## Research Article

# Boundedness and Compactness of the Mean Operator Matrix on Weighted Hardy Spaces 

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We investigate the boundedness and the compactness of the mean operator matrix acting on the weighted Hardy spaces.

## 1. Introduction

First in the following, we generalize the definitions coming in [1]. Let $\beta=\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0)=1$ and $1<p<\infty$. We consider the space of sequences $f=\{\widehat{f}(n)\}_{n=0}^{\infty}$ such that

$$
\begin{equation*}
\|f\|^{p}=\|f\|_{\beta}^{p}=\sum_{n=0}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p}<\infty . \tag{1.1}
\end{equation*}
$$

The notation

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \tag{1.2}
\end{equation*}
$$

will be used whether or not the series converges for any value of $z$. These are called formal power series and the set of such series is denoted by $H^{p}(\beta)$. Let $\widehat{f}_{k}(n)=\delta_{k}(n)$. So $f_{k}(z)=z^{k}$ and then $\left\{f_{k}\right\}_{k}$ is a basis such that $\left\|f_{k}\right\|=\beta(k)$. Recall that $H^{p}(\beta)$ is a reflexive Banach space with norm $\|\cdot\|_{\beta}$ and the dual of $H^{p}(\beta)$ is $H^{q}\left(\beta^{p / q}\right)$ where $1 / p+1 / q=1$ and $\beta^{p / q}=\left\{\beta(n)^{p / q}\right\}$ [2]. For some other sources on this topic see [1-12].

The study of weighted Hardy spaces lies at the interface of analytic function theory and operator theory. As a part of operator theory, research on weighted Hardy spaces is of fairly recent origin, dating back to valuable work of Allen Shields [1] in the mid- 1970s. The mean operator matrix has been the focus of attention for several decades and many of its properties have been studied. Some of basic and useful works in this area are due to Browein et al. [13-16], which are pretty large works that contain a number of interesting results and indeed they are mainly of auxiliary nature. Also, some properties of mean operator matrices have been studied recently by Lashkaripour on weighted sequence spaces [17-20]. In this paper, we have given conditions under which the mean operator matrix is bounded and compact as an operator acting on weighted Hardy spaces. More details of our works are as follows: the idea of Theorem 2.6 comes from [16]. In Theorem 2.9, we extend the method used in [20, Theorem 1.2] to show the boundedness of the mean operator matrix acting on the weighted Hardy spaces. Some inequalities are useful to find a bound for the mean operator matrix acting on weighted Hardy spaces [21-26]. For example the inequality proved in [26, Theorem 8] is used in the proof of Theorem 2.11.

## 2. Main Results

In this section we define an operator acting on $H^{p}(\beta)$ and then we will investigate its boundedness and compactness on $H^{p}(\beta)$.

Definition 2.1. Let $\left\{a_{n}\right\}$ be a sequence of positive numbers and define

$$
\begin{equation*}
A_{n}=\sum_{i=0}^{n} a_{i} \beta(i)^{p} \tag{2.1}
\end{equation*}
$$

The mean operator matrix associated with the sequence $\left\{a_{n}\right\}$ is represented by the matrix $A=\left[a_{n k}\right]_{n, k}$ and is defined by

$$
a_{n k}= \begin{cases}\frac{a_{k} \beta(n)^{p}}{A_{n}}, & 0 \leq k \leq n  \tag{2.2}\\ 0, & k>n\end{cases}
$$

From now on, by $A$ we denote the mean operator matrix associated with the fixed sequence $\left\{a_{n}\right\}$ as in Definition 2.1.

Theorem 2.2 (see [12, Theorem 1]). If $0<a_{n} \leq a_{n}+1$ for all integers $n \geq 0$, then $A$ is a bounded operator on $H^{p}(\beta)$.

Theorem 2.3 (see [12, Theorem 2]). Let $1 / p+1 / q=1$ and $b_{n}>0$ for $n=0,1, \ldots$ If

$$
\begin{align*}
& M_{1}=\sup _{n \geq 0} \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{b_{k}}{b_{n}}\right)^{1 / p}<\infty, \\
& M_{2}=\sup _{k \geq 0}^{\infty} \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{b_{n}}{b_{k}}\right)^{1 / q}<\infty, \tag{2.3}
\end{align*}
$$

then $A=\left[a_{n k}\right]_{n, k}$ is a bounded operator on $H^{p}(\beta)$ and $\|A\| \leq M_{1}^{1 / q} M_{2}^{1 / p}$.

Recall that if $a_{n}, b_{n}$ are two positive sequences, by $a_{n} \sim b_{n}$, we mean that $a_{n} / b_{n} \rightarrow 1$ whenever $n \rightarrow \infty$. Also, we write $a_{n}=o\left(b_{n}\right)$, if $a_{n} / b_{n} \rightarrow 0$ as $n \rightarrow \infty$.

Corollary 2.4. Let $\lim _{n \rightarrow \infty} n a_{n} / A_{n}$ be finite and $1 / p+1 / q=1$. If

$$
\begin{align*}
& \sup _{n \geq 0} \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{k}}{b_{n}}\right)^{1 / p}<\infty,  \tag{2.4}\\
& \sup _{k \geq 0}^{\infty} \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{n}}{b_{k}}\right)^{1 / q}<\infty,
\end{align*}
$$

then $A$ is a bounded operator on $H^{p}(\beta)$.
Proof. Put $\lim _{n \rightarrow \infty} n a_{n} / A_{n}=\beta$. Then $n a_{n} / \beta \sim A_{n}$ and so

$$
\begin{align*}
& \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{b_{k}}{b_{n}}\right)^{1 / p} \sim \beta \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{k}}{b_{n}}\right)^{1 / p} \quad \text { as } n \longrightarrow \infty  \tag{2.5}\\
& \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{b_{n}}{b_{k}}\right)^{1 / q} \sim \beta \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{n}}{b_{k}}\right)^{1 / q} \quad \text { as } k \longrightarrow \infty
\end{align*}
$$

On the other hand

$$
\begin{align*}
& \sup _{n \geq 0} \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{k}}{b_{n}}\right)^{1 / p}<\infty  \tag{2.6}\\
& \sup _{k \geq 0}^{\infty} \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{b_{n}}{b_{k}}\right)^{1 / q}<\infty
\end{align*}
$$

thus Theorem 2.3 implies that $A$ is a bounded operator on $H^{p}(\beta)$.
Lemma 2.5. Suppose that $n^{c} a_{n} / \beta(n)$ is eventually increasing when the constant $c>1-\gamma$, and eventually decreasing when $c<1-\gamma$. Let

$$
\begin{align*}
& S_{1}(n)=\frac{1}{n} \sum_{k=1}^{n} \frac{a_{k} \beta(n)}{a_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p} \\
& S_{2}(k)=k^{1 / q} \sum_{n=k}^{\infty} \frac{a_{k} \beta(n)}{a_{n} \beta(k)} \frac{1}{n^{1 /(q+1)}} \tag{2.7}
\end{align*}
$$

If $\gamma>1 / p$, then $\lim _{n \rightarrow \infty} S_{1}(n)=\lim _{k \rightarrow \infty} S_{2}(k)=1 /(\gamma-1 / p)$.

Proof. Let $1 / p+1 / q=1$ and $c_{2}<1-\gamma<c_{1}<1$. Then in either case there is a positive integer $N$ such that

$$
\begin{equation*}
\left(\frac{k}{n}\right)^{-c_{2}}<\frac{a_{k} \beta(n)}{a_{n} \beta(k)}<\left(\frac{k}{n}\right)^{-c_{1}} \tag{2.8}
\end{equation*}
$$

for $N \leq k \leq n$. Suppose first that $\gamma>1 / p$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{n^{1-1 / p} a_{n}}{\beta(n)}=\infty \tag{2.9}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{N-1} \frac{a_{k} \beta(n)}{a_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p}=0 \tag{2.10}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sup S_{1}(n) \leq \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^{n}\left(\frac{k}{n}\right)^{-c_{1}-1 / p}=\int_{0}^{1} x^{-c_{1}-1 / p} d x \tag{2.11}
\end{equation*}
$$

By calculus integral we get

$$
\begin{equation*}
\int_{0}^{1} x^{-c-1 / p} d x=\frac{1}{1-c-1 / p} ; \quad c \neq \frac{1}{q^{\prime}} \tag{2.12}
\end{equation*}
$$

and so

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \inf S_{1}(n) \geq \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^{n}\left(\frac{k}{n}\right)^{-c_{2}-\delta}=\int_{0}^{1} x^{-c_{2}-1 / p} d x=\frac{1}{1-c_{2}-1 / p} \tag{2.13}
\end{equation*}
$$

Letting $c_{1} \rightarrow 1-\gamma$ from the right and $c_{2} \rightarrow 1-\gamma$ from the left, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} S_{1}(n)=\frac{1}{r-1 / p} \tag{2.14}
\end{equation*}
$$

Also note that

$$
\begin{gather*}
\lim _{k \rightarrow \infty} \sup S_{2}(k) \leq \lim _{k \rightarrow \infty} k^{1 / q} \sum_{n=k}^{\infty}\left(\frac{k}{n}\right)^{-c_{1}} \frac{1}{n^{1 / q+1}} \\
\lim _{k \rightarrow \infty} k^{1 / q} \sum_{n=k}^{\infty}\left(\frac{k}{n}\right)^{-c_{1}} \frac{1}{n^{1 / q+1}}=\frac{1}{1 / q-c_{1}} \tag{2.15}
\end{gather*}
$$

If $c_{1} \rightarrow 1-\gamma$, then $1 / q-c_{1} \rightarrow \gamma-1 / p$ and similarly we get

$$
\begin{align*}
& \lim _{k \rightarrow \infty} \inf S_{2}(k) \geq \lim _{k \rightarrow \infty} k^{1 / q-c_{2}} \sum_{n=k}^{\infty}\left(\frac{1}{n}\right)^{1-1 / q-c_{2}} \\
& \lim _{k \rightarrow \infty} k^{1 / q-c_{2}} \sum_{n=k}^{\infty}\left(\frac{1}{n}\right)^{1-1 / q-c_{2}}=\frac{1}{1 / q-c_{2}} \tag{2.16}
\end{align*}
$$

If $c_{2} \rightarrow 1-\gamma$, then $1 / q-c_{2} \rightarrow \gamma-1 / p$. This completes the proof.
Theorem 2.6. Let $\lim _{n \rightarrow \infty} n a_{n} \beta(n)^{p} / A_{n}=\gamma, n^{c} a_{n} \beta(n)^{p}$ be eventually monotonic for any constant $c$, and $\{\beta(n)\}$ be bounded. Then $A$ is a bounded operator if $1 / \gamma<p$.

Proof. Let $\delta_{n}=n a_{n} \beta(n)^{p} / A_{n}$ and suppose first that $0 \leq \gamma<\infty$. Then

$$
\begin{equation*}
n\left(\log \left(A_{n}\right)-\log \left(A_{n-1}\right)\right)=-n \log \left(1-\frac{\delta_{n}}{n}\right) \longrightarrow \gamma \tag{2.17}
\end{equation*}
$$

as $n \rightarrow \infty$, and hence

$$
\begin{equation*}
\log \left(A_{n}\right)-\log \left(A_{1}\right)=-n \sum_{k=2}^{n} \log \left(1-\frac{\delta_{k}}{k}\right)=\epsilon_{n} \log n \tag{2.18}
\end{equation*}
$$

where $\epsilon_{n} \rightarrow \gamma$. Consequently $A_{n}=A_{1} n^{\epsilon_{n}}$. Now suppose that $\gamma=\infty$, then for $n \geq 2$,

$$
\begin{equation*}
\log \left(A_{n}\right)-\log \left(A_{n-1}\right)=-\log \left(1-\frac{\delta_{n}}{n}\right) \geq \frac{\delta_{n}}{n} \tag{2.19}
\end{equation*}
$$

since $\delta_{n} \rightarrow \infty$. If $M>0$, then there is $N_{1} \in \mathbb{N}$ such that $\delta_{n} \geq M+1$ for all $n \geq N_{1}$.
Without loss of the generality suppose that there is a positive real number $a>0$ such that $\delta_{n}>a$ for $n \leq N_{1}$. Note that

$$
\begin{equation*}
\sum_{k=2}^{N_{1}} \frac{1}{k}=\log N_{1}+c+o(1)-1 \tag{2.20}
\end{equation*}
$$

If $n>N_{1}$, then

$$
\begin{equation*}
\sum_{k=2}^{n} \frac{1}{k}=\log n+c+o(1)-1, \quad \sum_{k=N_{1}+1}^{n} \frac{1}{k}=\log n-\log N_{1} \tag{2.21}
\end{equation*}
$$

Also,

$$
\begin{align*}
\frac{\sum_{k=2}^{n} \delta_{k} / k}{\log n} & \geq \frac{a\left(\sum_{k=2}^{N_{1}} 1 / k\right)+(M+1)\left(\sum_{k=N_{1}+1}^{n} 1 / k\right)}{\log n}, \\
\frac{\sum_{k=2}^{N_{1}} 1 / k+\sum_{k=N_{1}+1}^{n} 1 / k}{\log n} & \geq \frac{M_{1}+1\left(\log n-\log N_{1}\right)+a\left(\log N_{1}+c+o(1)-1\right)}{\log n}  \tag{2.22}\\
& =M_{1}+1+\frac{\left(a-M_{1}-1\right) \log N_{1}+a(c+o(1)-1)}{\log n},
\end{align*}
$$

for large amount of $n$ last equality greater than $M_{1}$. Hence

$$
\begin{equation*}
\log A_{n} \geq \sum_{k=2}^{n} \frac{\delta_{k}}{k}=\gamma_{n} \log n \tag{2.23}
\end{equation*}
$$

where $\gamma_{n} \rightarrow \infty$. It follows that, for any real number $c, n^{c} A_{n}=n^{c+\gamma_{n}}$. Since

$$
\begin{equation*}
n^{c-1} A_{n} \sim \frac{1}{r} n^{c} a_{n} \beta(n)^{p} \tag{2.24}
\end{equation*}
$$

thus $n^{c} a_{n} \beta(n)^{p}$ is eventually increasing for $c>1-\gamma$, and eventually decreasing for $c<1-\gamma$. But $\{\beta(n)\}_{n}$ is bounded, so there are $M_{1}, M_{2}>0$ such that $M_{1}<\beta(n)<M_{2}$, and

$$
\begin{gather*}
\frac{n^{c} a_{n}}{\beta(n)}=\frac{n^{c} a_{n} \beta(n)^{p}}{\beta(n)^{p+1}}, \\
\frac{n^{c} a_{n} \beta(n)^{p}}{\beta(n)^{p+1}} \geq \frac{n^{c} a_{n} \beta(n)^{p}}{M_{1}^{p+1}} . \tag{2.25}
\end{gather*}
$$

This implies that $n^{c} a_{n} / \beta(n)$ is eventually increasing for $c>1-\gamma$. Similarly $n^{c} a_{n} / \beta(n)$ is eventually decreasing for $c<1-\gamma$. Thus

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p} \sim \frac{\gamma}{n} \sum_{k=1}^{n} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p} \tag{2.26}
\end{equation*}
$$

By Lemma 2.5

$$
\begin{equation*}
\frac{\gamma}{n} \sum_{k=1}^{n} \frac{a_{k} \beta(n)^{p+1}}{n a_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p} \tag{2.27}
\end{equation*}
$$

is bounded and so

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{k}{n}\right)^{-1 / p} \tag{2.28}
\end{equation*}
$$

is bounded. We can see that

$$
\begin{equation*}
\sum_{n=k}^{\infty} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}\left(\frac{k}{n}\right)^{1 / q} \tag{2.29}
\end{equation*}
$$

is also bounded. Now by Theorem 2.3, $A$ is a bounded operator and so the proof is complete.

Lemma 2.7. Let $\left\{a_{n}\right\},\left\{t_{n}\right\}$ be nonnegative sequences with $t_{-1}=0$. Then for all $n \in \mathbb{N}$ one has

$$
\begin{equation*}
\sum_{k=0}^{n}\left(t_{k} a_{k}\right) \leq\left\{\max _{0 \leq k \leq n}\left(\frac{1}{n-k+1} \sum_{j=K}^{n} a_{j}\right)\right\}\left(\sum_{k=1}^{n}(n-k+1)\left(t_{k}-t_{k-1}\right)^{+}+t_{0}(n+1)\right) \tag{2.30}
\end{equation*}
$$

Proof. Employing the summation by parts, we get

$$
\begin{align*}
\sum_{k=0}^{n}\left(t_{k} a_{k}\right) & =\sum_{k=0}^{n}\left(\sum_{j=k}^{n} a_{j}\right)\left(t_{k}-t_{k-1}\right)  \tag{2.31}\\
& \leq \sum_{k=0}^{n}\left(\sum_{j=k}^{n} a_{j} \frac{1}{n-k+1}\right)\left(t_{k}-t_{k-1}\right)^{+}(n-k+1)
\end{align*}
$$

So

$$
\begin{equation*}
\sum_{k=0}^{n}\left(t_{k} a_{k}\right) \leq\left\{\max _{0 \leq k \leq n}\left(\frac{1}{n-k+1} \sum_{j=k}^{n} a_{j}\right)\right\}\left(\sum_{k=1}^{n}(n-k+1)\left(t_{k}-t_{k-1}\right)^{+}+t_{0}(n+1)\right) \tag{2.32}
\end{equation*}
$$

and at this time the proof is complete.
Theorem 2.8 (see $[26$, Theorem 8$])$. Let $1 / p+1 / q=1,\left\{x_{n}\right\}$ be a positive sequence, then

$$
\begin{equation*}
\sum_{j=0}^{\infty} \max _{0 \leq i \leq j}\left(\frac{1}{j-i+1} \sum_{k=i}^{j} x_{k}\right)^{p} \leq q^{p}\left(\sum_{k=0}^{\infty} x_{k}^{p}\right) \tag{2.33}
\end{equation*}
$$

Theorem 2.9. Let $\left\{a_{n}\right\}$ be a positive sequence and

$$
\begin{equation*}
M_{3}=\sup _{n \geq 0}\left(\sum_{k=1}^{n} \frac{n-k+1}{A_{n}}\left(\frac{a_{k}}{\beta(k)}-\frac{a_{k-1}}{\beta(k-1)}\right)^{+} \beta(n)^{p+1}+\frac{(n+1) a_{0}}{A_{n} \beta(0)} \beta(n)^{p+1}\right) \tag{2.34}
\end{equation*}
$$

be finite. Then $A$ is bounded and $\|A\| \leq M_{3} q$.

Proof. Let $f(z)=\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in H^{p}(\beta)$, thus

$$
\begin{equation*}
A(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p}}{A_{n}} \hat{f}(k)\right) z^{n} . \tag{2.35}
\end{equation*}
$$

By definition of $\|\cdot\|_{\beta}$, we have

$$
\begin{equation*}
\sum_{n \geq 0} \beta(n)^{p}\left|\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p}}{A_{n}} \widehat{f}(k)\right|^{p} \leq \sum_{n \geq 0}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}|\widehat{f}(k)| \beta(k)\right)^{p} . \tag{2.36}
\end{equation*}
$$

In Lemma 2.7, consider $t_{k}=a_{k} / \beta(k)$ and $a_{j}=|\widehat{f}(j)| \beta(j)$. Then

$$
\begin{align*}
& \sum_{n \geq 0}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}|\widehat{f}(k)| \beta(k)\right)^{p} \\
& \leq \sum_{n \geq 0}\left\{\max _{0 \leq k \leq n} \frac{1}{n-k+1} \sum_{j=k}^{n}|\widehat{f}(j)| \beta(j)\right\}^{p}  \tag{2.37}\\
& \times\left(\sum_{k=1}^{n} \frac{n-k+1}{A_{n}}\left(\frac{a_{k}}{\beta(k)}-\frac{a_{k-1}}{\beta(k-1)}\right)^{+} \beta(n)^{p+1}+\frac{(n+1) a_{0}}{A_{n} \beta(0)} \beta(n)^{p+1}\right)^{p} .
\end{align*}
$$

Now, Theorem 2.8 implies that

$$
\begin{equation*}
\sum_{n \geq 0}\left\{\max _{0 \leq k \leq n}\left(\frac{1}{n-k+1} \sum_{j=k}^{n}|\widehat{f}(j)| \beta(j)\right)\right\}^{p} M_{3}^{p} \leq M_{3}^{p} q^{p} \sum_{k=1}^{\infty}|\widehat{f}(k)|^{p} \beta(k)^{p}, \tag{2.38}
\end{equation*}
$$

and so we get $\|A f\| \leq M_{3} q\|f\|_{\beta}$ for all $f \in H^{p}(\beta)$. Thus $A \in B\left(H^{p}(\beta)\right)$ and indeed $\|A\| \leq M_{3} q$. This completes the proof.

Corollary 2.10. Let $1 / p+1 / q=1, a_{k} / \beta(k) \geq a_{k-1} / \beta(k-1)$ and

$$
\begin{equation*}
M_{4}=\sup _{n \geq 0} \sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{\beta(k) A_{n}}<\infty . \tag{2.39}
\end{equation*}
$$

Then $A$ is a bounded operator on $H^{p}(\beta)$ and $\|A\| \leq M_{4}$.
Proof. Note that

$$
\begin{equation*}
\sum_{n \geq 0}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{A_{n} \beta(k)}|\widehat{f}(k)| \beta(k)\right)^{p} \leq \sum_{n \geq 0}\left\{\max _{0 \leq k \leq n} \frac{1}{n-k+1} \sum_{j=k}^{n}|\widehat{f}(j)| \beta(j)\right\}^{p}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p+1}}{\beta(k) A_{n}}\right)^{p} . \tag{2.40}
\end{equation*}
$$

Theorem 2.8 implies that

$$
\begin{equation*}
\sum_{n \geq 0}\left\{\max _{0 \leq k \leq n}\left(\frac{1}{n-k+1} \sum_{j=k}^{n}|\widehat{f}(j)| \beta(j)\right)\right\}^{p} M_{4}^{p} \leq M_{4}^{p} q^{p} \sum_{k=1}^{\infty}|\widehat{f}(k)|^{p} \beta(k)^{p} \tag{2.41}
\end{equation*}
$$

and so by Theorem 2.9 we obtain $\|A f\| \leq q M_{4}\|f\|_{\beta}$ for all $f \in H^{p}(\beta)$. Thus $A \in B\left(H^{p}(\beta)\right)$ and indeed $\|A\| \leq M_{4} q$. This completes the proof.

Now, we characterize compactness of subsets of $H^{p}(\beta)$ and then we will investigate compactness of the mean operator matrix on $H^{p}(\beta)$.

Theorem 2.11. Let $S$ be a nonempty subset of $H^{p}(\beta)$. Then $S$ is relatively compact if and only if the following hold:
(i) there exists $M>0$, such that for all $\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in S,|\widehat{f}(i) \beta(i)| \leq M$ for all $i \in \mathbb{N} \cup\{0\}$;
(ii) given $\epsilon>0$, there is $n_{0} \in \mathbb{N}$ such that $\sum_{n=n_{0}}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p}<\epsilon^{p}$ for all $\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in S$.

Proof. Let $S$ be relatively compact, thus there exist $g_{1}, \ldots, g_{k} \in H^{p}(\beta)$ such that

$$
\begin{equation*}
S \subseteq \bigcup_{i=1}^{k} B\left(g_{i}, 1\right) \tag{2.42}
\end{equation*}
$$

For every $f(z)=\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in S$, there is $g_{i}$ such that $f \in B\left(g_{i}, 1\right)$. By Minkowski inequality we get

$$
\begin{align*}
\sum_{n=0}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p} & \leq\left[\left(\sum_{n=0}^{\infty}\left|\widehat{f}(n)-\widehat{g}_{i}(n)\right|^{p} \beta(n)^{p}\right)^{1 / p}+\left(\sum_{n=0}^{\infty}\left|\widehat{g}_{i}(n)\right|^{p} \beta(n)^{p}\right)^{1 / p}\right]^{p} \\
& \leq\left(\left\|f-g_{i}\right\|+\left\|g_{i}\right\|\right)^{p}  \tag{2.43}\\
& \leq\left(1+\left\|g_{i}\right\|\right)^{p} \\
& \leq\left(1+\max \left\{\left\|g_{i}\right\|: i=1, \ldots, k\right\}\right)^{p} .
\end{align*}
$$

Thus for every $f \in S$ and $n \in \mathbb{N} \cup\{0\}$, we get

$$
\begin{equation*}
|\widehat{f}(n) \beta(n)| \leq 1+\max \left\{\left\|g_{i}\right\|: i=1, \ldots, k\right\} \tag{2.44}
\end{equation*}
$$

So (i) holds. Now suppose that $\epsilon$ is an arbitrary positive number. Since $S$ is relatively compact, thus there exist $h_{1}, \ldots, h_{k} \in H^{p}(\beta)$ such that

$$
\begin{equation*}
S \subseteq \bigcup_{i=1}^{k} B\left(h_{i}, \frac{\epsilon}{2}\right) \tag{2.45}
\end{equation*}
$$

Since $h_{i} \in H^{p}(\beta)$, there exists $N_{i} \in \mathbb{N}$ such that

$$
\begin{equation*}
\sum_{n=N_{i}}^{\infty}\left|\widehat{h}_{i}(n)\right|^{p} \beta(n)^{p}<\frac{\epsilon^{p}}{2^{p}} \tag{2.46}
\end{equation*}
$$

for $i=1, \ldots, k$. Put

$$
\begin{equation*}
N_{0}=\max \left\{N_{i}: i=1, \ldots, k\right\} \tag{2.47}
\end{equation*}
$$

and consider $f \in S$. Then there exists $i \in\{1, \ldots, k\}$, such that $f \in B\left(h_{i}, \epsilon / 2\right)$. Hence we get

$$
\begin{align*}
\sum_{n=N_{0}}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p} & \leq\left[\left(\sum_{n=N_{0}}^{\infty}\left|\widehat{f}(n)-\widehat{h}_{i}(n)\right|^{p} \beta(n)^{p}\right)^{1 / p}+\left(\sum_{n=N_{0}}^{\infty}\left|\widehat{h}_{i}(n)\right|^{p} \beta(n)^{p}\right)^{1 / p}\right]^{p}  \tag{2.48}\\
& \leq\left(\left\|f-h_{i}\right\|+\frac{\epsilon}{2}\right)^{p} \\
& \leq \epsilon^{p}
\end{align*}
$$

So (ii) holds.
Conversely, assume that $\epsilon>0$ be given and let (i) and (ii) hold. By condition (ii), there exists $n_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\sum_{n=n_{0}}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p}<\frac{\epsilon^{p}}{2} \tag{2.49}
\end{equation*}
$$

for all $f \in S$. Let $M_{n_{0}}$ be the closed linear span of the set $\left\{1, z, \ldots, z^{n_{0}-1}\right\}$ in $H^{p}(\beta)$. Consider $\mathbb{C}^{n_{0}}$ and $M_{n_{0}}$ with norms

$$
\begin{equation*}
\left\|\left(z_{1}, \ldots, z_{n_{0}}\right)\right\|=\left(\sum_{n=1}^{n_{0}}\left|z_{n}\right|^{p} \beta(n)^{p}\right)^{1 / p} \tag{2.50}
\end{equation*}
$$

for all $\left(z_{i}\right)_{i=1}^{n_{0}} \in \mathbb{C}^{n_{0}}$, and

$$
\begin{equation*}
\left\|\sum_{i=0}^{n_{0}-1} a_{i} z^{i}\right\|=\left(\sum_{i=0}^{n_{0}-1}\left|a_{i}\right|^{p} \beta(i)^{p}\right)^{1 / p} \tag{2.51}
\end{equation*}
$$

for all $\sum_{i=0}^{n_{0}-1} a_{i} z^{i} \in M_{n_{0}}$. Define $L: M_{n_{0}} \rightarrow \mathbb{C}^{n_{0}}$, by

$$
\begin{equation*}
L\left(\sum_{i=0}^{n_{0}-1} a_{i} z^{i}\right)=\left(a_{0}, \ldots, a_{n_{0}-1}\right) \tag{2.52}
\end{equation*}
$$

Clearly, we can see that $L$ is a bounded linear operator. Now, consider the compact subset

$$
\begin{equation*}
\left\{\left(z_{i}\right)_{i=1}^{n_{0}}: \sum_{i=1}^{n_{0}}\left|z_{i}\right|^{p} \beta(i)^{p} \leq n_{0} M^{p}\right\} \tag{2.53}
\end{equation*}
$$

in $\mathbb{C}^{n_{0}}$. Then we have

$$
\begin{equation*}
\left\{\sum_{i=0}^{n_{0}-1} \widehat{f}(i) z^{i}: \sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in S\right\} \subseteq L^{-1}\left\{\left(z_{i}\right)_{i=1}^{n_{0}}: \sum_{i=1}^{n_{0}}\left|z_{i}\right|^{p} \beta(i)^{p} \leq n_{0} M^{p}\right\} \tag{2.54}
\end{equation*}
$$

Since

$$
\begin{equation*}
L^{-1}\left\{\left(z_{i}\right)_{i=1}^{n_{0}}: \sum_{i=1}^{n_{0}}\left|z_{i}\right|^{p} \beta(i)^{p} \leq n_{0} M^{p}\right\} \tag{2.55}
\end{equation*}
$$

is a compact subspace of $M_{n_{0}}$, so there exist $g_{1}, \ldots, g_{k} \in M_{n_{0}}$ such that

$$
\begin{equation*}
L^{-1}\left\{\left(z_{i}\right)_{i=1}^{n_{0}}: \sum_{i=1}^{n_{0}}\left|z_{i}\right|^{p} \beta(i)^{p} \leq n_{0} M^{p}\right\} \in \bigcup_{i=1}^{k} B\left(g_{i}, \frac{\epsilon}{2^{1 / p}}\right) \tag{2.56}
\end{equation*}
$$

Hence for every

$$
\begin{equation*}
f \in\left\{\sum_{i=0}^{n_{0}-1} \widehat{f}(i) z^{i}: f(z)=\sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in S\right\} \tag{2.57}
\end{equation*}
$$

there is $i \in\{1, \ldots, k\}$ satisfying

$$
\begin{equation*}
\sum_{n=0}^{n_{0}-1}\left|\widehat{f}(n)-\widehat{g}_{i}(n)\right|^{p} \beta(n)^{p} \leq \frac{\epsilon^{p}}{2} \tag{2.58}
\end{equation*}
$$

Also, we have

$$
\begin{align*}
\left(\left\|f-g_{i}\right\|_{\beta}\right)^{p} & \leq \sum_{n=0}^{n_{0}-1}\left|\widehat{f}(n)-\widehat{g}_{i}(n)\right|^{p} \beta(n)^{p}+\sum_{n=n_{0}}^{\infty}|\widehat{f}(n)|^{p} \beta(n)^{p} \\
& \leq \frac{\epsilon^{p}}{2}+\frac{\epsilon^{p}}{2}  \tag{2.59}\\
& \leq \epsilon^{p}
\end{align*}
$$

Thus, $S$ is relatively compact and so the proof is complete.

Theorem 2.12. Let the mean matrix operator $A$ be bounded on $H^{p}(\beta)$, and

$$
\begin{equation*}
\lim _{m \rightarrow \infty}\left(\sum_{n=m}^{\infty} \frac{\beta(n)^{p^{2}+p}}{A_{n}^{p}}\right)^{1 / p}\left(\sum_{k=0}^{m}\left(\frac{a_{k}}{\beta(k)}\right)^{q}\right)^{1 / q}=0 \tag{2.60}
\end{equation*}
$$

where $1 / p+1 / q=1$. Then $A$ is a compact operator on $H^{p}(\beta)$.
Proof. Let $B_{H^{p}(\beta)}$ be the closed unit ball of $H^{p}(\beta)$. Define $S=A\left(B_{H^{p}(\beta)}\right)$ and note that $S$ is a bounded subset of $H^{p}(\beta)$. Put $r_{n}=|\widehat{f}(n)| a_{n}, u_{n}=\beta(n)^{p^{2}+p} / A_{n}^{p}, v_{k}=\left(\beta(k) / a_{k}\right)^{p}$, and

$$
\begin{equation*}
E_{m}=\left(\sum_{n=m}^{\infty} u_{n}\right)^{1 / p}\left(\sum_{k=0}^{m} v_{k}^{1-q}\right)^{1 / q} \tag{2.61}
\end{equation*}
$$

Note that $\lim _{m \rightarrow \infty} E_{m}=0$. So for every $\epsilon>0$, there exists $m_{0} \in \mathbb{N}$ such that $E_{m}<\epsilon /\left(q^{p-1} p\right)^{1 / p}$ for all $m \geq m_{0}$. Note that if

$$
\begin{equation*}
f(z)=\sum_{k=0}^{\infty} \widehat{f}(k) z^{k} \in B_{H^{p}(\beta)} \tag{2.62}
\end{equation*}
$$

then

$$
\begin{equation*}
A f(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{a_{k} \beta(n)^{p} \widehat{f}(k)}{A_{n}}\right) z^{n} \in S . \tag{2.63}
\end{equation*}
$$

Since $\|f\|_{\beta}^{p} \leq 1$, we have

$$
\begin{align*}
\sum_{n=m}^{\infty}|\widehat{A} f(n)|^{n} \beta(n)^{p} & \leq \sum_{n=m}^{\infty} \frac{\beta(n)^{p^{2}+p}}{A_{n}^{p}}\left(\sum_{k=0}^{n} a_{k}|\hat{f}(k)|\right)^{p} \\
& =\sum_{n=m}^{\infty} u_{n}\left(\sum_{k=0}^{n} r_{k}\right)^{p} \\
& \leq \epsilon^{p} \sum_{k=0}^{\infty}\left(r_{k}\right)^{p} v_{k}  \tag{2.64}\\
& \leq \epsilon^{p} \sum_{k=0}^{\infty}|\widehat{f}(k)|^{p} \beta(k)^{p} \\
& \leq \epsilon^{p}
\end{align*}
$$

Thus by Theorem $2.11, S$ is compact and so the proof is complete.

## References

[1] A. L. Shields, "Weighted shift operators and analytic function theory," in Topics in Operator Theory, pp. 49-128, Amer. Math. Soc., Providence, RI, USA, 1974.
[2] B. Yousefi, "On the space $l^{p}(\beta), "$ Rendiconti del Circolo Matematico di Palermo. Serie II, vol. 49, no. 1, pp. 115-120, 2000.
[3] B. Yousefi, "Unicellularity of the multiplication operator on Banach spaces of formal power series," Studia Mathematica, vol. 147, no. 3, pp. 201-209, 2001.
[4] B. Yousefi, "Bounded analytic structure of the Banach space of formal power series," Rendiconti del Circolo Matematico di Palermo. Serie II, vol. 51, no. 3, pp. 403-410, 2002.
[5] B. Yousefi and S. Jahedi, "Composition operators on Banach spaces of formal power series," Bollettino della Unione Matematica Italiana, vol. 6, no. 2, pp. 481-487, 2003.
[6] B. Yousefi, "Strictly cyclic algebra of operators acting on Banach spaces $H^{p}(\beta)$," Czechoslovak Mathematical Journal, vol. 54(129), no. 1, pp. 261-266, 2004.
[7] B. Yousefi, "Composition operators on weighted Hardy spaces," Kyungpook Mathematical Journal, vol. 44, no. 3, pp. 319-324, 2004.
[8] B. Yousefi and Y. N. Dehghan, "Reflexivity on weighted Hardy spaces," Southeast Asian Bulletin of Mathematics, vol. 28, no. 3, pp. 587-593, 2004.
[9] B. Yousefi, "On the eighteenth question of Allen Shields," International Journal of Mathematics, vol. 16, no. 1, pp. 37-42, 2005.
[10] B. Yousefi and A. I. Kashkuli, "Cyclicity and unicellularity of the differentiation operator on Banach spaces of formal power series," Mathematical Proceedings of the Royal Irish Academy, vol. 105A, no. 1, pp. 1-7, 2005.
[11] B. Yousefi and A. Farrokhinia, "On the hereditarily hypercyclic operators," Journal of the Korean Mathematical Society, vol. 43, no. 6, pp. 1219-1229, 2006.
[12] B. Yousefi and L. Bagheri, "Boundedness of an operator acting on spaces of formal power series," International Journal of Applied Mathematics, vol. 20, no. 6, pp. 821-825, 2007.
[13] D. Borwein and A. Jakimovski, "Matrix operators on $l^{p}$, " The Rocky Mountain Journal of Mathematics, vol. 9, no. 3, pp. 463-477, 1979.
[14] D. Borwein, "Simple conditions for matrices to be bounded operators on $l p$," Canadian Mathematical Bulletin, vol. 41, no. 1, pp. 10-14, 1998.
[15] D. Borwein and X. Gao, "Matrix operators on $l_{p}$ to $l_{q}$," Canadian Mathematical Bulletin, vol. 37, no. 4, pp. 448-456, 1994.
[16] D. Borwein, "Weighted mean operators on $l_{p}$," Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques, vol. 43, no. 4, pp. 406-412, 2000.
[17] R. Lashkaripour, "Weighted mean matrix on weighted sequence spaces," WSEAS Transactions on Mathematics, vol. 3, no. 4, pp. 789-793, 2004.
[18] R. Lashkaripour, "Transpose of the weighted mean matrix on weighted sequence spaces," WSEAS Transactions on Mathematics, vol. 4, no. 4, pp. 380-385, 2005.
[19] R. Lashkaripour and D. Foroutannia, "Inequalities involving upper bounds for certain matrix operators," Proceedings of the Indian Academy of Sciences. Mathematical Sciences, vol. 116, no. 3, pp. 325336, 2006.
[20] R. Lashkaripour and D. Foroutannia, "Extension of Hardy inequality on weighted sequence spaces," Journal of Sciences. Islamic Republic of Iran, vol. 20, no. 2, pp. 159-166, 2009.
[21] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, UK, 1952.
[22] J. M. Cartlidge, Weighted mean matrices as operators on $l^{p}$, Ph.D. thesis, Indiana University, 1978.
[23] A. Kufner, L. Maligranda, and L.-E. Persson, The Hardy Inequality, About Its History and Some Related Results, Vydavatelsky Servis, Plzeň, Czech Republic, 2007.
[24] A. Kufner and L.-E. Persson, Weighted Inequalities of Hardy Type, World Scientific, River Edge, NJ, USA, 2003.
[25] G. Bennett, "Some elementary inequalities," The Quarterly Journal of Mathematics. Oxford. Second Series, vol. 38, no. 152, pp. 401-425, 1987.
[26] G. H. Hardy and J. E. Littlewood, "A maximal theorem with function-theoretic applications," Acta Mathematica, vol. 54, no. 1, pp. 81-116, 1930.


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