Research Article

Solutions of Unified Fractional Schrödinger Equations

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We obtain the solution of a unified fractional Schrödinger equation. The solution is derived by the application of the Laplace and Fourier transforms in closed form in terms of the Mittag-Leffler function. The result obtained here is quite general in nature and capable of yielding a very large number of results (new and known) hitherto scattered in the literature. Most of results obtained are in a form suitable for numerical computation.

1. Introduction

Recent applications of fractional differential equations to number of systems such as those exhibiting enormously slow diffusion or subdiffusion have given opportunity for physicists to study even more complicated systems. These systems include charge transport in amorphous semiconductors: the relaxation in polymer systems, fluid mechanics, viscoelasticity, and Hall effect. The generalized diffusion equation is studied to describe complex systems with anomalous behavior in much the same way as simpler systems. Fractional calculus is now considered as practical techniques in many branches of applied sciences and engineering. Several authors notably Hilfer [1], Beyer and Kempfle [2], Kempfle and Gaul [3], Schneider and Wyss [4], and Debnath [5–7] have discussed many examples of homogeneous fractional ordinary differential equations and homogeneous fractional diffusion and wave equations.

Laskin [8–10] constructed space fractional quantum mechanics by using Feynman path integrals, the only difference being that Lévy distributions are employed instead of Gaussians distributions for the set of possible paths. The Schrödinger equation thus obtained

contains fractional derivatives. Naber [11] has investigated certain properties of time fractional Schrödinger equation by writing the Schrödinger equation in terms of fractional derivatives as dimensionless objects. In recent work, solutions of fractional Schrödinger equations are investigated by Bhatti [12], Chaurasia and Singh [13], Saxena et al. [14], among others.

In the present paper, we obtain the solution of a unified fractional Schrödinger equation. The result obtained here provides an elegant extension of the results given earlier by Bhatti [12], Chaurasia and Singh [13], Saxena et al. [14], and Debnath [5].

The Riemann-Liouville fractional integral of order ν is defined by (Miller and Ross [15, page 45]; Kilbas et al. [16]),

$${}_0D_t^{-\nu} f(x, t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(x, u) du, \quad (1.1)$$

where $\text{Re}(\nu) > 0$.

The following fractional derivative of order $\alpha > 0$ is introduced by Caputo [17]; see also Kilbas et al. [16] in the form

$$\begin{aligned} {}_0D_t^\alpha f(x, t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(x, \tau) d\tau}{(t-\tau)^{\alpha+1-m}}, \quad m-1 < \alpha \leq m, \text{Re}(\alpha) > 0, m \in \mathbb{N}, \\ &= \frac{\partial^m f(x, t)}{\partial t^m}, \quad \text{if } \alpha = m, \end{aligned} \quad (1.2)$$

where $\partial^m f(x, t)/\partial t^m$ is the m th partial derivative of $f(x, t)$ with respect to t .

The Laplace transform of the Caputo derivative is given by Caputo [17]; see also Kilbas et al. [16] in the form

$$L\{{}_0D_t^\alpha f(x, t)\} = s^\alpha \bar{f}(x, s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(x, 0+), \quad (m-1 < \alpha \leq m). \quad (1.3)$$

We also need the Weyl fractional operator, defined by

$${}_{-\infty}D_x^\mu f(x, t) = \frac{1}{\Gamma(n-\mu)} \left(\frac{\partial}{\partial x} \right)^n \int_{-\infty}^x \frac{f(\tau, t)}{(x-\tau)^{\mu-n+1}} d\tau, \quad (1.4)$$

where $n = [\mu]$ is an integral part of $\mu > 0$.

Its Fourier transform is (Metzler and Klafter [18, page 59, A.11])

$$F\{{}_{-\infty}D_x^\mu f(x, t)\} = (ik)^\mu f^*(k, t), \quad (1.5)$$

where $f^*(k, t)$ is the Fourier transform of $f(x, t)$ with respect to the variable x of $f(x, t)$.

Following the convention initiated by Compte [19], we suppress the imaginary unit in Fourier space by adopting the slightly modified form of above result in our investigations (Metzler and Klafter, [18, p.59, A.12])

$$F\left\{-\infty D_x^\mu f(x, t)\right\} = -|k|^\mu f^*(k, t) \quad (1.6)$$

instead of (1.5).

2. Unified Fractional Schrödinger Equation

In this section, we will derive the solution of the unified fractional Schrödinger equation (2.1). The result is as follows.

Theorem 2.1. *Consider the following unified fractional Schrödinger equation:*

$$\begin{aligned} & {}_0D_t^\alpha \psi(x, t) + a {}_0D_t^\beta \psi(x, t) + b {}_0D_t^\gamma \psi(x, t) \\ &= \frac{i\hbar}{2m} {}_{-\infty}D_x^\mu \psi(x, t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 < \gamma \leq 1, \quad \mu > 0, \end{aligned} \quad (2.1)$$

with the initial conditions:

$$\begin{aligned} \psi(x, 0) &= \psi_0(x), \quad -\infty < x < \infty, \\ \psi(x, t) &\longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \quad t > 0, \end{aligned} \quad (2.2)$$

where $\hbar$ is Planck's constant divided by 2π , m is the mass, and $\psi(x, t)$ is a wave function of the particle. Then, for the solution of (2.1), subject to the initial conditions (2.2), there holds the formula

$$\begin{aligned} \psi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\gamma)r + (\gamma-\beta)\ell} \\ &\quad \times E_{\alpha, (\alpha-\gamma)r + (\gamma-\beta)\ell + 1}^{r+1} (-\eta |k|^\mu t^\alpha) e^{-ikx} dk \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} a \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\gamma)r + (\gamma-\beta)\ell + \alpha - \beta} \\ &\quad \times E_{\alpha, (\alpha-\gamma)r + (\gamma-\beta)\ell + \alpha - \beta + 1}^{r+1} (-\eta |k|^\mu t^\alpha) e^{-ikx} dk \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} b \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\gamma)r + (\gamma-\beta)\ell + \alpha - \gamma} \\ &\quad \times E_{\alpha, (\alpha-\gamma)r + (\gamma-\beta)\ell + \alpha - \gamma + 1}^{r+1} (-\eta |k|^\mu t^\alpha) e^{-ikx} dk, \end{aligned} \quad (2.3)$$

where $\eta = i\hbar/2m$ and $E_{\alpha, \beta}^\gamma(z)$ is the generalized Mittag-Leffler function [20].

Proof. Applying the Laplace transform with respect to the time variable t on both the sides of (2.1) and using the initial conditions (2.2), we get

$$\begin{aligned} s^\alpha \bar{\psi}(x, s) - s^{\alpha-1} \psi_0(x) + as^\beta \bar{\psi}(x, s) - as^{\beta-1} \psi_0(x) + bs^\gamma \bar{\psi}(x, s) - bs^{\gamma-1} \psi_0(x) \\ = \eta {}_{-\infty}D_x^\mu \bar{\psi}(x, s), \end{aligned} \quad (2.4)$$

where $\eta = i\hbar/2m$.

If we apply the Fourier transform with respect to variable x and use the formula (1.6), it yields

$$\begin{aligned} s^\alpha \bar{\psi}^*(k, s) - s^{\alpha-1} \psi_0^*(k) + as^\beta \bar{\psi}^*(k, s) - as^{\beta-1} \psi_0^*(k) + bs^\gamma \bar{\psi}^*(k, s) - bs^{\gamma-1} \psi_0^*(k) \\ = -\eta |k|^\mu \bar{\psi}^*(k, s). \end{aligned} \quad (2.5)$$

Solving for $\bar{\psi}^*(k, s)$, it gives

$$\begin{aligned} \bar{\psi}^*(k, s) = \frac{s^{\alpha-1} \psi_0^*(k)}{s^\alpha + as^\beta + bs^\gamma + \eta |k|^\mu} + \frac{as^{\beta-1} \psi_0^*(k)}{s^\alpha + as^\beta + bs^\gamma + \eta |k|^\mu} \\ + \frac{bs^{\gamma-1} \psi_0^*(k)}{s^\alpha + as^\beta + bs^\gamma + \eta |k|^\mu}. \end{aligned} \quad (2.6)$$

On taking the inverse Laplace transform of (2.6) and applying the formula (Saxena et al. [21]), it is seen that

$$\begin{aligned} \psi^*(k, t) = \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\gamma)r+(\gamma-\beta)\ell} \\ \times E_{\alpha, (\alpha-\gamma)r+(\gamma-\beta)\ell+1}^{r+1} (-\eta |k|^\mu t^\alpha) \\ + a \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\gamma)r+(\gamma-\beta)\ell+\alpha-\beta} \\ \times E_{\alpha, (\alpha-\gamma)r+(\gamma-\beta)\ell+\alpha-\beta+1}^{r+1} (-\eta |k|^\mu t^\alpha) \\ + b \psi_0^*(k) \sum_{r=0}^{\infty} (-1)^r \sum_{\ell=0}^r \binom{r}{\ell} a^\ell b^{r-\ell} t^{(\alpha-\beta)r+(\gamma-\beta)\ell+\alpha-\gamma} \\ \times E_{\alpha, (\alpha-\gamma)r+(\gamma-\beta)\ell+\alpha-\gamma+1}^{r+1} (-\eta |k|^\mu t^\alpha). \end{aligned} \quad (2.7)$$

Finally, the required solution (2.3) is obtained by taking inverse Fourier transform of (2.7). $\square$

3. Special Cases

If we take $b = 0$ in (2.1), then we obtain the following result.

Corollary 3.1. *Consider the following fractional Schrödinger equation*

$$\begin{aligned} {}_0D_t^\alpha \psi(x, t) + a {}_0D_t^\beta \psi(x, t) \\ = \frac{i\hbar}{2m} {}_{-\infty}D_x^\mu \psi(x, t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad \mu > 0, \end{aligned} \quad (3.1)$$

with the initial conditions:

$$\begin{aligned} \psi(x, 0) &= \psi_0(x), \\ \psi(x, t) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty, \quad t > 0, \end{aligned} \quad (3.2)$$

where $\hbar$ is Planck's constant divided by 2π , m is the mass, and $\psi(x, t)$ is the wave function of the particle. Then, for the solution of (3.1), subject to the initial conditions (3.2), there holds the formula

$$\begin{aligned} \psi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_0^*(k) \sum_{r=0}^{\infty} (-b)^r E_{\alpha, (\alpha-\beta)r+1}^{r+1} (-\eta|k|^\mu t^\alpha) e^{-ikx} dk \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} a\psi_0^*(k) \sum_{r=0}^{\infty} (-b)^r E_{\alpha, (\alpha-\beta)r+\alpha-\beta+1}^{r+1} (-\eta|K|^\mu t^\alpha) e^{-ikx} dk. \end{aligned} \quad (3.3)$$

If we set $a = 0 = b$ in (2.1), then we arrive at the following result recently obtained by Saxena et al. [14].

Corollary 3.2. *The solution of the following fractional Schrödinger equation:*

$${}_0D_t^\alpha \psi(x, t) = \frac{i\hbar}{2m} {}_{-\infty}D_x^\mu \psi(x, t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha \leq 1, \quad \mu > 0, \quad (3.4)$$

with the initial conditions

$$\begin{aligned} \psi(x, 0) &= \psi_0(x), \\ \psi(x, t) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty, \quad t > 0, \end{aligned} \quad (3.5)$$

where $\hbar$ is the Planck's constant divided by 2π , m is the mass, and $\psi(x, t)$ is a wave function of the particle, is given by

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_0^*(k) t^\alpha E_{\alpha, 1}(-\eta|k|^\mu t^\alpha) e^{-ikx} dk. \quad (3.6)$$

Finally, on taking $a = 0 = b$ and $\mu = 2$ in (2.1), then we arrive at the following result given by Bhatti [12].

Corollary 3.3. Consider the following fractional Schrödinger equation:

$${}_0D_t^\alpha \psi(x, t) = \frac{i\hbar}{2m} \nabla^2 \psi(x, t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (3.7)$$

with the initial conditions:

$$\begin{aligned} \psi(x, 0) &= \psi_0(x), \\ \psi(x, t) &\longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty, \quad t > 0, \end{aligned} \quad (3.8)$$

where $\hbar$ is Planck's constant divided by 2π , m is the mass, and $\psi(x, t)$ is a wave function of the particle. Then, for the solution of (3.7), under the initial conditions (3.8), there holds the relation:

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_0^*(k) t^\alpha E_{\alpha,1}(-\eta|k|^2 t^\alpha) e^{-ikx} dk. \quad (3.9)$$

4. Conclusion

In this paper, we have introduced a unified fractional Schrödinger equation and established solution for the same. The solution has been developed in terms of the generalized Mittag-Leffler function in a compact and elegant form with the help of Laplace and Fourier transforms and their inverses. All the results derived in this paper are in a form suitable for numerical computation. The fractional Schrödinger equation discussed in the present article contains a number of known (may be new also) fractional Schrödinger equations. The results obtained in the present paper provide an extension of the results given by Bhatti [12], Chaurasia and Singh [13], Debnath [5], and Saxena et al. [14].

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