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Research Article

On The Solution n-Dimensional of the Composite $\Diamond^k$ Operator and $\Diamond_B^k$ Operator

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Firstly, we study the solution of the equation $\Diamond^k \Diamond_B^k u(x) = f(x)$, where $\Diamond^k \Diamond_B^k$ is the composite of the diamond operator and Bessel diamond operator. Finally, we study of the nonlinear equation $\Diamond^k \Diamond_B^k u(x) = f(x, \triangle^{k-1} \Box^k \Diamond_B^k)$. It was found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\triangle^{k-1} \Box^k \Diamond_B^k u(x)$. Moreover, such equation $u(x)$ is related to the elastic wave equation.

1. Introduction

Let $\Box^k$ be ultrahyperbolic operator iterated k-times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n, n$ is the dimension of space $\mathbb{R}^n$ and k is a nonnegative integer.

Consider the linear differential equation of the form

$$\Box^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized function and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [1, pages 279–282] first introduced the fundamental solution of (1.2) which is complicated form. Later Trione [2] has shown that the generalized function

$R_{2k}(x)$ which is defined by (2.2) is the unique fundamental solution of (1.2) and Aguirre Tellez [3] also proved that $R_{2k}(x)$ exists only in case p is odd and n is odd or even and $p+q=n$.

In 1996, Kananthai [4] has been the first to introduce the operator $\lozenge^k$ which is named as the diamond operator iterated k-times and is defined by

$$\lozenge^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \tag{1.3}$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ and k is a nonnegative integer. The operator $\lozenge^k$ can be expressed in the form

$$\lozenge^k = \triangle^k \square^k = \square^k \triangle^k, \tag{1.4}$$

where $\triangle^k$ is the Laplace operator defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k, \tag{1.5}$$

and $\square^k$ is the ultrahyperbolic operator iterated k-times and is defined by (1.1). Tellez and Kananthai [5, lemma 3.1, page 46] have shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is a fundamental solution of the operator $\lozenge^k$, where $R_{2k}^e(x)$ and $R_{2k}^H(x)$ are defined by (2.8) and (2.2), respectively. That is,

$$\lozenge^k \left\{ (-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right\} = \delta(x). \tag{1.6}$$

Furthermore, Yildirim et al. [6] first introduced the Bessel diamond operator $\lozenge_B^k$ iterated k-times defined by

$$\lozenge_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k, \tag{1.7}$$

where $B_{x_i} = \partial^2/\partial x_i^2 + (2\upsilon_i/x_i)\ (\partial/\partial x_i)$, $2\upsilon_i = 2\alpha_i + 1,\ \alpha_i > -1/2,\ x_i > 0$. The operator $\lozenge_B^k$ can be expressed by $\lozenge_B^k = \triangle_B^k \square_B^k = \square_B^k \triangle_B^k$, where

$$\triangle_B^k = \left(\sum_{i=1}^{p} B_{x_i} \right)^k, \tag{1.8}$$

$$\square_B^k = \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right]^k. \tag{1.9}$$

Yildirim et al. [6] have shown that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique fundamental solution of the operator $\Diamond_B^k$, that is,

$$\Diamond_B^k\Big((-1)^k S_{2k}(x) * R_{2k}(x)\Big) = \delta, \tag{1.10}$$

where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.11) and (2.15) with $\alpha = \gamma = 2k$, respectively.

Now, firstly the purpose of this paper is to study the following equation:

$$\Diamond^k \Diamond_B^k u(x) = f(x), \tag{1.11}$$

where the operator $\Diamond^k$ defined by (1.3) and $\Diamond_B^k$ defined by (1.7) with $f(x)$ is a generalized function and $u(x)$ is an unknown function.

Finally, we will study the nonlinear of the form

$$\Diamond^k \Diamond_B^k u(x) = f\Big(x, \triangle^{k-1}\Box^k \Diamond_B^k u(x)\Big), \tag{1.12}$$

with f defined and having continuous first derivative for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω, and f is bounded on Ω, that is, $|f| \leq N, N$ is constant. We can find the solution $u(x)$ of (1.12) which is unique under the boundary condition $\triangle^{k-1}\Box^k \Diamond_B^k u(x) = 0$ for $x \in \partial\Omega$, and we obtain the solution related to the elastic wave equation.

Before going to that point, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denote by

$$\upsilon = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

the nondegenerated quadratic form and $p + q = n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } \upsilon > 0\}$ and $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R_\alpha^H(\upsilon) = \begin{cases} \dfrac{\upsilon^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.3}$$

The function $R_\alpha^H(v)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki [7].

It is well known that $R_\alpha^H(v)$ is a function of $\mathrm{Re}(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(v)$ denote the support of $R_\alpha^H(v)$ and suppose supp $R_\alpha^H(v) \subset \overline{\Gamma}_+$, that is, supp $R_\alpha^H(v)$, is compact.

From Trione [2, page 11], $R_{2k}^H(v)$ is a fundamental solution of the operator $\Box^k$, that is,

$$\Box^k R_{2k}^H(v) = \delta(x). \tag{2.4}$$

By putting $p = 1$ in $R_{2k}(v)$ and taking into account Legendre's duplication formula for $\Gamma(z)$

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.5}$$

then the formula (2.1) reduces to

$$M_\alpha^H(u) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.6}$$

and $u = x_1^2 - x_2^2 - \cdots - x_n^2$, where

$$H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma\left(\frac{\alpha - n + 2|\nu|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.7}$$

$M_\alpha(u)$ is the hyperbolic kernel of Riesz [8, page 31].

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $|x| = x_1^2 + x_2^2 + \cdots + x_n^2$ the function $R_\alpha^e(x)$ denoted by the elliptic kernel of Marcel Riesz which is defined by

$$R_\alpha^e(x) = \frac{|x|^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{2.8}$$

where

$$W_n(\alpha) = \frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}. \tag{2.9}$$

α is a complex parameter and n is the dimension of the space $\mathbb{R}^n$.

Let α and β be complex numbers such that $\alpha+\beta\neq n+2r, r=0.1,2,\ldots$ The function $R_\alpha^e(x)$ has the following properties [9]:

$$\begin{gathered} R_0(x)=\delta(x), \\ R_{-2k}(x)=(-1)^k\triangle^k\delta(x), \\ \triangle^k\{R_\alpha(x)\}=(-1)^k R_{\alpha-2k}(x), \\ R_\alpha(x)*R_\beta(x)=R_{\alpha+\beta}(x). \end{gathered} \tag{2.10}$$

Definition 2.3. Let $x=(x_1,x_2,\ldots,x_n), v=(v_1,v_2,\ldots,v_n)\in\mathbb{R}_n^+$. For any complex number α, we define the distribution family $S_\alpha(x)$ by

$$S_\alpha(x)=\frac{|x|^{\alpha-n-2|v|}}{w_n(\alpha)}, \tag{2.11}$$

where $|x|=x_1^2+x_2^2+\cdots+x_n^2$, $|v|=v_1+v_2+\cdots+v_n$ and

$$w_n(\alpha)=\frac{\prod_{i=1}^n 2^{v_i-1/2}\Gamma(v_i+1/2)}{2^{n+2|v|-2\alpha}\Gamma((n+2|v|-\alpha)/2)}. \tag{2.12}$$

Definition 2.4. Let $x=(x_1,x_2,\ldots,x_n), v=(v_1,v_2,\ldots,v_n)\in\mathbb{R}_n^+$, and denote by

$$V=x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-x_{p+2}^2-\cdots-x_{p+q}^2 \tag{2.13}$$

the nondegenerated quadratic form. Denote the interior of the forward cone by

$$\Gamma_+=\{x\in\mathbb{R}_n^+ : x_1>0, x_2>0,\ldots,x_n>0, V>0\}, \tag{2.14}$$

and $\overline{\Gamma}_+$ denotes its closure. For any complex number γ the distribution family $R_\gamma(x)$ is defined by

$$R_\gamma(x)=\begin{cases}\dfrac{V^{(\gamma-n-2|v|)/2}}{K_n(\gamma)}, & \text{for } x\in\Gamma_+,\\ 0, & \text{for } x\notin\Gamma_+,\end{cases} \tag{2.15}$$

where

$$K_n(\gamma)=\frac{\pi^{(n+2|v|-1)/2}\Gamma((2+\gamma-n-2|v|)/2)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((2+\gamma-p-2|v|)/2)\Gamma((p-\gamma)/2)}, \tag{2.16}$$

where γ is a complex number.

By putting $p = 1$ in $R_{2k}(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.17}$$

we obtain

$$N_\gamma(x) = \frac{u^{(\gamma-n-2|\nu|)/2}}{E_n(\gamma)}, \tag{2.18}$$

and $u = x_1^2 - x_2^2 - \cdots - \cdots - x_n^2$, where

$$E_n(\gamma) = \pi^{(n+2|\nu|-1)/2}2^{\gamma-1}\Gamma\left(\frac{2+\gamma-n-2|\nu|}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.19}$$

Lemma 2.5. *Given the equation $\triangle_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is defined by* (1.8)*, then*

$$u(x) = (-1)^k S_{2k}(x), \tag{2.20}$$

where $S_{2k}(x)$ is defined by (2.11)*, with $\alpha = 2k$.*

Proof. See [6, page 379]. □

Lemma 2.6. *Given the equation $\square_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\square_B^k$ is defined by* (1.9)*. Then*

$$u(x) = R_{2k}(x), \tag{2.21}$$

where $R_{2k}(x)$ is defined by (2.15)*, with $\gamma = 2k$.*

Proof. See [6, page 379]. □

Lemma 2.7. *Let $S_\alpha(x)$and $R_\beta(x)$ be the function defined by* (2.11) *and* (2.15)*, respectively. Then*

$$\begin{gathered} S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x), \\ (-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = (-1)^{2k} S_{-2k+2k}(x) = S_0(x) = \delta(x), \end{gathered} \tag{2.22}$$

where α and β are a positive even number.

Proof. See [10, pages 171–190]. □

Lemma 2.8. *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverse in the convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively, that is,*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x),$$
$$R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x). \tag{2.23}$$

Proof. See [6]. □

Lemma 2.9. *Given P is a hyper-function, then*

$$P\delta^k(P) + k\delta^{(k-1)}(P) = 0, \tag{2.24}$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives and

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2. \tag{2.25}$$

Proof. See [1, page 233]. □

Lemma 2.10. *Given the following equation:*

$$\triangle^k u(x) = 0, \tag{2.26}$$

where $\triangle^k$ is defined by (1.5) *and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then*

$$u(x) = \delta^{(m)}\left(r^2\right) \tag{2.27}$$

or

$$u(x) = \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R^e_{-2(m-n/2+1)}(x) \tag{2.28}$$

is a homogeneous solution of (2.26) *with $m = n/2-k-1$ for $k = 1, 2, 3, \ldots$. The function $R^e_{-2(m-n/2+1)}$ is defined by* (2.8) *and $\alpha = -2(m-n/2+1)$.*

Proof. We first need to show that the generalized function $u(x) = \delta^{(m)}(r^2)$, where $r^2 = |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, and

$$\triangle\, u(x) = 0, \tag{2.29}$$

where $\triangle = (\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2)$ is a Laplace operator. In fact,

$$\begin{aligned} \frac{\partial}{\partial x_i}\delta^{(m)}\left(r^2\right) &= 2x_i\delta^{(m+1)}\left(r^2\right), \\ \frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2\right) &= 2\delta^{(m+1)}r^2 + 4x_i^2\delta^{(m+2)}\left(r^2\right). \end{aligned} \tag{2.30}$$

Thus

$$\begin{aligned} \triangle\ \delta^{(m)}\left(r^2\right) &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2\right) \\ &= 2n\delta^{(m+1)}\left(r^2\right) + 4r^2\delta^{(m+2)}\left(r^2\right) \\ &= 2n\delta^{(m+1)}\left(r^2\right) - 4(m+2)\delta^{(m+1)}\left(r^2\right) \\ &= (2n - 4(m+2))\delta^{(m+1)}\left(r^2\right), \\ \triangle^2\delta^{(m)}\left(r^2\right) &= (2n - 4(m+2))\ \triangle\ \delta^{(m+1)}\left(r^2\right) \\ &= (2n - 4(m+2))(2n - 4(m+3))\delta^{(m+2)}\left(r^2\right), \\ \triangle^3\delta^{(m)}\left(r^2\right) &= (2n - 4(m+2))(2n - 4(m+3))(2n - 4(m+4))\delta^{(m+3)}\left(r^2\right), \\ \triangle^k\delta^{(m)}\left(r^2\right) &= (2n - 4(m+2))(2n - 4(m+3))(2n - 4(m+4)) \\ &\quad \cdots (2n - 4(m+k+1))\delta^{(m+k)}\left(r^2\right) \\ &= 2^2\left(\frac{n}{2} - (m+2)\right)2^2\left(\frac{n}{2} - (m+3)\right)2^2\left(\frac{n}{2} - (m+4)\right) \\ &\quad \cdots 2^2\left(\frac{n}{2} - (m+k+1)\right)\delta^{(m+k)}\left(r^2\right). \end{aligned} \tag{2.31}$$

Thus

$$\begin{aligned} \triangle^k\delta^{(m)}\left(r^2\right) &= 2^{2k}\left[\left(\frac{n}{2} - m\right) - 2\right]\left[\left(\frac{n}{2} - m\right) - 3\right]\left[\left(\frac{n}{2} - m\right) - 4\right] \\ &\quad \cdots \left[\left(\frac{n}{2} - m\right) - (k+1)\right]\delta^{(m+k)}\left(r^2\right). \end{aligned} \tag{2.32}$$

Using the following formula:

$$(z-1)(z-2)\cdots(z-k) = \frac{(-1)^k\Gamma(-z+k+1)}{\Gamma(m+1)}, \tag{2.33}$$

the above expression can be written in the following form:

$$\begin{aligned}\triangle^k \delta^{(m)}\left(r^2\right) &= \frac{2^{2k}(-1)^k \Gamma(n/2-m-2+1)}{\Gamma(n/2-m-2-k+1)}\delta^{(m+k)}\left(r^2\right)\\ &= \frac{2^{2k}(-1)^k \Gamma(n/2-m-1)}{\Gamma(n/2-m-k-1)}\delta^{(m+k)}\left(r^2\right).\end{aligned} \tag{2.34}$$

If we put $m = n/2 - k - 1$ for $k = 1, 2, 3, \ldots$ in (2.34), we obtain

$$\triangle^k \delta^{(m)}\left(r^2\right) = 0\delta^{(m+k)}\left(r^2\right) = 0. \tag{2.35}$$

It follows that

$$u(x) = \delta^{(m)}\left(r^2\right) \tag{2.36}$$

is homogeneous solution of the equation $\triangle^k u(\mathrm{x}) = 0$. On the other hand, by Aguirre Tellez [11], we have

$$\begin{aligned}\delta^{(m)}\left(r^2\right) &= \frac{(-1)^m \pi^{n/2}}{\Gamma(m-n/2+1)4^{m-n/2+1}}\triangle^{m-n/2+1}\delta(x)\\ &= \frac{(-1)^m \pi^{\frac{n}{2}}(-1)^{m-\frac{n}{2}+1}}{\Gamma\left(m-\frac{n}{2}+1\right)4^{m-\frac{n}{2}+1}} R^e_{-2\left(m-\frac{n}{2}+1\right)}(x).\end{aligned} \tag{2.37}$$

If we put $m = n/2 - k - 1$ in (2.37), we obtain

$$\begin{aligned}\delta^{(n/2-k-1)}\left(r^2\right) &= \frac{(-1)^{n/2-k-1}\pi^{n/2}(-1)^{n/2-k-1-n/2+1}}{\Gamma(n/2-k-1-n/2+1)4^{n/2-k-1-n/2+1}} R^e_{-2(n/2-k-1-n/2+1)}(x)\\ &= \frac{(-1)^{n/2-k-1}\pi^{n/2}(-1)^{-k}}{\Gamma(-k)4^{-k}} R^e_{2k}(x)\\ &= 0.\end{aligned} \tag{2.38}$$

By (2.36) and (2.37), we conclude

$$u(x) = \delta^{(m)}\left(r^2\right) \tag{2.39}$$

or

$$u(x) = \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R^e_{-2(m-n/2+1)}(x) \tag{2.40}$$

is a homogeneous solution of the equation $\triangle^k u(x) = 0$. This completes the proof. □

Lemma 2.11. *Given the following equation:*

$$\Diamond^k \Diamond_B^k u(x) = 0, \tag{2.41}$$

where $\Diamond^k$ *and* $\Diamond_B^k$ *are diamond operator and Bessel diamond operator iterated* k*-times defined by* (1.3) *and* (1.7)*, respectively,* $u(x)$ *is an unknown function, we obtain*

$$u(x) = R_{2k}^H(u) * (-1)^k S_{2k}(x) * R_{2k}(v) * \delta^{(m)}\left(r^2\right) \tag{2.42}$$

or

$$u(x) = R_{2k}^H(u) * (-1)^k S_{2k}(x) * R_{2k}(v) * \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R^e_{-2(m-n/2+1)}(x) \tag{2.43}$$

with $m = n/2 - k - 1$ *as a homogeneous solution of* (2.41).

Proof. Since

$$\Diamond^k = \triangle^k \Box^k, \qquad \Diamond_B^k = \triangle_B^k \Box_B^k. \tag{2.44}$$

Consider the following homogeneous equation:

$$\Diamond^k \Diamond_B^k u(x) = 0. \tag{2.45}$$

The above equation can be written as

$$\triangle^k \Box^k \triangle_B^k \Box_B^k u(x) = 0. \tag{2.46}$$

By Lemma 2.10, we have

$$\Box^k \triangle_B^k \Box_B^k u(x) = \delta^{(m)}\left(r^2\right). \tag{2.47}$$

Convolving both sides by $R_{2k}^H(u) * (-1)^k S_{2k}(x) * T_{2k}(x)$, we obtain

$$R_{2k}^H(v) * (-1)^k S_{2k}(x) * R_{2k}(x) * \Box^k \triangle_B^k \Box_B^k u(x) = R_{2k}^H(v) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right). \tag{2.48}$$

By properties of convolution, we have

$$\Box R_{2k}^H(v) * \triangle_B^k (-1)^k S_{2k}(x) * \Box_B^k R_{2k}(x) * u(x) = R_{2k}^H(v) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right). \tag{2.49}$$

By (2.4), Lemmas 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * u(x) = R_{2k}^{H}(\upsilon) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right). \tag{2.50}$$

Thus

$$u(x) = R_{2k}^{H}(\upsilon) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right) \tag{2.51}$$

or

$$u(x) = R_{2k}^{H}(\upsilon) * (-1)^k S_{2k}(x) * R_{2k}(x) * \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R^{e}_{-2(m-n/2+1)}(x) \tag{2.52}$$

is a homogeneous solution of (2.41). □

Lemma 2.12. *Consider the following:*

$$\triangle\, u(x) = f(x, u(x)), \tag{2.53}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n, and $\partial\Omega$ is the boundary of Ω. Assume that f is bounded, that is, $|f(x,u)| \leq N$, and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (2.53).

Proof. We can prove the existence of the solution $u(x)$ of (2.53) by the method of iterations and the Schuder's estimates. The details of the proof are given by Courant and Hilbert, [12, pages 369–372]. □

3. Main Results

Theorem 3.1. *Given the following equation:*

$$\Diamond^k \Diamond_B^k u(x) = f(x), \tag{3.1}$$

where $\Diamond^k$ and $\Diamond_B^k$ are defined by (1.3) *and* (1.7)*, respectively, $f(x)$ is the generalized function, $u(x)$ is an unknown function $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $m = n/2 - k - 1$. We obtain*

$$\begin{aligned} u(x) = {} & R_{2k}^{H}(\upsilon) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right) \\ & + R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x) \end{aligned} \tag{3.2}$$

or

$$u(x) = R_{2k}^{H}(\upsilon) * (-1)^k S_{2k}(x) * R_{2k}(x) * \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R_{-2(m-n/2+1)}^{e}(x) + R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x) \tag{3.3}$$

as a general solution of (3.1).

Proof. Consider the following equation:

$$\Diamond^k \Diamond_B^k u(x) = f(x), \tag{3.4}$$

or

$$\Box^k \triangle^k \triangle_B^k \Diamond_B^k u(x) = f(x). \tag{3.5}$$

Convolving both sides of (3.1) by $R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * R_{2k}(x) * (-1)^k S_{2k}(x) * \Box^k \triangle^k \Box_B^k \triangle_B^k u(x) = R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * R_{2k}(x) * (-1)^k S_{2k}(x) * f(x). \tag{3.6}$$

By properties of convolution, we have

$$\Box^k R_{2k}^{H}(\upsilon) * \triangle^k (-1)^k R_{2k}^{e}(x) * \triangle_B^k (-1)^k S_{2k}(x) * \Box_B^k R_{2k}(x) * u(x) = R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * R_{2k}(x) * (-1)^k S_{2k}(x) * f(x). \tag{3.7}$$

By (2.4), Lemmas 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * \delta(x) * u(x) = R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.8}$$

Thus

$$u(x) = R_{2k}^{H}(\upsilon) * (-1)^k R_{2k}^{e}(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.9}$$

Consider the following homogeneous equation:

$$\Diamond^k \Diamond_B^k u(x) = 0. \tag{3.10}$$

By Lemma 2.10, we have a homogeneous solution as

$$u(x) = R_{2k}^{H}(u) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right). \tag{3.11}$$

Thus, the general solution of (3.1) is

$$\begin{aligned} u(x) &= R_{2k}^{H}(x) * (-1)^{k} S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^{2}\right) \\ &\quad + R_{2k}^{H}(\upsilon) * (-1)^{k} R_{2k}^{e}(x) * (-1)^{k} S_{2k}(x) * R_{2k}(x) * f(x), \end{aligned} \tag{3.12}$$

or

$$\begin{aligned} u(x) &= R_{2k}^{H}(\upsilon) * (-1)^{k} S_{2k}(x) * R_{2k}(x) * \frac{(-1)^{2m-n/2+1}\pi^{n/2}}{\Gamma(m-n/2+2)4^{m-n/2+1}} R_{-2(m-n/2+1)}^{e}(x) \\ &\quad + R_{2k}^{H}(\upsilon) * (-1)^{k} R_{2k}^{e}(x) * (-1)^{k} S_{2k}(x) * R_{2k}(x) * f(x). \end{aligned} \tag{3.13}$$

The proof is complete. □

Theorem 3.2. *Consider the following nonlinear equation:*

$$\Diamond^{k}\Diamond_{B}^{k} u(x) = f\left(x, \triangle^{k-1}\Box^{k}\Diamond_{B}^{k} u(x)\right), \tag{3.14}$$

where $\Diamond^{k}, \Diamond_{B}^{k}, \triangle^{k-1}$, and $\Box^{k}$ are defined by (1.3), (1.7), (1.5), *and* (1.1), *respectively. Let f be defined and having continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^{n}$ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Suppose f is bounded, that is,*

$$\left| f(x), \triangle^{k-1}\Box^{k}\Diamond_{B}^{k} u(x) \right| \leq N, \tag{3.15}$$

and, the boundary condition for all $x \in \partial\Omega$ let be

$$\triangle^{k-1}\Box^{k}\Diamond_{B}^{k} u(x) = 0. \tag{3.16}$$

We can assume $\triangle^{k-1}\Box^{k}\Diamond_{B}^{k} u(x) = U(x)$ and $U(x)$ is a continuous function for $x \in \partial\Omega$, then we obtain

$$u(x) = (-1)^{k-1} R_{2(k-1)}^{e}(x) * R_{2k}^{H}(\upsilon) * (-1)^{k} S_{2k}(x) * R_{2k}(x) * U(x) \tag{3.17}$$

as a solution of (3.14) *with the boundary condition as*

$$u(x) = \delta^{(m)}\left(r^{2}\right) * R_{2k}^{H}(\upsilon) * (-1)^{k} S_{2k}(x), \tag{3.18}$$

for all $x \in \partial\Omega$ and $m = n/2 - k$. The function $S_{2k}(x), R_{2k}(x), R_{2(k-2)}^{e}(x)$, and $R_{2k}^{H}(\upsilon)$ are given by (2.11), (2.15), (2.8), *and* (2.2), *respectively. Moreover,*

$$W(x) = (-1)^{k-1} R_{2(1-k)}^{e}(x) * (-1)^{k} S_{-2k}(x) * U(x) \tag{3.19}$$

is a solution of the following equation:

$$\square^k \square_B^k W(x) = U(x), \tag{3.20}$$

where $\square^k, \square_B^k$ *are defined by* (1.1), (1.9), *respectively, and* $U(x)$ *is obtained from* (3.11). *Furthermore, if we put* $p = k = 1$, *then* $W(x)$ *is reduced to*

$$W(x) = M_2^H(u) * N_2^H(u) * U(x), \tag{3.21}$$

which is a solution of the following inhomogeneous elastic wave equation:

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot (B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}) W(x) = U(x). \tag{3.22}$$

Proof. We have

$$\begin{aligned} \Diamond^k \Diamond_B^k u(x) &= \triangle\ \triangle^{k-1} \square^k \Diamond_B^k u(x) \\ &= f\left(x, \triangle^{k-1} \square^k \Diamond_B^k u(x)\right). \end{aligned} \tag{3.23}$$

Since $u(x)$ has continuous derivative up to order $4p$ for $k = 1, 2, 3, \ldots$. thus we can assume

$$\triangle^{k-1} \square^k \Diamond_B^k u(x) = U(x), \quad \forall x \in \Omega. \tag{3.24}$$

Then (3.17) can be written in the following form:

$$\Diamond^k \Diamond_B^k u(x) = \triangle\ U(x) = f(x, U(x)). \tag{3.25}$$

By (3.2), we have

$$\left| f(x, U(x)) \right| \leq N, \quad \forall x \in \Omega, \tag{3.26}$$

For $U(x) = 0$ or

$$\triangle^{k-1} \square^k \Diamond_B^k u(x) = 0 \quad \forall x \in \partial\Omega, \tag{3.27}$$

Convolving both sides of (3.24) by

$$(-1)^k R_{2(k-1)}^e(x) * R_{2k}^H(x) * (-1)^k S_{2k}(x) * R_{2k}(x), \tag{3.28}$$

we obtain

$$\begin{aligned}&\left((-1)^k R^e_{2(k-1)}(x) * R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) \triangle^{k-1} \square^k \Diamond^k_B u(x)\\ &\quad = \left((-1)^k R^e_{2(k-1)}(x) * R^H_{2k}(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * U(x).\end{aligned} \tag{3.29}$$

By properties of convolution, we have

$$\begin{aligned}&\left(\triangle^{k-1}(-1)^k R^e_{2(k-1)}(x)\right) * \left(\square^k R^H_{2k}(v)\right) * \left(\Diamond^k_B (-1)^k S_{2k}(x) * R_{2k}(x)\right) * u(x)\\ &\quad = \left((-1)^k R^e_{2(k-1)}(x) * R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * U(x).\end{aligned} \tag{3.30}$$

By Lemma 2.8, we obtain

$$\delta * \delta * \delta * u(x) = \left((-1)^k R^e_{2(k-1)}(x) * R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * U(x). \tag{3.31}$$

Thus

$$u(x) = \left((-1)^k R^e_{2(k-1)}(x) * R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * U(x), \tag{3.32}$$

as a solution (3.14).

Now, considering the boundary condition we have

$$\triangle^{k-1} \square^k \Diamond^k_B u(x) = 0. \tag{3.33}$$

By Lemma 2.10, we obtain

$$\square^k \Diamond^k_B u(x) = \delta^{(m)}\left(r^2\right), \tag{3.34}$$

with $m = n/2 - k$. Convolving both sides of (3.34) by $R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x) \square^k \Diamond^k u(x) = R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x) * \delta^{(m)}\left(r^2\right) \tag{3.35}$$

or

$$u(x) = R^H_{2k}(v) * (-1)^k S_{2k}(x) * R_{2k}(x) * (-1)^{m+k-2} \frac{(-1)^m \pi^{n/2}}{\Gamma(m - n/2 + 1) 4^{m-n/2+1}} R^e_{-2(m-n/2+1)}(x), \tag{3.36}$$

for $x \in \partial\Omega$.

Lastly, convolving both sides of (3.36) by $(-1)^{k-1}R^{e}_{2(1-k)}(x) * (-1)^{k}S_{-2k}(x)$, we obtain

$$(-1)^{k-1}R^{e}_{2(1-k)}(x) * (-1)^{k}S_{-2k}(x) * u(x) = R^{H}_{2k}(x) * R_{2k}(x) * U(x). \tag{3.37}$$

Setting

$$W(x) = (-1)^{k-1}R^{e}_{2(1-k)}(x) * (-1)^{k}S_{-2k}(x) * u(x). \tag{3.38}$$

By Lemmas 2.8 and 2.5, we obtain $W(x)$ as a solution of the following equation:

$$\square^{k}\square^{k}_{B}W(x) = U(x). \tag{3.39}$$

If we put $p = 1$, then $R^{H}_{2k}(v)$ and $R_{2k}(x)$ are reduced to $M^{H}_{2k}(u)$ and $N_{2k}(u)$ and are defined by (2.6) and (2.18), respectively. Moreover, if we put $p = k = 1$, then the operator $\square^{k}$ and $\square^{k}_{B}$ is reduced to

$$\begin{gathered}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \\ B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n},\end{gathered} \tag{3.40}$$

respectively, and the solution $W(x)$ is reduced to

$$W(x) = M^{H}_{2}(u) * N^{H}_{2}(u) * U(x), \tag{3.41}$$

which is solution of the following inhomogeneous elastic wave equation:

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot (B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n})W(x) = U(x). \tag{3.42}$$

The proof is complete. □

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