International Scholarly Research Network
ISRN Geometry
Volume 2012, Article ID 970682, 18 pages
doi:10.5402/2012/970682

Research Article

Curvature Properties and η-Einstein (k, μ)-Contact Metric Manifolds

H. G. Nagaraja and C. R. Premalatha

Department of Mathematics, Bangalore University, Central College Campus, Bangalore 560 001, India

Correspondence should be addressed to H. G. Nagaraja, hgnagaraj1@gmail.com

Received 16 September 2012; Accepted 2 October 2012

Academic Editors: G. Martin, C. Qu, and A. Viña

We study curvature properties in (k, μ)-contact metric manifolds. We give the characterization of η-Einstein (k, μ)-contact metric manifolds with associated scalars.

1. Introduction

The class of (k, μ)-contact manifolds [1] is of interest as it contains both the classes of Sasakian and non-Sasakian cases. The contact metric manifolds for which the characteristic vector field ξ belongs to (k, μ)-nullity distribution are called (k, μ) contact metric manifolds. Boeckx [2] gave a classification of (k, μ)-contact metric manifolds. Sharma [3], Papantoniou [4], and many others have made an investigation of (k, μ)-contact metric manifolds. A special class of (k, μ)-contact metric manifolds called $N(k)$-contact metric manifolds was studied by authors [5, 6] and others. In this paper we study (k, μ)-contact metric manifolds by considering different curvature tensors on it (Table 1). We characterize η-Einstein (k, μ)-contact metric manifolds with associated scalars by considering symmetry, ϕ-symmetry, semisymmetry, ϕ-recurrent, and flat conditions on (k, μ)-contact metric manifolds. The paper is organized as follows: In Section 2, we give some definitions and basic results. In Section 3, we consider conharmonically symmetric, conharmonically semisymmetric, ϕ-conharmonically flat, ξ-conharmonically flat, and ϕ-recurrent (k, μ)-contact metric manifolds and we prove that such manifolds are η-Einstein or η-parallel or cosymplectic depending on the conditions. In Section 4, we prove that ξ-conformally flat (k, μ)-contact metric manifold reduces to $N(k)$-contact metric manifold if and only if it is an η-Einstein manifold. Further we prove conformally Ricci-symmetric and ϕ-conformally flat (k, μ)-contact metric manifolds are η-Einstein. In Section 5, we prove that pseudoprojectively symmetric and pseudoprojectively Ricci-symmetric (k, μ)-contact metric manifolds are η-Einstein. In Section 6 we consider Ricci-semisymmetric (k, μ)-contact metric manifolds and prove that such manifolds are η-Einstein.

Table 1: Comparison of the results for different curvature tensors in $M(k,\mu)$.

Curvature tensor	Condition	Result
$\widetilde{C}(X,Y,Z,W)$	$(\nabla_W \widetilde{C})(X,Y)Z=0$	$M(k,\mu)$ is cosymplectic $\Leftrightarrow$ $\mu=2(1-n)\Leftrightarrow$ $M(k,\mu)$ is η-Einstein
$\widetilde{C}(X,Y,Z,W)$	$R\cdot\widetilde{C}=0$	$\Leftrightarrow$ η-Einstein with $\alpha=-2nk$ and $\beta=\dfrac{4nk}{\mu}(k+\mu)$
$\widetilde{C}(X,Y,Z,W)$	$\widetilde{C}(\phi X,\phi Y,\phi Z,\phi W)=0$	Ricci tensor is η-parallel and $\mu=\dfrac{2(n-1)}{n}$
$\widetilde{C}(X,Y,Z,W)$	$\widetilde{C}(X,Y)\xi=0$	η-Einstein with $\beta=-\alpha-2nk$, $\alpha=k-\dfrac{(2n-1)\mu[2n\mu+2(n-1)](k-1)}{2(n-1)-n\mu-k}$
$\widetilde{C}(X,Y,Z,W)$	$M(k,\mu)$ is ϕ-recurrent	η-Einstein with $\beta=2nk-\alpha$, $\alpha=-k+\dfrac{\mu(2n-1)(k-1)}{a+k}[b-\mu(2n-1)]$
$C(X,Y)Z$	$C(X,Y)\xi=0$	$M(k,\mu)$ reduces to $N(k)$-c.m.m $\Leftrightarrow$ $M(k,\mu)$ is η-Einstein
$C(X,Y)Z$	$C\cdot S=0$	$M(1,-2)$ is η-Einstein with $\alpha=\dfrac{2n(2n-1)}{5n-2}$, $\beta=\dfrac{2n(1-n)}{5n-2}$
$C(X,Y)Z$	${}'C(\phi X,\phi Y,\phi Z,\phi W)=0$	η-Einstein with $\alpha=\dfrac{r(4n-1)+4n^2k}{2n}$, $\beta=\dfrac{-r(4n-1)}{2n}$
$\widetilde{P}(X,Y)Z$	$R\cdot\widetilde{P}=0$	η-Einstein
$\widetilde{P}(X,Y)Z$	$\widetilde{P}\cdot S=0$	η-Einstein
$R(X,Y)Z$	$R\cdot S=0$	η-Einstein

In all the cases where (k,μ)-contact metric manifold is an η-Einstein manifold, we obtain associated scalars in terms of k and μ.

2. Preliminaries

A $(2n+1)$ dimensional C^∞-differentiable manifold M is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations [7, 8]

$$\phi^2=-I+\eta\otimes\xi,\qquad \eta(\xi)=1,\qquad \phi\xi=0,\qquad \eta\circ\phi=0, \tag{2.1}$$

$$\begin{gathered} g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\\ g(X,\phi Y)=-g(\phi X,Y),\qquad g(X,\phi X)=0,\qquad g(X,\xi)=\eta(X), \end{gathered} \tag{2.2}$$

where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form, and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X,\phi Y)=d\eta(X,Y), \tag{2.3}$$

for all vector fields X, Y.

The (1,1) tensor field h defined by $h = (1/2)L_\xi \phi$, where L denotes the Lie differentiation, is a symmetric operator and satisfies $h\phi = -\phi h$, $\operatorname{tr} h = \operatorname{tr} \phi h = 0$, and $h\xi = 0$. Further we have [1]

$$\nabla_X \xi = -\phi X - \phi h X, \qquad (\nabla_X \eta)Y = g(X + hX, \phi Y), \tag{2.4}$$

where ∇ denotes the Riemannian connection of g.

The (k,μ)-nullity distribution $N(k,\mu)$ of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution [1]

$$\begin{aligned} N(k,\mu) &: p \longrightarrow N_p(k,\mu) \\ &= \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{2.5}$$

for any vector fields X and Y on M.

Definition 2.1. A contact metric manifold is said to be

(i) Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and S is the Ricci tensor,

(ii) η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y)$, where α and β are smooth functions.

A contact metric manifold with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. In a (k,μ)-contact metric manifold, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

If $k = 1$, $\mu = 0$, then the manifold becomes Sasakian [1], and if $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to k-nullity distribution [9]. If $k = 0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$. In a (2n+1)-dimensional (k,μ)-contact metric manifold, we have the following [1]:

$$h^2 = (k-1)\phi^2, \quad k \le 1, \tag{2.7}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$

$$\begin{aligned} QX = {} & [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ & + [2(1-n) + n(2k+\mu)]\eta(X)\xi, \quad n \ge 1, \end{aligned} \tag{2.9}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ & + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \quad n \ge 1, \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$r = 2n(2n - 2 + k - n\mu), \tag{2.12}$$

$$(\nabla_X h)(Y) = [(1-k)g(X,\phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi h X) - \mu\eta(X)\phi h Y, \tag{2.13}$$

where Q is the Ricci operator and r is the scalar curvature of M.

Throughout this paper $M(k,\mu)$ denotes (2n+1)-dimensional (k,μ)-contact metric manifold.

3. Conharmonic Curvature Tensor in (k,μ)-Contact Metric Manifolds

The conharmonic curvature tensor in $M(k,\mu)$ is given by [10]

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right]. \tag{3.1}$$

A (k,μ)-contact metric manifold is said to be

(1) conharmonically symmetric if

$$\left(\nabla_W \widetilde{C}\right)(X,Y)Z = 0, \quad \text{where } X,Y,Z,W \in T(M), \tag{3.2}$$

(2) conharmonically semisymmetric if

$$\begin{aligned}\left(R(U,X)\cdot\widetilde{C}\right)(Y,Z,W) = {} & R(U,X)\widetilde{C}(Y,Z)W - \widetilde{C}(R(U,X)Y,Z)W \\ & - \widetilde{C}(Y,R(U,X)Z)W - \widetilde{C}(Y,Z)R(U,X)W = 0.\end{aligned} \tag{3.3}$$

3.1. Conharmonically Symmetric (k,μ)-Contact Metric Manifolds

Differentiating (3.1) covariantly with respect to W, we obtain

$$\begin{aligned}\left(\nabla_W \widetilde{C}\right)(X,Y)Z = {} & (\nabla_W R)(X,Y)Z - \frac{1}{2n-1}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ & + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)].\end{aligned} \tag{3.4}$$

If $M(k,\mu)$ is conharmonically symmetric, then, from (3.4), we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)Z = {} & \frac{1}{2n-1}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ & + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)].\end{aligned} \tag{3.5}$$

Differentiating (2.6) covariantly with respect to W and using (2.4), we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = {} & k\left[g(W+hW,\phi Y)X - g(W+hW,\phi X)Y\right] \\ & + \mu\left[g(W+hW,\phi Y)hX - g(W+hW,\phi X)hY\right].\end{aligned} \tag{3.6}$$

Differentiating (2.10) covariantly with respect to W and using (2.11), (2.4), we have

$$\begin{aligned}(\nabla_W S)(X,Y) = b\big[&(1-k)g(W,\phi X)\eta(Y) + g(W,h\phi X)\eta(Y) + \eta(X)g(h\phi W,Y)\\ &+\eta(X)g(h\phi hW,Y) - \mu\eta(W)g(\phi hX,Y)\big]\\ &+ c\big[g(W,\phi X)\eta(Y) + g(hW,\phi X)\eta(Y) + \eta(X)g(W,\phi Y) + \eta(X)g(hW,\phi Y)\big],\end{aligned} \tag{3.7}$$

where

$$b = 2(n-1) + \mu, \qquad c = 2(1-n) + n(2k+\mu). \tag{3.8}$$

From (3.7), we obtain

$$\begin{aligned}(\nabla_W Q)(X) = {} & \big[(1-k+n)\mu + 2k\big], \big[g(W,\phi X)\xi - \eta(X)(\phi W)\big]\\ &+ \big[\mu(1+n) + 2nk\big], \big[g(W,h\phi X)\xi + \eta(X)(h\phi W)\big]\\ &- \big[2(n-1) + \mu\big]\eta(W)(\phi hX).\end{aligned} \tag{3.9}$$

Taking $Z = \xi$ in (3.5) and using (3.6), (3.7), and (3.9), we obtain

$$\begin{aligned}(2n-1)\big[&k\big(g(W+hW,\phi Y)X - g(W+hW,\phi X)Y\big)\\ &+\mu\big(g(W+hW,\phi Y)hX - g(W+hW,\phi X)hY\big)\big]\\ = {} & l\big[g(W,\phi Y) + g(W,h\phi Y)\big]X - l\big[g(W,\phi X) + g(W,h\phi X)\big]Y\\ &+ \eta(Y)\big(mg(W,\phi X)\xi + l\big[g(W,h\phi X)\xi + \eta(X)(h\phi W)\big] - b\eta(W)(\phi hX)\big)\\ &- \eta(X)\big(mg(W,\phi Y)\xi + l\big[g(W,h\phi Y)\xi + \eta(Y)(h\phi W)\big] - b\eta(W)(\phi hY)\big),\end{aligned} \tag{3.10}$$

where

$$l = \mu(1+n) + 2nk, \qquad m = (1-k+n)\mu + 2k. \tag{3.11}$$

Contracting (3.10) with ξ and using (2.1), we obtain

$$\begin{aligned}k\big[&\big(g(W,\phi Y)\eta(X) - g(W,\phi X)\eta(Y)\big)(1-\mu)\\ &+\big(g(hW,\phi Y)\eta(X) - g(hW,\phi X)\eta(Y)\big)(2n-1)\big] = 0.\end{aligned} \tag{3.12}$$

From (3.12), we get either $k = 0$ or

$$\begin{aligned}\big[&(1-\mu)\big(g(W,\phi Y)\eta(X) - g(W,\phi X)\eta(Y)\big)\\ &+\big(g(hW,\phi Y)\eta(X) - g(hW,\phi X)\eta(Y)\big)(2n-1)\big] = 0.\end{aligned} \tag{3.13}$$

Taking $Y = \phi Y$ in (3.13) and using (2.1), we obtain

$$(1-\mu)\big(g(Y,W)-\eta(Y)\eta(W)\big)+(2n-1)g(Y,hW)=0. \tag{3.14}$$

Taking $Y = \phi Y$ in (3.14), we obtain

$$(1-\mu)g(\phi Y,W)+(2n-1)g(\phi Y,hW)=0. \tag{3.15}$$

Since $\mu \neq 1$, from (3.15), it follows that $\mu = 2(1-n)$ if and only if

$$g(\phi Y,W)+g(\phi Y,hW)=0. \tag{3.16}$$

In view of (2.4), the above equation gives that $M(k,\mu)$ reduces to a cosymplectic manifold. Thus we have $M(k,\mu)$ is cosymplectic if and only if $\mu = 2(1-n)$.

Further from (2.10) and Definition 2.1, we have $M(k,\mu)$ is η-Einstein with $\alpha = 2(n^2-1)$, $\beta = 2((1-n^2)+nk)$ if and only if $\mu = 2(1-n)$. Thus we have the following.

Theorem 3.1. *In a conharmonically symmetric (k,μ)-contact metric manifold $M(k,\mu)$, the following statements are equivalent.*

(1) *$M(k,\mu)$ is cosymplectic.*

(2) *$M(k,\mu)$ is η-Einstein with $\alpha = 2(n^2-1), \beta = 2((1-n^2)+nk)$.*

(3) *$\mu = 2(1-n)$.*

3.2. Conharmonically Semisymmetric (k,μ)-Contact Metric Manifolds

Suppose $R \cdot \widetilde{C} = 0$. Then from (3.3), we have

$$R(\xi,X)\widetilde{C}(Y,Z)W-\widetilde{C}(R(\xi,X)Y,Z)W-\widetilde{C}(Y,R(\xi,X)Z)W-\widetilde{C}(Y,Z)R(\xi,X)W=0. \tag{3.17}$$

Using (2.5), (2.9), (2.6), (2.10), and (2.11) in (3.17) and taking $W = Y = \xi$, we get

$$\begin{aligned}
&k^2[\eta(Z)\eta(X)-g(Z,X)]\xi-k\mu g(hZ,X)\xi\\
&\quad-\frac{k}{2n-1}(4nk\eta(Z)\eta(X)-2nkg(X,Z)-S(Z,X))\xi\\
&\quad-k\mu g(Z,hX)\xi-\mu^2g(hZ,hX)\xi-\frac{\mu}{2n-1}[-2nkg(hX,Z)-S(Z,hX)]\xi\\
&\quad-k^2\eta(X)[\eta(Z)\xi-Z]+\frac{k}{2n-1}\eta(X)[4nk\eta(Z)\xi-2nkZ-QZ]\\
&\quad+kR(X,Z)\xi-\frac{k}{2n-1}[2nk\eta(Z)X-2nk\eta(X)Z+\eta(Z)QX-\eta(X)QZ]
\end{aligned}$$

$$
\begin{aligned}
&+\mu R(hX,Z)\xi-\frac{\mu}{2n-1}\left[2nk\eta(Z)X-2nk\eta(X)Z+\eta(Z)QX-QZ\right]\\
&+k\eta(Z)R(\xi,X)\xi-\frac{k}{2n-1}\eta(Z)\left[4nk\eta(X)\xi-2nkX-QX\right]\\
&+\mu\eta(Z)R(\xi,hX)\xi-\frac{\mu}{2n-1}\eta(Z)[-2nk(hX)-Q(hX)]\\
&-k^{2}\eta(X)\left[\eta(Z)\xi-Z\right]+\frac{k}{2n-1}\eta(X)[-2nkZ-QZ]\\
&+kR(\xi,Z)X-\frac{k}{2n-1}\left[S(Z,X)\xi-2nk\eta(X)Z+2nkg(Z,X)\xi-\eta(X)QZ\right]\\
&+\mu R(\xi,Z)hX-\frac{\mu}{2n-1}\left[S(Z,hX)\xi-S(hX,\xi)Z+2nkg(Z,hX)\xi\right]=0.
\end{aligned}
\tag{3.18}
$$

Taking $X=\xi$ and using (2.9), (2.6), and (2.11), we obtain

$$
QZ=\frac{4nk}{\mu}(k+\mu)\eta(Z)\xi-2nkZ. \tag{3.19}
$$

That is, $M(k,\mu)$ is an η-Einstein manifold.

Conversely, suppose in $M(k,\mu)$ the relation (3.19) holds. Then we have

$$
\begin{aligned}
R\cdot\widetilde{C}&=R(\xi,X)\widetilde{C}(Y,Z)W-\widetilde{C}(R(\xi,X)Y,Z)W-\widetilde{C}(Y,R(\xi,X)Z)W-\widetilde{C}(Y,Z)R(\xi,X)W\\
&=\eta(Z)\xi\left(-\frac{4nk}{2n-1}(k+\mu)\right)+\frac{\mu}{2n-1}(2nkZ+QZ).
\end{aligned}
\tag{3.20}
$$

Using (3.19) in (3.20), we get $R\cdot\widetilde{C}=0$ which implies that $M(k,\mu)$ is conharmonically semisymmetric. Thus we have the following.

Theorem 3.2. *A (k,μ)-contact metric manifold is conharmonically semisymmetric if and only if it is η-Einstein with $\alpha=-2nk$ and $\beta=(4nk/\mu)(k+\mu)$.*

3.3. ϕ-Conharmonically Flat (k,μ)-Contact Metric Manifolds

Suppose $M(k,\mu)$ is ϕ-conharmonically flat, that is, $\widetilde{C}(\phi X,\phi Y,\phi Z,\phi W)=0$ for all vector fields X, Y, Z, W. Then from (3.1), we obtain

$$
\begin{aligned}
\widetilde{R}(\phi X,\phi Y,\phi Z,\phi W)=\frac{1}{(2n-1)}&\left[g(\phi Y,\phi Z)S(\phi X,\phi W)-g(\phi X,\phi Z)S(\phi Y,\phi W)\right.\\
&\left.+S(\phi Y,\phi Z)g(\phi X,\phi W)-S(\phi X,\phi Z)g(\phi Y,\phi W)\right].
\end{aligned}
\tag{3.21}
$$

Let $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of the tangent space $T_P(M)$ at each P in $M(k,\mu)$. Then in $M(k,\mu)$, the following relations hold:

$$\sum_{i=1}^{2n} g(e_i, e_i) = 2n,$$
$$\sum_{i=1}^{2n} S(e_i, e_i) = r - 2nk, \tag{3.22}$$

$$\sum_{i=1}^{2n} g(e_i, Z) S(Y, e_i) = S(Y, Z) - 2nk\eta(Y)\eta(Z),$$
$$\sum_{i=1}^{2n} g(e_i, \phi Z) S(Y, e_i) = S(Y, \phi Z). \tag{3.23}$$

Taking $X = W = e_i$ in (3.21) and summing up from 1 to $2n$, we have

$$\begin{aligned}\sum_{i=1}^{2n} \widetilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) &= \frac{1}{(2n-1)} \sum_{i=1}^{2n} \big[g(\phi Y, \phi Z) S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) \\ &\quad + S(\phi Y, \phi Z) g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z) g(\phi Y, \phi e_i)\big].\end{aligned} \tag{3.24}$$

Using (2.13), (3.22), in (3.24), we obtain

$$S(\phi Y, \phi Z) = (r - 2nk) g(\phi Y, \phi Z). \tag{3.25}$$

Replacing Y by ϕY and Z by ϕZ in (3.25) and using (2.1), we have

$$S(Y, Z) = (r - 2nk) g(Y, Z) + (4nk - r)\eta(Y)\eta(Z). \tag{3.26}$$

Taking $Y = Z = e_i$ in (3.26) and taking summation over $i = 1$ to $(2n+1)$, we obtain $r = 2nk$.

Substituting this in (3.26) and taking the covariant derivative with respect to X, we obtain

$$\nabla_X S(\phi Y, \phi Z) = 0. \tag{3.27}$$

That is, S is η-parallel.

Further substituting $r = 2nk$ in (2.12), we obtain

$$\mu = \frac{2n-2}{n}. \tag{3.28}$$

Thus from the above discussions we can state the following.

Theorem 3.3. *In a $(2n+1)$-dimensional ϕ-conharmonically flat (k,μ)-contact metric manifold, Ricci tensor is η-parallel and $\mu = 2(n-1)/n$.*

3.4. ξ-Conharmonically Flat (k,μ)-Contact Metric Manifolds

Suppose $M(k,\mu)$ is ξ-conharmonically flat, that is, $\widetilde{C}(X,Y)\xi = 0$.

Then from (3.1), we obtain

$$R(X,Y)\xi = \frac{1}{2n-1}\left(S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY\right). \tag{3.29}$$

Using (2.9), (2.6), and (2.11) in (3.29), we obtain

$$(2n-1)\mu\left[\eta(Y)hX - \eta(X)hY\right] - k\left[\eta(Y)X - \eta(X)Y\right] - \left[\eta(Y)QX - \eta(X)QY\right] = 0. \tag{3.30}$$

Taking $Y = \xi$ in (3.30) and using (2.1), we obtain

$$QX = kX - (k+2nk)\eta(X)\xi - (2n-1)\mu hX. \tag{3.31}$$

Contracting (3.31) with W, we obtain

$$S(X,W) = kg(X,W) - (k+2nk)\eta(X)\eta(W) - (2n-1)\mu g(hX,W). \tag{3.32}$$

Replacing X by hX in (3.32) and using (2.7) and (2.10), we obtain

$$g(hX,W) = \frac{\left[(2n-1)\mu + 2(n-1) + \mu\right](k-1)\left[g(X,W) - \eta(X)\eta(W)\right]}{2(n-1) - n\mu - k}. \tag{3.33}$$

Tha above equation with (3.32) yields

$$S(X,W) = \alpha g(X,W) + \beta\eta(X)\eta(W), \tag{3.34}$$

where

$$\begin{gathered}\alpha = k - \frac{(2n-1)\mu\left[2n\mu + 2(n-1)\right](k-1)}{2(n-1) - n\mu - k},\\ \beta = -\alpha - 2nk.\end{gathered} \tag{3.35}$$

Hence $M(k,\mu)$ reduces to an η-Einstein manifold.

Thus we have the following.

Theorem 3.4. *A $(2n+1)$-dimensional ξ-conharmonically flat (k,μ)-contact metric manifold is an η-Einstein manifold.*

3.5. ϕ-Recurrent (k,μ)-Contact Metric Manifolds

A $(2n+1)$-dimensional (k,μ)-contact metric manifold $M(k,\mu)$ is said to be ϕ-recurrent if and only if there exists a nonzero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z. \tag{3.36}$$

Differentiating (3.1) covariantly with respect to W, we obtain

$$\begin{aligned}\left(\nabla_W \widetilde{C}\right)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{1}{2n-1}((\nabla_W S)(X,Y)Z - (\nabla_W S)(X,Z)Y \\ +g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)).\end{aligned} \tag{3.37}$$

Suppose $M(k,\mu)$ is ϕ-recurrent. Then from (3.37), we have

$$-\left(\nabla_W \widetilde{C}\right)(X,Y)Z + \eta\left(\left(\nabla_W \widetilde{C}\right)(X,Y)Z\right)\xi = A(W)\widetilde{C}(X,Y)Z. \tag{3.38}$$

Contracting (3.38) with ξ, we obtain

$$A(W)\eta\left(\widetilde{C}(X,Y)Z\right) = 0. \tag{3.39}$$

Since A is a nonzero 1-form, we have $\eta(\widetilde{C}(X,Y)Z) = 0$.

Using (3.1), the above equation yields

$$\eta(R(X,Y)Z) - \frac{1}{2n-1}\left[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)\right] = 0. \tag{3.40}$$

Using (2.6) and (2.9) in (3.40), we obtain

$$\begin{aligned}k\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right] + \mu\left[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\right] \\ = \frac{1}{2n-1}\left[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + 2nkg(Y,Z)\eta(X) - 2nkg(X,Z)\eta(Y)\right].\end{aligned} \tag{3.41}$$

Taking $X = \xi$ in (3.41), we get

$$S(Y,Z) = -kg(Y,Z) + (2n+1)\eta(Y)\eta(Z) + (2n-1)\mu g(hY,Z). \tag{3.42}$$

Replacing Y by hY in (3.42), we obtain

$$S(hY,Z) = -kg(hY,Z) - (2n-1)(k-1)\mu\left(g(Y,Z) - \eta(Y)\eta(Z)\right). \tag{3.43}$$

Replacing Y by hY in (2.10) and comparing the resulting equation with (3.43), we obtain

$$g(hY,Z)=\frac{(k-1)\big(b-\mu(2n-1)\big)}{a}\big(g(Y,Z)-\eta(Y)\eta(Z)\big), \tag{3.44}$$

where $a=2(n-1)-n\mu,\ b=2(n-1)+\mu$.

Using (3.44) in (3.42), we get

$$S(Y,Z)=\alpha g(Y,Z)+\beta\eta(Y)\eta(Z), \tag{3.45}$$

where $\alpha=-k+(\mu(2n-1)(k-1)/(a+k))[b-\mu(2n-1)]$ and $\beta=2nk-\alpha$.

That is, $M(k,\mu)$ is an η-Einstein manifold.

Thus we have the following.

Theorem 3.5. *A ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold.*

4. Conformal Curvature Tensor in (k,μ)-Contact Metric Manifolds

The conformal curvature tensor C in $M(k,\mu)$ is defined by [11]

$$\begin{aligned}C(X,Y)Z={}&R(X,Y)Z-\frac{1}{2n-1}\left[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\right]\\&+\frac{1}{2n(2n-1)}\left[g(Y,Z)X-g(X,Z)Y\right].\end{aligned} \tag{4.1}$$

Definition 4.1. A (k,μ)-contact metric manifold $M(k,\mu)$ is

(1) ξ-conformally flat if $C(X,Y)\xi=0$,

(2) conformally Ricci symmetric if $C\cdot S=0$,

(3) ϕ-conformally flat if $'C(\phi X,\phi Y,\phi Z,\phi W)=0$ for all X, Y, Z, and $W\in T(M)$.

4.1. *ξ-Conformally Flat (k,μ)-Contact Metric Manifolds*

Suppose that (k,μ)-contact metric manifold $M(k,\mu)$ is ξ-conformally flat. Then from (4.1), we obtain

$$\begin{aligned}R(X,Y)\xi={}&\frac{1}{2n-1}\left[S(Y,\xi)X-S(X,\xi)Y+g(Y,\xi)QX-g(X,\xi)QY\right]\\&-\frac{1}{2n(2n-1)}\left[g(Y,\xi)X-g(X,\xi)Y\right].\end{aligned} \tag{4.2}$$

Using (2.1), (2.6), and (2.11) in (4.2), we obtain

$$\begin{aligned}&\left(k-\frac{2nk}{2n-1}+\frac{1}{2n(2n-1)}\right)(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY)\\&\quad-\frac{1}{(2n-1)}(\eta(Y)QX-\eta(X)QY)=0.\end{aligned}\tag{4.3}$$

Putting $Y=\xi$ in (4.3) and using (2.1) and (2.10), we obtain

$$\begin{aligned}QX&=(2n-1)\left(\frac{2nk}{2n-1}-\left(k-\frac{2nk}{2n-1}+\frac{1}{2n(2n-1)}\right)\right)\eta(X)\xi\\&\quad+(2n-1)\left(k-\frac{2nk}{2n-1}+\frac{1}{2n(2n-1)}\right)X+(2n-1)\mu(hX).\end{aligned}\tag{4.4}$$

Contraction of the above with Y yields

$$S(X,Y)=\left(\frac{2nk(2n-1)-1}{2n}\right)\eta(X)\eta(Y)+\frac{1-2nk}{2n}g(X,Y)+(2n-1)\mu g(hX,Y).\tag{4.5}$$

From (4.5), we have the following.

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y),\tag{4.6}$$

with $\alpha=(1-2nk)/n$ and $\beta=(2nk(2n+1)-1)/2n$ if and only if $\mu=0$.

Thus we have the following.

Theorem 4.2. *A ξ-conformally flat (k,μ)-contact metric manifold reduces to $N(k)$-contact metric manifold if and only if it is an η-Einstein manifold.*

4.2. Conformally Ricci-Symmetric (k,μ)-Contact Metric Manifolds

If $C\cdot S=0$, then we have

$$S(C(\xi,X)Y,Z)+S(Y,C(\xi,X)Z)=0.\tag{4.7}$$

Taking $Z=\xi$ in (4.7) and using (4.1), (2.6), (2.9) to (2.12), we obtain

$$\begin{aligned}&\left(2nk^2-\frac{4n^2k^2}{2n-1}+\frac{2nk(2n-2+k-n\mu)}{2n-1}+b\mu(k-1)-\frac{b^2(k-1)}{2n-1}\right)g(X,Y)\\&\quad+\left(2nk\mu-a\mu+\frac{ab}{2n-1}\right)g(hY,X)\end{aligned}$$

$$+\left(\frac{(a+c)2nk}{2n-1}+\frac{2cnk}{2n-1}-\frac{4n^2k^2}{2n-1}-b\mu(k-1)+\frac{b^2(k-1)}{2n-1}\right)\eta(Y)\eta(X)$$
$$=\left(k-\frac{a}{2n-1}+\frac{(2n-2+k-n\mu)}{2n-1}\right)S(X,Y), \tag{4.8}$$

where

$$a=2(n-1)-n\mu, \qquad b=2(n-1)+\mu, \qquad c=2(1-n)+n(2k+\mu). \tag{4.9}$$

Taking $Y=\xi$ in (4.8), we obtain $\mu=-2k$.

If $k=1$, then $h=0$, $\mu=-2$.

Thus for $k=1$, (4.8) reduces to

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y), \tag{4.10}$$

where

$$\alpha=\frac{2n(2n-1)}{5n-2}, \qquad \beta=\frac{2n(1-n)}{5n-2}, \tag{4.11}$$

that is, $M(k,\mu)$ reduces to η-Einstein.

Thus we have the following.

Theorem 4.3. *A conformally Ricci-symmetric* $(1,-2)$*-contact metric manifold is an* η*-Einstein manifold.*

4.3. ϕ*-Conformally Flat* (k,μ)*-Contact Metric Manifolds*

Suppose $M(k,\mu)$ is ϕ-conformally flat, that is, $'C(\phi X,\phi Y,\phi Z,\phi W)=0$ for all vector fields X, Y, Z, and W. Then from (4.1), we obtain

$$\begin{aligned}
&'R(\phi X,\phi Y,\phi Z,\phi W)\\
&\quad=\frac{1}{2n-1}\big(S(\phi Y,\phi Z)g(\phi X,\phi W)-S(\phi X,\phi Z)g(\phi X,\phi W)\\
&\qquad\qquad+g(\phi Y,\phi Z)S(\phi X,\phi W)-g(\phi X,\phi Z)S(\phi Y,\phi W)\big)\\
&\qquad+\frac{r}{2n(2n-1)}\big(g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\big).
\end{aligned} \tag{4.12}$$

Let $\{e_1,e_2,\dots e_{2n},\xi\}$ be a local orthonormal basis of the tangent space $T_P(M)$ at P in $M(k,\mu)$.

Taking $X = W = e_i$ in (4.12) and summing up from 1 to $2n$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2n}{}'R(\phi e_i, \phi Y, \phi Z, \phi e_i) \\
&\quad = \frac{1}{2n-1}\sum_{i=1}^{2n}\big(S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)\big) \\
&\quad - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\
&\quad + \frac{r}{2n(2n-1)}\sum_{i=1}^{2n}\big(g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\big).
\end{aligned} \tag{4.13}$$

Using (3.22) in (4.13), we obtain

$$S(\phi Y, \phi Z) = \frac{r(4n-1) + 4n^2k}{2n}g(\phi Y, \phi Z). \tag{4.14}$$

Replacing Y by ϕY and Z by ϕZ in (4.14) and using (2.1), we have

$$S(Y, Z) = \alpha g(Y, Z) + \beta\eta(Y)\eta(Z), \tag{4.15}$$

where

$$\alpha = \frac{r(4n-1) + 4n^2k}{2n}, \qquad \beta = \frac{-r(4n-1)}{2n}. \tag{4.16}$$

From the relation (4.15), we conclude that $M(k, \mu)$ is an η-Einstein manifold.

Hence we can state the following.

Theorem 4.4. *A ϕ-conformally flat (k, μ)-contact metric manifold is an η-Einstein manifold with $\alpha = (r(4n-1) + 4n^2k)/2n, \beta = -r(4n-1)/2n$.*

5. Pseudoprojective Curvature Tensor in (k, μ)-Contact Metric Manifolds

In $M(k, \mu)$, the pseudoprojective curvature tensor $\widetilde{P}$ is given by [11]

$$\begin{aligned}
\widetilde{P}(X, Y)Z = {}& aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\
& - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(Y, Z)X - g(X, Z)Y],
\end{aligned} \tag{5.1}$$

where a and b are constants such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci tensor, r is the scalar curvature.

5.1. Pseudoprojectively Symmetric (k,μ)-Contact Metric Manifolds

Suppose $R\cdot\widetilde{P}=0$ holds in $M(k,\mu)$. Then we have

$$R(\xi,X)\widetilde{P}(Y,Z)W-\widetilde{P}(R(\xi,X)Y,Z)W-\widetilde{P}(Y,R(\xi,X)Z)W-\widetilde{P}(Y,Z)R(\xi,X)W=0. \tag{5.2}$$

Taking $Y=W=\xi$ in (5.2) and using (5.1), (2.5), (2.10), and (2.11), we have

$$\begin{aligned}
&\left(ak^2+2bnk^2-\frac{kr}{2n+1}\left(\frac{a}{2n}+b\right)+b\mu(2(n-1)+\mu)(k-1)\right)[\eta(Z)\eta(X)-g(Z,X)]\xi\\
&\quad-[ak\mu+2nkb\mu-\mu-b\mu(2(n-1)-n\mu)]g(Z,hX)\xi\\
&\quad+\left(\frac{2kr}{2n+1}\left(\frac{a}{2n}+b\right)2ak^2-4nk^{2b}\right)\eta(X)(\eta(Z)\xi-Z)\\
&\quad+\left(ak^2+2bnk^2-\frac{kr}{2n+1}\left(\frac{a}{2n}+b\right)-a\mu^2(k-1)\right)\eta(Z)(\eta(X)\xi-X)\\
&\quad+\eta(Z)hX(-ak\mu+ak\mu\eta(Y))+\eta(X)hZ(-2ak\mu-\mu-ak\mu\eta(Y))\\
&\quad+\left(ak^2+2bnk^2-\frac{kr}{2n+1}\left(\frac{a}{2n}+b\right)\right)\eta(Y)[\eta(Z)X-\eta(X)Z]\\
&\quad+\mu^2(k-1)[\eta(Z)\eta(X)\xi-\eta(Z)X]+ak^2[g(X,Z)\xi-\eta(X)Z]+kbS(Z,X)\xi\\
&\quad-k\left(2nkb+\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right)\eta(X)Z-\frac{kr}{2n+1}\left(\frac{a}{2n}+b\right)g(Z,X)\xi=0.
\end{aligned}\tag{5.3}$$

Contracting the above with ξ, we obtain

$$\begin{aligned}
S(Z,X)=\frac{1}{kb}\Bigg(&\left[2nk^2b+b\mu((2n-1)+\mu)(k-1)\right]g(Z,X)\\
&+\left[\frac{2kr}{2n+1}\left(\frac{a}{2n}+b\right)-b\mu((2n-1)+\mu)(k-1)\right]\eta(X)\eta(Z)\\
&+[2nkb\mu-b\mu((2n-1)-n\mu)(k-1)]g(Z,hX)\Bigg).
\end{aligned}\tag{5.4}$$

Replacing X by hX in (5.4) and using (2.7) and (2.10), we obtain

$$g(Z,hX)=\frac{2nkb\mu-b\mu l-kbm}{kbl-(2nk^2b+b\mu m(k-1))}(k-1)[\eta(X)\eta(Z)-g(X,Z)], \tag{5.5}$$

where

$$\begin{aligned}
l&=2(n-1)-n\mu,\\
m&=2(n-1)+\mu.
\end{aligned}\tag{5.6}$$

Substituting for $g(Z,hX)$ in (5.4), we obtain

$$S(Z,X)=\alpha g(Z,X)+\beta\eta(X)\eta(Z), \tag{5.7}$$

where

$$\alpha=\frac{1}{kb}\left(2nk^2b+b\mu m(k-1)-\frac{(2nkb\mu-b\mu l-kbm)(2nkb\mu-b\mu l)}{kbl-(2nk^2b+b\mu m(k-1))}(k-1)\right),$$
$$\beta=\frac{1}{kb}\left(\frac{2kr}{2n+1}\left(\frac{a}{2n}+b\right)-b\mu m(k-1)+\frac{(2nkb\mu-b\mu l-kbm)(2nkb\mu-b\mu l)}{kbl-(2nk^2b+b\mu m(k-1))}(k-1)\right). \tag{5.8}$$

From relation (5.7), we conclude that $M(k,\mu)$ is an η-Einstein manifold.

Hence we can state the following.

Theorem 5.1. *A pseudoprojective symmetric (k,μ)-contact metric manifold is an η-Einstein manifold.*

5.2. Pseudoprojective Ricci-Symmetric (k,μ)-Contact Metric Manifolds

If $\widetilde{P}\cdot S=0$, then we have

$$S\left(\widetilde{P}(\xi,X)Y,Z\right)+S\left(Y,\widetilde{P}(\xi,X)Z\right)=0. \tag{5.9}$$

Taking $Y=\xi$ in (5.9) and using (5.1), (2.1), (2.5), (2.10), and (2.11), we obtain

$$\begin{aligned}S(X,Z)=\frac{1}{m-ak}\Big(&\left[2mnk-a\mu(k-1)q-a2nk^2\right]g(X,Z)\\&+\left[a\mu l-a\mu 2nk\right]g(hX,Z)+a\mu(k-1)q\eta(X)\eta(Z)\Big),\end{aligned} \tag{5.10}$$

where

$$l=2(n-1)-n\mu,\quad q=2(n-1)+\mu,\quad m=\frac{r}{2n+1}\left(\frac{a}{2n}+b\right). \tag{5.11}$$

Replacing X by hX in (5.10) and using (2.7) and (2.10), we obtain

$$g(hX,Z)=\frac{\left[a\mu l-a\mu 2nk-q(m-ak)\right](k-1)}{l(m-ak)-\left[2mnk-a\mu(k-1)q-a2nk^2\right]}\left[\eta(X)\eta(Z)-g(X,Z)\right]. \tag{5.12}$$

Now substituting for $g(hX,Z)$ in (5.10), we obtain

$$S(X,Z)=\alpha g(X,Z)+\beta\eta(X)\eta(Y), \tag{5.13}$$

where

$$\alpha = \frac{1}{m-ak}\left(2mnk - a\mu(k-1)q - a2nk^2 - \frac{\left[a\mu l - a\mu 2nk - q(m-ak)\right](k-1)}{l(m-ak) - \left[2mnk - a\mu(k-1)q - a2nk^2\right]}\right),$$
$$\beta = \frac{1}{m-ak}\left(a\mu(k-1)q + \frac{\left[a\mu l - a\mu 2nk - q(m-ak)\right](k-1)}{l(m-ak) - \left[2mnk - a\mu(k-1)q - a2nk^2\right]}\right). \tag{5.14}$$

From (5.13), we have that $M(k,\mu)$ is an η-Einstein manifold.

Hence we can state the following.

Theorem 5.2. *A (k,μ)-contact metric manifold is an η-Einstein manifold if $\widetilde{P}\cdot S = 0$.*

6. Ricci Semisymmetric (k,μ)-Contact Metric Manifolds

If a $(2n+1)$-dimensional (k,μ)-contact metric manifold is Ricci semisymmetric, then $R\cdot S = 0$. That is,

$$S(R(W,X)Y,Z) + S(Y,R(W,X)Z) = 0. \tag{6.1}$$

Taking $W = Y = \xi$ in (6.1) and using (2.5), (2.7), and (2.11), we obtain

$$\begin{aligned} S(X,Z) = \frac{1}{k}\Big[&\Big(2nk^2 + b\mu(k-1)\Big)g(X,Z) + \left(2nk\mu - a\mu\right)g(hX,Z) \\ &- b\mu(k-1)\eta(X)\eta(Z)\Big], \end{aligned} \tag{6.2}$$

where

$$a = 2(n-1) - n\mu, \qquad b = 2(n-1) + \mu. \tag{6.3}$$

Replacing Z by hZ in (6.2) and using (2.7) and (2.10), we obtain

$$g(X,hZ) = \frac{\left(2nk\mu - a\mu - bk\right)(k-1)}{ak - 2nk^2 - b\mu(k-1)}\left(\eta(X)\eta(Z) - g(X,Z)\right). \tag{6.4}$$

Then (6.2) reduces to

$$S(X,Z) = \alpha g(X,Z) + \beta\eta(X)\eta(Z), \tag{6.5}$$

where

$$a = 2(n-1) - n\mu, \qquad b = 2(n-1) + \mu,$$

$$\alpha = \frac{1}{k}\left(2nk^2 + b\mu(k-1) - \frac{(2nk\mu - a\mu)(2nk\mu - a\mu - bk)(k-1)}{ak - 2nk^2 - b\mu(k-1)}\right), \tag{6.6}$$

$$\beta = \frac{1}{k}\left(\frac{(2nk\mu - a\mu)(2nk\mu - a\mu - bk)(k-1)}{ak - 2nk^2 - b\mu(k-1)} - b\mu(k-1)\right).$$

From relation (6.5), we conclude that the manifold is an η-Einstein manifold.

Hence we can state the following.

Theorem 6.1. *A Ricci semisymmetric (k,μ)-contact metric manifold is an η-Einstein manifold.*

References

[1] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, "Contact metric manifolds satisfying a nullity condition," *Israel Journal of Mathematics*, vol. 91, no. 1–3, pp. 189–214, 1995.
[2] E. Boeckx, "A full classification of contact metric (k,μ)-spaces," *Illinois Journal of Mathematics*, vol. 44, no. 1, pp. 212–219, 2000.
[3] R. Sharma, "Certain results on K-contact and (k,μ)-contact manifolds," *Journal of Geometry*, vol. 89, no. 1-2, pp. 138–147, 2008.
[4] B. J. Papantoniou, "Contact manifolds, harmonic curvature tensor and (k,μ)-nullity distribution," *Commentationes Mathematicae Universitatis Carolinae*, vol. 34, no. 2, pp. 323–334, 1993.
[5] U. C. De and A. K. Gazi, "On ϕ-recurrent N(k)-contact metric manifolds," *Mathematical Journal of Okayama University*, vol. 50, pp. 101–112, 2008.
[6] H. G. Nagaraja, "On N(k)-mixed quasi Einstein manifolds," *European Journal of Pure and Applied Mathematics*, vol. 3, no. 1, pp. 16–25, 2010.
[7] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, vol. 509 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.
[8] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, vol. 203 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 2002.
[9] S. Tanno, "Ricci curvatures of contact Riemannian manifolds," *The Tohoku Mathematical Journal*, vol. 40, no. 3, pp. 441–448, 1988.
[10] U. C. De and A. A. Shaikh, *Differential Geometry of Manifolds*, Alpha Science International Ltd, Oxford, UK, 2007.
[11] M. M. Tripathi and P. Gupta, "On τ-curvature tensor in k-contact and Sasakian manifolds," *International Electronic Journal of Geometry*, vol. 4, no. 1, pp. 32–47, 2011.