\Delta u|^2 dx &= \int (u \cdot \nabla) u \cdot \Delta u dx \\ &\quad - \sum_{i,k} \int \Delta d_k \cdot \partial_i \nabla d_k \cdot \nabla u_i dx - \sum_{i,k} \int \partial_i d_k \cdot \nabla \Delta d_k \cdot \nabla u_i dx. \end{aligned} \quad (2.8)$$

Applying Δ to (1.2), testing by Δd , and using (1.3), we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 dx + \int |\nabla \Delta d|^2 dx &= \sum_{i,k} \int \partial_i d_k \cdot \nabla \Delta d_k \cdot \nabla u_i dx \\ &\quad - \sum_{i,k} \int \partial_i \partial_j d_k \cdot \partial_j \nabla d_k \cdot \nabla u_i dx - \int \Delta f(d) \cdot \Delta d dx. \end{aligned} \quad (2.9)$$

Summing up (2.8) and (2.9), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 + |\Delta d|^2 dx + \int |\Delta u|^2 + |\nabla \Delta d|^2 dx \\ &= \int (u \cdot \nabla) u \cdot \Delta u dx - \sum_{i,k} \int \Delta d_k \cdot \partial_i \nabla d_k \cdot \nabla u_i dx \\ &\quad - \sum_{i,k} \int \partial_i \partial_j d_k \cdot \partial_j \nabla d_k \cdot \nabla u_i dx - \int \Delta f(d) \Delta d dx =: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (2.10)$$

By using (2.6), I_4 is simply bounded as

$$I_4 \leq C \|\Delta d\|_{L^2}^2. \quad (2.11)$$

By using the inequalities [9]

$$\begin{aligned} \|u \cdot \nabla u\|_{L^2} &\leq C \|u\|_{\dot{B}_{\infty,\infty}^{-s}} \|\nabla u\|_{\dot{B}_{2,2}^s}, \\ \|\nabla u\|_{\dot{B}_{2,2}^s} &\leq C \|\nabla u\|_{L^2}^{1-s} \|\Delta u\|_{L^2}^s. \end{aligned} \quad (2.12)$$

I_1 can be bounded as follows:

$$\begin{aligned}
I_1 &\leq \|u \cdot \nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\leq C \|u\|_{\dot{B}_{\infty,\infty}^{-s}} \|u\|_{\dot{B}_{2,2}^{1+s}} \|\Delta u\|_{L^2} \\
&\leq C \|u\|_{\dot{B}_{\infty,\infty}^{-s}} \|\nabla u\|_{L^2}^{1-s} \|\Delta u\|_{L^2}^{1+s} \\
&\leq \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)} \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.13}$$

We bound I_2 and I_3 as follows:

$$\begin{aligned}
I_2, I_3 &\leq C \|\nabla u\|_{L^2} \|\Delta d\|_{L^4}^2 \\
&\leq C \|\nabla u\|_{L^2} \|\nabla \Delta d\|_{L^2} \\
&\leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.14}$$

Here we used the Gagliardo-Nirenberg inequality

$$\|\Delta d\|_{L^4}^2 \leq C \|d\|_{L^\infty} \|\nabla \Delta d\|_{L^2}. \tag{2.15}$$

Inserting the above estimates into (2.10), we derive

$$\begin{aligned}
&\frac{d}{dt} \int |\nabla u|^2 + |\Delta d|^2 dx + \int |\Delta u|^2 + |\nabla \Delta d|^2 dx \\
&\leq C \left(\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)} + 1 \right) \left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right) \\
&\leq C \frac{1 + \|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)}}{1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}})} \left(1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}}) \right) \left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right) \\
&\leq C \frac{1 + \|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)}}{1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}})} (1 + \log(e + y)) \left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right).
\end{aligned} \tag{2.16}$$

Due to (1.6), one concludes that for any small constant $\epsilon > 0$, there exists $T_* < T$ such that

$$\int_{T_*}^T \frac{\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)}}{1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}})} dt < \epsilon. \tag{2.17}$$

For any $T_* < t \leq T$, we set

$$y(t) := \sup_{T_* \leq \tau \leq t} \left\| \Lambda^3(u, \nabla d)(\cdot, \tau) \right\|_{L^2} \quad \text{with } \Lambda := (-\Delta)^{1/2}. \tag{2.18}$$

Applying Gronwall's inequality to (2.16) in the interval $[T_*, t]$, one has

$$\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2 \leq C(1 + y)^{C_0\epsilon}. \quad (2.19)$$

Now, we derive a bound on $y(t)$ defined by (2.18). To this end, we will use the following commutator and product estimates due to Kato-Ponce [10]:

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leq C\left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{\alpha-1} g\|_{L^{q_1}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right), \quad (2.20)$$

$$\|\Lambda^\alpha(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^\alpha g\|_{L^{q_1}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \quad (2.21)$$

with $\alpha > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Applying Λ^3 to (1.1), testing by $\Lambda^3 u$, and using (1.3), (2.20), (2.21) and (2.19), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\Lambda^3 u|^2 dx + \int |\Lambda^4 u|^2 dx \\ &= - \int \left(\Lambda^3(u \cdot \nabla u) - u \nabla \Lambda^3 u \right) \Lambda^3 u dx + \int \Lambda^3(\nabla d \odot \nabla d) : \Lambda^3 \nabla u dx \\ &\leq C \|\nabla u\|_{L^3} \|\Lambda^3 u\|_{L^3}^2 + C \|\nabla d\|_{L^\infty} \|\Lambda^4 d\|_{L^2} \|\Lambda^4 u\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2}^{3/4} \|\Lambda^3 u\|_{L^2}^{1/4} \cdot \|\nabla u\|_{L^2}^{1/3} \|\Lambda^4 u\|_{L^2}^{5/3} + \frac{1}{16} \|\Lambda^4 u\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Lambda^4 d\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^{13/2} \|\Lambda^3 u\|_{L^2}^{3/2} + C \|\Delta d\|_{L^2}^{3/2} \|\Lambda^4 d\|_{L^2}^{1/2} \cdot \|\Delta d\|_{L^2}^{2/3} \|\Lambda^5 d\|_{L^2}^{4/3} \\ &\leq \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C \|\nabla u\|_{L^2}^{13/2} \|\Lambda^3 u\|_{L^2}^{3/2} + C \|\Delta d\|_{L^2}^{13/2} \|\Lambda^4 d\|_{L^2}^{3/2}. \end{aligned} \quad (2.22)$$

Here we have used the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned} \|\nabla u\|_{L^3} &\leq C \|\nabla u\|_{L^2}^{3/4} \|\Lambda^3 u\|_{L^2}^{1/4}, \\ \|\Lambda^3 u\|_{L^3} &\leq C \|\nabla u\|_{L^2}^{1/6} \|\Lambda^4 u\|_{L^2}^{5/6}, \\ \|\nabla d\|_{L^\infty} &\leq C \|\Delta d\|_{L^2}^{3/4} \|\Lambda^4 d\|_{L^2}^{1/4}, \\ \|\Lambda^4 d\|_{L^2} &\leq C \|\Delta d\|_{L^2}^{1/3} \|\Lambda^5 d\|_{L^2}^{2/3}. \end{aligned} \quad (2.23)$$

Taking Λ^4 to (1.2), testing by $\Lambda^4 d$, and using (1.3), (2.20), (2.23), and (2.6), we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int |\Lambda^4 d|^2 dx + \int |\Lambda^5 d|^2 dx \\
&= - \int (\Lambda^4 (u \cdot \nabla d) - u \nabla \Lambda^4 d) \Lambda^4 d dx - \int \Lambda^4 f(d) \cdot \Lambda^4 d dx \\
&\leq C \|\nabla u\|_{L^3} \|\Lambda^4 d\|_{L^6} \|\Lambda^4 d\|_{L^2} + C \|\nabla d\|_{L^\infty} \|\Lambda^4 u\|_{L^2} \|\Lambda^4 d\|_{L^2} + C \|\Lambda^4 d\|_{L^2}^2 \\
&\leq C \|\nabla u\|_{L^2}^{3/4} \|\Lambda^3 u\|_{L^2}^{1/4} \cdot \|\Delta d\|_{L^2}^{1/3} \|\Lambda^5 d\|_{L^2}^{5/3} \\
&\quad + \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Lambda^4 d\|_{L^2}^2 + C \|\Lambda^4 d\|_{L^2}^2 \\
&\leq \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C \|\nabla u\|_{L^2}^{9/2} \|\Delta d\|_{L^2}^2 \|\Lambda^3 u\|_{L^2}^{3/2} \\
&\quad + C \|\Delta d\|_{L^2}^{13/2} \|\Lambda^4 d\|_{L^2}^{3/2} + C \|\Lambda^4 d\|_{L^2}^2.
\end{aligned} \tag{2.24}$$

Summing up (2.22) and (2.24) and taking ϵ small enough, we arrive at

$$\begin{aligned}
& \|u\|_{L^\infty(0,T;H^3)} + \|u\|_{L^2(0,T;H^4)} \leq C, \\
& \|d\|_{L^\infty(0,T;H^4)} + \|d\|_{L^2(0,T;H^5)} \leq C.
\end{aligned} \tag{2.25}$$

This completes the proof.

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Since it is easy to prove that there are $T_0 > 0$ and a unique smooth solution (u, p, d) to the problem (1.7)–(1.10) in $[0, T_0]$, we only need to prove a priori estimates.

First, as in the previous section, we still have (2.1).

Testing (1.9) by $-\Delta d$, using $d\Delta d = -|\nabla d|^2$ and $|d| = 1$, we see that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d|^2 dx \\
&= \int (u \cdot \nabla d) \Delta d dx + \int (d \cdot \Delta d)^2 dx \\
&\leq \int (u \cdot \nabla d) \Delta d dx + \int |\Delta d|^2 dx.
\end{aligned} \tag{3.1}$$

Summing up (2.1) and (3.1), we find that

$$\int \frac{1}{2} u^2 + \frac{1}{2} |\nabla d|^2 dx + \int_0^T \int |\nabla u|^2 dx dt \leq \int \frac{1}{2} u_0^2 + |\nabla d_0|^2 dx. \tag{3.2}$$

Similarly to (2.10), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 + |\Delta d|^2 dx + \int |\Delta u|^2 + |\nabla \Delta d|^2 dx \\ & =: I_1 + I_2 + I_3 + I_5. \end{aligned} \quad (3.3)$$

Here I_1, I_2 , and I_3 are the same as that in (2.10) and can be bounded as in the previous section. The corresponding last term I_5 is written and bounded as

$$\begin{aligned} I_5 &:= - \sum_k \int \partial_k (|\nabla d|^2 d) \cdot \partial_k \Delta d dx \\ &= - \sum_k \int \partial_k d \cdot |\nabla d|^2 \cdot \partial_k \Delta d dx - \sum_k \int d \partial_k |\nabla d|^2 \cdot \partial_k \Delta d dx \\ &= \sum_k \int \partial_k (d \cdot |\nabla d|^2) \cdot \Delta d dx - \sum_k \int d \partial_k |\nabla d|^2 \cdot \partial_k \Delta d dx \\ &\leq C \|\nabla d\|_{L^4}^2 \|\Delta d\|_{L^4}^2 + \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 \\ &\leq C \|\Delta d\|_{L^2} \cdot \|\nabla d\|_{\dot{B}_{\infty,\infty}^0} \|\nabla \Delta d\|_{L^2} + \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 \\ &\leq \frac{1}{2} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2 \|\Delta d\|_{L^2}^2. \end{aligned} \quad (3.4)$$

Here we have used the following inequality [11, 12]:

$$\|\Delta d\|_{L^4}^2 \leq C \|\nabla d\|_{\dot{B}_{\infty,\infty}^0} \|\nabla \Delta d\|_{L^2} \quad (3.5)$$

and the Gagliardo-Nirenberg inequality

$$\|\nabla d\|_{L^4}^2 \leq C \|d\|_{L^\infty} \|\Delta d\|_{L^2}. \quad (3.6)$$

Substituting the above estimates into (3.3), we obtain

$$\begin{aligned} & \frac{d}{dt} \int |\nabla u|^2 + |\Delta d|^2 dx \\ & \leq C \left(\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2 \right) \left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right) \\ & \leq C \frac{\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})} (1 + \log(e + y)) \left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right). \end{aligned} \quad (3.7)$$

Due to (1.12), one concludes that for any small constant $\epsilon > 0$, there exists $T_* < T$ such that

$$\int_{T_*}^T \frac{\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{2/(1-s)} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{1 + \log(e + \|u\|_{\dot{B}_{\infty,\infty}^{-s}} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})} dt < \epsilon. \quad (3.8)$$

Applying Gronwall's inequality to (3.7) in the interval $[T_*, t]$, one has (2.19).

As in the previous section, we still have (2.22).

Similarly to (2.24), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\Lambda^4 d|^2 dx + \int |\Lambda^5 d|^2 dx \\ &= - \int (\Lambda^4 (u \cdot \nabla d) - u \nabla \Lambda^4 d) \Lambda^4 d dx + \int \Lambda^4 (d |\nabla d|^2) \Lambda^4 d dx =: J_1 + J_2. \end{aligned} \quad (3.9)$$

J_1 is bounded as that in (2.24);

$$\begin{aligned} J_2 &\leq C \|\nabla d\|_{L^\infty} \|\Lambda^4 d\|_{L^2}^2 + C \|\Lambda^4 (|\nabla d|^2)\|_{L^2} \|\Lambda^4 d\|_{L^2}, \\ &\leq C \|\nabla d\|_{L^\infty}^2 \|\Lambda^4 d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty} \|\Lambda^5 d\|_{L^2} \|\Lambda^4 d\|_{L^2}, \\ &\leq \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Lambda^4 d\|_{L^2}^2, \end{aligned} \quad (3.10)$$

then J_2 can be bounded as that in (2.24).

Combining (2.22) and (3.9) and taking ϵ small enough, we conclude that

$$\begin{aligned} \|u\|_{L^\infty(0,T;H^3)} + \|u\|_{L^2(0,T;H^4)} &\leq C, \\ \|\nabla d\|_{L^\infty(0,T;H^3)} + \|\nabla d\|_{L^2(0,T;H^4)} &\leq C. \end{aligned} \quad (3.11)$$

This completes the proof.

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