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Research Article

The Cyclic Graph of a Finite Group

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The cyclic graph Γ_G of a finite group G is as follows: take G as the vertices of Γ_G and join two distinct vertices x and y if $\langle x, y\rangle$ is cyclic. In this paper, we investigate how the graph theoretical properties of Γ_G affect the group theoretical properties of G. First, we consider some properties of Γ_G and characterize certain finite groups whose cyclic graphs have some properties. Then, we present some properties of the cyclic graphs of the dihedral groups D_{2n} and the generalized quaternion groups Q_{4n} for some n. Finally, we present some parameters about the cyclic graphs of finite noncyclic groups of order up to 14.

1. Introduction and Results

Recently, study of algebraic structures by graphs associated with them gives rise to many interesting results. There are many papers on assigning a graph to a group and algebraic properties of group by using the associated graph; for instance, see [1–4].

Let G be a group with identity element e. One can associate a graph to G in many different ways. Abdollahi and Hassanabadi introduced a graph (called the noncyclic graph of a group; see [4]) associated with a group by the cyclicity of subgroups. It is a graph whose vertex set is the set $G\setminus \mathrm{Cyc}(G)$, where $\mathrm{Cyc}(G) = \{x \in G \mid \langle x, y\rangle$ is cyclic for all $y \in G\}$ and x is adjacent y if $\langle x, y\rangle$ is not a cyclic subgroup. They established some graph theoretical properties (such as regularity) of this graph in terms of the group ones.

In this paper, we consider the converse. We associate a graph Γ_G with G (called the cyclic graph of G) as follows: take G as the vertices of Γ_G and two distinct vertices x and y are adjacent if and only if $\langle x, y\rangle$ is a cyclic subgroup of G. For example, Figure 1 is the cyclic graph of $Z_2 \times Z_2$, and Figure 2 is Γ_{S_3}. For any group G, it is easy to see that the cyclic graph Γ_G is simple and undirected with no loops and multiple edges. By the definition, we shall explore how the graph theoretical properties of Γ_G affect the group theoretical properties of G. In particular, the structure of the group by some graph theoretical properties of the associated graph is determined.

The outline of this paper is as follows. In Section 2, we introduce a lot of basic concepts and notations of group and graph theory which will be used in the sequel. In Section 3, we give some properties of the cyclic graph of a group on diameter, planarity, partition, clique number, and so forth and characterize a finite group whose cyclic graph is complete (planar, a star, regular, etc.). For example, the cyclic graph of any group is always connected whose diameter is at most 2 and the girth is either 3 or ∞; the cyclic graph Γ_G of group G is complete if and only if G is cyclic and is a star if and only if G is an elementary abelian 2-group. In particular, for a finite group G, $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G)$ if and only if $G \cong Z_2 \times Z_2$, the Klein group. In Section 4, we present some properties of the cyclic graphs of the dihedral groups D_{2n}, including degrees of vertices, traversability (Eulerian and Hamiltonian), planarity, coloring, and the number of edges and cliques. Furthermore, we get the automorphism group of D_{2n} for all $n \geq 3$. Particularly, for all $n > 2$, if G is a group with $\Gamma_G \cong \Gamma_{D_{2n}}$, then $G \cong D_{2n}$. Similar to Section 4, we discuss the properties of the cyclic graphs on the generalized quaternion groups Q_{4n} in Section 5. In Section 6, we obtain some parameters on the cyclic graphs of finite noncyclic groups of order up to 14.

2. Preliminaries

In this paper, we consider simple graphs which are undirected, with no loops or multiple edges. Let Γ be a graph. We

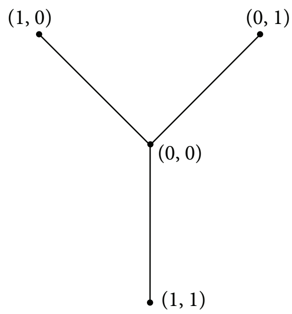

Figure 1: $\Gamma_{Z_2 \times Z_2}$.

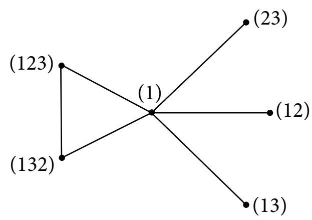

Figure 2: Γ_{S_3}.

will denote $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges of Γ, respectively. Γ is, respectively called empty and complete if $V(\Gamma)$ is empty and every two distinct vertices in $V(\Gamma)$ are adjacent. A complete graph of order n is denoted by K_n. The degree of a vertex v in Γ, denoted by $\deg_\Gamma(v)$, is the number of edges which are incident to v. A subset Ω of $V(\Gamma)$ is called a clique if the induced subgraph of Ω is complete. The order of the largest clique in Γ is its clique number, which is denoted by $\omega(\Gamma)$. A k-vertex coloring of Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The chromatic number $\chi(\Gamma)$ of Γ is the minimum k for which Γ has a k-vertex coloring. If $u, v \in V(\Gamma)$, then $d(u, v)$ denotes the length of the shortest path between u and v. The largest distance between all pairs of $V(\Gamma)$ is called the diameter of Γ and denoted by $\operatorname{diam}(\Gamma)$. The length of the shortest cycle in the graph Γ is called girth of Γ; if Γ does not contain any cycles, then its girth is defined to be infinity (∞). For a vertex v of Γ, denote by $N_\Gamma(v)$ the set of vertices in Γ which are adjacent to v. A vertex v of Γ is a cutvertex if $\Gamma - \{v\}$ is disconnected. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic; the closed interval $I[x, y]$ of x and y is the set of those vertices belonging to at least one $x - y$ geodesic. A set U of $V(\Gamma)$ is called a geodetic set for Γ if $I[U] = V(\Gamma)$, where $I[U] = \bigcup_{x,y\in U} I[x, y]$. A geodetic set of minimum cardinality in Γ is called a minimum geodetic set and this cardinality is the geodetic number. A set S of vertices of Γ is a dominating set of Γ if every vertex in $V(\Gamma) \setminus S$ is adjacent to some vertex in S; the cardinality of a minimum dominating set is called the domination number of Γ and is denoted by $\gamma(\Gamma)$. Γ is a bipartite graph means that $V(\Gamma)$ can be partitioned into two subsets U and W, called partite sets, such that every edge of Γ joins a vertex of U and a vertex of W. If every vertex of U is adjacent to every vertex of W, Γ is called a complete bipartite graph, where U and W are independent. A complete bipartite graph with $|U| = s$ and $|W| = t$ is denoted by $K_{s,t}$. For more information, the reader can refer to [5].

In this paper, all groups considered are finite. Let G be a finite group with identity element e. The number of elements of G is called its order and is denoted by $|G|$. The order of an element x of G is the smallest positive integer n such that $x^n = e$. The order of an element x is denoted by $|x|$. For more notations and terminologies in group theory consult [6].

3. Some Properties of the Cyclic Graphs

Definition 1. In group theory, a locally cyclic group is a group in which every finitely generalized subgroup is cyclic. A group is locally cyclic if and only if every pair of elements in the group generates a cyclic group. It is a fact that every finitely generalized locally cyclic group is cyclic. So a finite locally cyclic group is cyclic.

Definition 2 (see [7, 8]). Let G be a group. The cyclicizer of an element x of G, denoted $\operatorname{Cyc}_G(x)$, is defined by

$$\operatorname{Cyc}_G(x) = \{y \in G \mid \langle x, y\rangle \text{ is cyclic}\}. \tag{1}$$

In general, the cyclicizer $\operatorname{Cyc}_G(x)$ of x is not a subgroup of G. For example, let $G = Z_4 \times Z_2$, then $\operatorname{Cyc}_G((2, 0)) = \{(0, 0), (1, 0), (1, 1), (2, 0), (3, 0), (3, 1)\}$ is not a subgroup.

Definition 3. The cyclicizer $\operatorname{Cyc}(G)$ of G is defined as follows:

$$\operatorname{Cyc}(G) = \bigcap_{x\in G} \operatorname{Cyc}_G(x) = \{y \in G \mid \langle x, y\rangle \text{ is cyclic } \forall x \in G\}. \tag{2}$$

By [9, Theorem 1], $\operatorname{Cyc}(G)$ is a normal subgroup of G and $\operatorname{Cyc}(G) \leq Z(G)$.

Definition 4. Let G be a group. The cyclic graph Γ_G of G is a graph with $V(\Gamma_G) = G$ and two distinct vertices x, y are adjacent in Γ_G if and only if $\langle x, y\rangle$ is a cyclic subgroup of G.

Proposition 5. *For any group G, $\deg_{\Gamma_G}(x) = |\operatorname{Cyc}_G(x)| - 1$, where $x \in G$.*

Proof. By Definitions 2 and 4, it is straightforward. □

Proposition 6. *Let G be a group with the identity element e. Then $\operatorname{diam}(\Gamma_G) \leq 2$. In particular, Γ_G is connected and the girth of Γ_G is either 3 or ∞.*

Proof. Suppose that x and y are two distinct vertices of Γ_G. If $\langle x, y\rangle$ is a cyclic subgroup of G, then x is adjacent to y, and hence $d(x, y) = 1$. Thus we may assume that $\langle x, y\rangle$ is not cyclic. Note that both $\langle e, x\rangle$ and $\langle e, y\rangle$ are cyclic and the vertices x and y are adjacent to e; hence we get $d(x, y) = 2$. This means that Γ_G is connected and $\operatorname{diam}(\Gamma_G) \leq 2$. If there exist $x \neq e$, $y \neq e$ such that x and y are joined by some edge, then $\{x, y, e\}$ is a cycle of order 3 of Γ_G and so the girth of Γ_G is 3. Otherwise, every two vertices (nonidentity elements of G) of Γ_G are not adjacent; that is, Γ_G is a star, which implies that the girth of Γ_G is equal to ∞. □

The following proposition is obvious; we omit its proof.

Proposition 7. *Let G be a group with $|G| > 2$. Then $\{e\}$ is a dominating set of order 1 of Γ_G. In particular, $\gamma(\Gamma_G) = 1$ and* $\deg_{\Gamma_G}(e) = |G| - 1$.

Corollary 8. *Let G be a group. Then $\{x\}$ is a dominating set if and only if $x \in \mathrm{Cyc}(G)$. Moreover, the number of the dominating sets of size 1 is $|\mathrm{Cyc}(G)|$.*

Theorem 9. *Let G be a nontrivial group. Then $\mathrm{diam}(\Gamma_G) = 1$ (or equivalently Γ_G is complete) if and only if G is a cyclic group.*

Proof. Let x and y be two arbitrary elements of G. Suppose that $\mathrm{diam}(\Gamma_G) = 1$. Then $\langle x, y\rangle$ is a cyclic subgroup of G. By Definition 1, G is a cyclic group as G is finite. For the converse, if G is a cyclic group, then $\langle x, y\rangle$ is a cyclic subgroup of G. Thus $\mathrm{diam}(\Gamma_G) = 1$, as desired. □

Corollary 10. *Let G be noncyclic group. Then Γ_G is not regular.*

Theorem 11. *Let G be a group with the identity element e. Then $\Gamma_G \cong K_{1,|G|-1}$ (or equivalently Γ_G is a star) if and only if G is an elementary abelian 2-group.*

Proof. Assume that Γ_G is a star. Let x be a nonidentity element of G. If $|x| \geq 3$, then x and x^{-1} are adjacent since $\langle x, x^{-1}\rangle$ is a cyclic subgroup of G, which is contrary to Γ_G being a star. Hence $|x| = 2$. It follows that the order of every element of G is 2. If x and y are two elements of G, then $(xy)^2 = xyxy = xxyy = e$, and hence $xy = yx$. It means that G is an abelian group and $\exp(G) = 2$. It follows that G is an elementary abelian 2-group.

Conversely, suppose that G is an elementary abelian 2-group. Then the order of every cyclic subgroup of G is 2. Let x is a nonidentity element of G. If there exists an element y such that $\langle x, y\rangle$ is cyclic, then $\langle x, y\rangle = \langle x\rangle$, which implies $y \in \langle x\rangle$. Note that x is an element of order 2; then $y = e$ as $x \neq y$. It follows that the unique element e is adjacent to x in Γ_G. So $\Gamma_G \cong K_{1,|G|-1}$. □

Corollary 12. *Let G be an elementary abelian 2-group. Then* $\mathrm{Aut}(\Gamma_G)$ *is isomorphic to the symmetric $S_{|G|-1}$ on $|G| - 1$ letters.*

Corollary 13. *Let G be group. Then Γ_G is a tree if and only if G is an elementary abelian 2-group.*

Corollary 14. *Let G be group with $|G| > 2$. If Γ_G is bipartite, then* $\mathrm{Cyc}(G) = \{e\}$.

Proof. Assume, on the contrary, $\mathrm{Cyc}(G) \neq \{e\}$. Then there exist two adjacent vertices x and y such that $x, y \in \mathrm{Cyc}(G)$. Since $|G| > 2$, there is an element z such that $z \neq x, z \neq y$. By Definition 3, $\{x, y, z\}$ is a cycle of length 3 and so the subgraph of Γ_G induced by $\{x, y, z\}$ is an odd cycle, which is a contradiction to Γ_G being bipartite (see [5, Theorem 1.12, page 22]). □

Remark 15. Let $G = Z_2$. Then Γ_G is a bipartite graph, while $\mathrm{Cyc}(G) \neq \{e\}$.

Corollary 16. *Let G be group. Then Γ_G is bipartite if and only if G is an elementary abelian 2-group.*

Proposition 17. *Let G_1 and G_2 be two groups. If $G_1 \cong G_2$, then $\Gamma_{G_1} \cong \Gamma_{G_2}$.*

Proof. Let ϕ be an isomorphism from G_1 to G_2. Obviously, ϕ is a one-to-one correspondence between Γ_{G_1} and Γ_{G_2}. Let x and y be two vertices of Γ_{G_1}. If $\langle x, y\rangle = \langle g\rangle$ is cyclic, then there exist two positive integers n, m such that $x = g^n$ and $y = g^m$, so $x^\phi = (g^\phi)^n$ and $y^\phi = (g^\phi)^m$; It means that $x^\phi, y^\phi \in \langle g^\phi\rangle$, that is, $\langle x^\phi, y^\phi\rangle$ is a subgroup of $\langle g^\phi\rangle$. Thus $\langle x^\phi, y^\phi\rangle$ is cyclic. Note that ϕ is invertible. It follows that x and y are adjacent in Γ_{G_1} if and only if x^ϕ is adjacent to y^ϕ in Γ_{G_2}. Consequently, ϕ is a graph automorphism from Γ_{G_1} to Γ_{G_2}, namely, $\Gamma_{G_1} \cong \Gamma_{G_2}$. □

Remark 18. The converse of Proposition 17 is not true in general. Let G_1 be the modular group of order 16 (a group is called a modular group if its lattice of subgroups is modular) with presentation

$$\left\langle s, t : s^8 = r^2 = e, st = ts^5 \right\rangle. \tag{3}$$

Clearly, $G_1 = \{s^k t^m \mid k = 0, 1, \ldots, 7, m = 0, 1\}$. Let $G_2 = Z_2 \times Z_8$. For G_1, this is the same subgroup lattice structure as for the lattice of subgroups of G_2. It is easy to see that $\Gamma_{G_1} \cong \Gamma_{G_2}$, however, $G_1 \ncong G_2$ because G_1 is not abelian.

Theorem 19. *Let G be a group and let a be an element of G. If $|g| \leq |a|$ for all $g \in G$, then $\omega(\Gamma_G) = |a|$.*

Proof. Let $|a| = n$. Then the induced subgraph of $\{a, a^2, \ldots, a^{n-1}, e\}$ is complete; hence $\{a, a^2, \ldots, a^{n-1}, e\}$ is a clique of Γ_G. On the other hand, if $\omega(\Gamma_G) = m$, then there exists a subset C of $V(\Gamma_G)$ such that the subgraph of Γ_G induced by C is complete and $|C| = m$. Note that the order of the largest clique is m; e must be an element of C. If $x \in C$, then we have $\langle x, g\rangle$ being cyclic for every g in $C \setminus \{x\}$. Clearly $\langle x, g\rangle = \langle x^{-1}, g\rangle$; that is, $x^{-1} \in C$. Let x, y be two arbitrary elements of C. So $\langle x, y\rangle$ is cyclic. Since $\langle xy, x\rangle \leq \langle x, y\rangle$ and $\langle xy, y\rangle \leq \langle x, y\rangle$, xy and x are adjacent in Γ_G; yet, xy is adjacent to y. Suppose $z \in C \setminus \{x, y\}$, it is easy to see that $\langle xy, z\rangle \leq \langle x, y, z\rangle$. Since $\langle x, y, z\rangle$ is a locally cyclic group by Definition 1, $\langle x, y, z\rangle$ is a cyclic group; namely, $\langle xy, z\rangle$ is cyclic. Consequently, xy and z are joined by an edge of Γ_G. From what we have mentioned above, we can see that C is a group of G. Again, C is a cyclic subgroup of G by Definition 1. Let $C = \langle b\rangle$, where b is an element of G. It follows that $|b| \leq |a|$ from the hypothesis; that is, $\omega(\Gamma_G) = |a|$. □

Corollary 20. *Let G be group. If $C = \{x, x^2, \ldots, x^{|x|-1}, e\} = \langle x\rangle$, then C is a clique of Γ_G. Converse holds only when C is the largest clique.*

Corollary 21. *Let $n \geq 3$. Then S_n and A_n are planar if and only if $n = 3$ or 4.*

Theorem 22. *Let G be a group. Then $\deg_{\Gamma_G}(x) = |x| - 1$ for all $x \in V(\Gamma_G) \setminus \{e\}$ if and only if every element of $G \setminus \{e\}$ is of prime order.*

Proof. Assume that $\deg_{\Gamma_G}(x) = |x| - 1$ for every nonidentity element x of G. If there exists an element x of $G \setminus \{e\}$ such that x is not of prime order, then we may choose t such that t divides the order of x and $1 < t < |x|$. Thus $x^t \neq e$ and x is adjacent to x^t, since $\langle x, x^t \rangle = \langle x \rangle$. while $x \notin \langle x^t \rangle$ (otherwise, $\langle x \rangle = \langle x^t \rangle$, a contradiction). so $N_{\Gamma_G}(x^t) = \{x\} \cup (\langle x^t \rangle \setminus \{x^t\})$. This is contrary to $\deg_{\Gamma_G}(x^t) = |x^t| - 1$.

For the converse, suppose every nonidentity element x of G is of prime order. If $\langle x, y \rangle$ is a cyclic subgroup of G, then $|\langle x, y \rangle|$ is a prime number. Thereby, $\langle x, y \rangle = \langle x \rangle$, and so $y \in \langle x \rangle$. That is, $\mathrm{Cyc}_G(x) = \langle x \rangle$. Hence the theorem follows. □

Theorem 23. *Let G be a group. Then $N_{\Gamma_G}(x) \cup \{x\}$ is a cyclic subgroup for all $x \in G \setminus \{e\}$ if and only if every element x of $G \setminus \{e\}$ is contained in precisely one maximal cyclic subgroup of G.*

Proof. Assume that every element of $G \setminus \{e\}$ is contained in exactly one maximal cyclic subgroup of G. If x is an element of $G \setminus \{e\}$, then there is a maximal cyclic subgroup $\langle y \rangle$ such that $x \in \langle y \rangle$. Let $a \in \mathrm{Cyc}_G(x)$. Since $\langle x, a \rangle$ is cyclic, $\langle x, a \rangle = \langle z \rangle$. If $\langle x, a \rangle \nleq \langle y \rangle$, then there exists a maximal cyclic subgroup $\langle w \rangle$ such that $z \in \langle w \rangle$ as $z \neq e$. However $x \in \langle w \rangle$; this gives a contradiction to $\langle y \rangle$ being the precisely one maximal cyclic subgroup of containing x. Consequently $\langle x, a \rangle \leq \langle y \rangle$, and so $a \in \langle y \rangle$. Also, if $b \in \langle y \rangle$, then $\langle x, b \rangle \leq \langle y \rangle$, so x and b are adjacent in Γ_G; it means that $b \in \mathrm{Cyc}_G(x)$. Thus $\mathrm{Cyc}_G(x) = \langle y \rangle$; that is, $\mathrm{Cyc}_G(x)$ is cyclic. In other words, $N_{\Gamma_G}(x) \cup \{x\}$ is a cyclic subgroup of G.

Conversely, let x be an element of $G \setminus \{e\}$ such that $x \in \langle y \rangle$ and $x \in \langle z \rangle$, where $\langle y \rangle$ and $\langle z \rangle$ are two maximal cyclic subgroups of G. Assume that $N_{\Gamma_G}(x) \cup \{x\}$ is cyclic. Since x is adjacent to y, we have $\langle y \rangle \leq N_{\Gamma_G}(x) \cup \{x\}$, so $\langle y \rangle = N_{\Gamma_G}(x) \cup \{x\}$. Similarly, $\langle z \rangle = N_{\Gamma_G}(x) \cup \{x\}$, and thus $\langle y \rangle = \langle z \rangle$; that is, x is contained in precisely one maximal cyclic subgroup of G. □

Theorem 24. *Let G be a group. Then $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G)$ if and only if G is isomorphic to the Klein group $Z_2 \times Z_2$.*

Proof. First we suppose that $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G)$ for group G. We shall show that G is isomorphic to the Klein group $Z_2 \times Z_2$ by the following steps.

Step 1 (G is abelian). Let ψ be an automorphism of Γ_G. Then ψ is an automorphism of group G, so $(xy)^\psi = x^\psi y^\psi$ for all $x, y \in G$. Now we define the mapping α: $x^\alpha = x^{-1}$ for all x in $V(\Gamma_G)$. It is well known that α is a bijection and $\langle a, b \rangle$ is cyclic if and only if $\langle a^{-1}, b^{-1} \rangle$ is cyclic; that is, ab is an edge of Γ_G if and only if $a^\alpha b^\alpha$ is an edge of Γ_G. Thus $\alpha \in \mathrm{Aut}(\Gamma_G)$. By hypothesis, $\alpha \in \mathrm{Aut}(G)$, so $(xy)^\alpha = x^\alpha y^\alpha = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$; namely, $xy = yx$, and hence G is a abelian group.

Step 2 (G is not a cyclic group). If G is a cyclic group, then we can see that Γ_G is isomorphic to the complete graph $K_{|G|}$ by Theorem 9, and hence $\mathrm{Aut}(\Gamma_G)$ is isomorphic to the symmetric group $S_{|G|}$. Since $|G| \geq 3$ (if $|G| = 2$, then $\mathrm{Aut}(G) = \{e\}$, but $\mathrm{Aut}(\Gamma_G) \cong Z_2$, a contradiction), $\mathrm{Aut}(\Gamma_G)$ is nonabelian. However, $\mathrm{Aut}(G)$ must be abelian as G is cyclic, a contradiction.

Step 3 (G is an elementary abelian 2-group). By Step 1, we have $G = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$, where $|a_i| \mid |a_{i+1}|$ for all $i = 1, 2, \ldots, r-1$. It is clear that $r > 1$ by Step 2. Obviously, there exists a graph automorphism ψ such that $a_1^\psi = a_1, a_r^\psi = a_r, (a_1a_r)^\psi = (a_1a_r)^{-1}$, and $((a_1a_r)^{-1})^\psi = a_1a_r$. Since $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G)$, we have $\psi \in \mathrm{Aut}(G)$ and $(a_1a_r)^\psi = a_1^\psi a_r^\psi$. It follows that $a_1^2 = a_r^{-2} \in \langle a_1 \rangle \cap \langle a_r \rangle = \{e\}$. Furthermore, $|a_1| = |a_r| = 2$. In particular, $|a_2| = |a_3| = \cdots = |a_{r-1}| = 2$. It follows that G is an elementary abelian 2-group.

Step 4 (finishing the proof). Let $|G| = 2^n$ for some positive integer n. By Step 3 and Theorem 11, Γ_G is isomorphic to the star $K_{1,2^n-1}$. So $\mathrm{Aut}(\Gamma_G)$ is the symmetric group S_{2^n-1} of degree $2^n - 1$, while $\mathrm{Aut}(G)$ is isomorphic to the general linear group $\mathrm{GL}(n, 2)$. Thus $n = 2$ as $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G)$. That is, $G \cong Z_2 \times Z_2$.

For the converse, we suppose that $G \cong Z_2 \times Z_2$. Then we have $\mathrm{Aut}(G) = S_3$. On the other hand, $\Gamma_G \cong K_{1,3}$, and so $\mathrm{Aut}(\Gamma_G) = \mathrm{Aut}(G) = S_3$. □

Remark 25. Suppose $G = Z_2 \times Z_2$, then $\Gamma_G \cong K_{1,3}$. Let $Z_2 \times Z_2 = \{e, a, b, ab \mid a^2 = b^2 = e, ab = ba\}$. Then $|\mathrm{Aut}(G)| = 6$, more specifically,

$$\varphi_1 = \begin{cases} a \longmapsto a, \\ b \longmapsto ab, \\ ab \longmapsto b, \\ e \longmapsto e, \end{cases} \qquad \varphi_2 = \begin{cases} a \longmapsto b, \\ b \longmapsto ab, \\ ab \longmapsto a, \\ e \longmapsto e, \end{cases}$$

$$\varphi_3 = \begin{cases} a \longmapsto ab, \\ b \longmapsto b, \\ ab \longmapsto b, \\ e \longmapsto e, \end{cases} \qquad \varphi_4 = \begin{cases} a \longmapsto b, \\ b \longmapsto a, \\ ab \longmapsto ab, \\ e \longmapsto e, \end{cases} \tag{4}$$

$$\varphi_5 = \begin{cases} a \longmapsto ab, \\ b \longmapsto a, \\ ab \longmapsto b, \\ e \longmapsto e, \end{cases} \qquad \varphi_6 = e,$$

here $\varphi_i \in \mathrm{Aut}(G)$ for $i = 1, 2, \ldots, 6$. Clearly, we can see that $\mathrm{Aut}(G)$ is nonabelian; that is, $\mathrm{Aut}(G) \cong S_3$.

Proposition 26. *Let G be an elementary abelian p-group for some prime integer p. Then Γ_G is isomorphic to Figure 3.*

Proof. Let x be an element of G and $x \neq e$. Since G is an elementary abelian p-group, we conclude that the order of x is p. It follows that the subgraph of Γ_G induced by

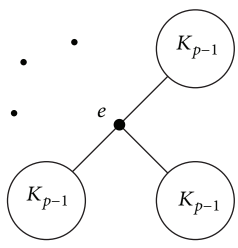

FIGURE 3: The cyclic graph of the elementary abelian p-group.

$\{x, x^2, \ldots, x^{p-1}\}$ is isomorphic to the complete graph K_{p-1} of order $p-1$. Let y be an element such that $y \notin \langle x \rangle$. If x and y are adjacent, then $\langle x, y \rangle$ is a cyclic subgroup of G, which implies $\langle x, y \rangle = \langle x \rangle = \langle y \rangle$ since $\langle x, y \rangle$ is a cyclic subgroup of order 5, and this gives a contradiction to $y \notin \langle x \rangle$. Thus x is uniquely adjacent to every vertex of $\{e, x, x^2, \ldots, x^{p-1}\}$. This completes the proof. □

Remark 27. Let p be composite. Then, in general, $Z_p \times Z_p \times \cdots \times Z_p$ is not isomorphic to Figure 3. For example, let $G = Z_4 \times Z_4$, then $\deg_{\Gamma_G}((2,0)) = 5 > 4$. In fact, $N_{\Gamma_G}((2,0)) = \{(1,2), (0,0), (3,2), (1,0), (3,0)\}$.

4. The Cyclic Graphs of the Dihedral Groups

For $n \geq 3$, the dihedral group D_{2n} is an important example of finite groups. As is well known, $D_{2n} = \langle r, s : s^2 = r^n = e, s^{-1}rs = r^{-1} \rangle$. As a list,

$$D_{2n} = \left\{ r^1, r^2, \ldots, r^n = e, sr^1, sr^2, \ldots, sr^n \right\}. \tag{5}$$

Theorem 28. *Let $\Gamma_{D_{2n}}$ be the cyclic graph of D_{2n} and $n \geq 3$. Then*

(1) $\deg_{\Gamma_{D_{2n}}}(sr^i) = 1$ *for any* $1 \leq i \leq n$;

(2) $\deg_{\Gamma_{D_{2n}}}(r^i) = n-1$ *for any* $1 \leq i < n$;

(3) $\Gamma_{D_{2n}}$ *is not Eulerian;*

(4) $\Gamma_{D_{2n}}$ *is not Hamiltonian;*

(5) $\Gamma_{D_{2n}}$ *is planar if and only if* $n = 3$ *or* 4;

(6) $\Gamma_{D_{2n}}$ *is a split graph;*

(7) $\mathrm{Aut}(\Gamma_{D_{2n}}) \cong S_n \times S_{n-1}$.

Proof. (1) Clearly, the order of sr^i is 2 for all $1 \leq i \leq n$ by the definition of D_{2n}. Since every cyclic subgroup of G has a uniquely cyclic subgroup of order 2, $\langle sr^i, sr^j \rangle$ is noncyclic; that is, sr^i and sr^j are not adjacent to each other. If sr^i is adjacent to r^j, where $j \neq n$, then $\langle sr^i, r^j \rangle$ is cyclic and hence $\langle sr^i, r^j \rangle = \langle r^k \rangle$, which is a contradiction. Thus, e is the unique element of G which is adjacent to sr^i, as required.

(2) It is easy to see that $\deg_{\Gamma_{D_{2n}}}(r^i) \geq n-1$ for any $1 \leq i < n$. Now (1) completes the proof.

(3) Since $\deg_{\Gamma_{D_{2n}}}(s)$ is an odd integer by (1), $\Gamma_{D_{2n}}$ is not Eulerian (see [5, Theorem 6.1, page 137]).

(4) In view of (1) and (2), $\Gamma_{D_{2n}}$ contains a cut-vertex e. In the light of [5, Theorem 6.5, page 145] , we conclude that $\Gamma_{D_{2n}}$ cannot be Hamiltonian.

(5) If $n = 3$ or 4, then it is easy to see that $\Gamma_{D_{2n}}$ is planar. Now suppose that $\Gamma_{D_{2n}}$ is planar. Since the complete graph of order 5 is not planar, we have $\omega(\Gamma_{D_{2n}}) < 5$. Since the subgraph of $\Gamma_{D_{2n}}$ induced by $\{r, r^2, \ldots, r^{n-1}, r^n\}$ is complete, we have $n < 5$. That is, $n = 3$ or 4, as desired.

(6) By (1) and (2), the vertex set of $\Gamma_{D_{2n}}$ can be partitioned into the clique $\{r, r^2, \ldots, r^{n-1}, e\}$ and the independent set $\{sr^1, sr^2, \ldots, sr^{n-1}, s\}$, and hence $\Gamma_{D_{2n}}$ is a split graph.

(7) It is straightforward. □

Corollary 29. *Let $\Gamma_{D_{2n}}$ be the cyclic graph of D_{2n} and $n \geq 3$. Then $\Gamma_{D_{2n}}$ is not bipartite.*

Corollary 30. *Let $n \geq 3$. Then $|E(\Gamma_{D_{2n}})| = n(n+1)/2$.*

Corollary 31. *Let $n \geq 3$. Then $\omega(\Gamma_{D_{2n}}) = \chi(\Gamma_{D_{2n}}) = n$.*

Theorem 32. *Let $n > 2$ be an integer. If G is a group with $\Gamma_G \cong \Gamma_{D_{2n}}$, then $G \cong D_{2n}$.*

Proof. We have $|G| = 2n$ by Definition 4. In view of Theorem 28, we can see that $\omega(\Gamma_G) = n$. It follows from Theorem 19 that there exists an element $r \in G$ such that $\langle r \rangle$ is a cyclic subgroup of order n. Note that $|G : \langle r \rangle| = 2$; we have $\langle r \rangle$ being a normal subgroup of G. Since there are n vertices in Γ_G such that the degrees equal 1, there exist n elements of order 2 in G. Now we choose an involution s of order 2 of G such that $s \notin \langle r \rangle$. It is easy to see that $G = \langle r \rangle \rtimes \langle s \rangle$; that is, $G \cong Z_n \rtimes Z_2$. By the definition of dihedral group, Z_2 acts on Z_n by inversion. This implies that $G \cong D_{2n}$, as required. □

5. The Cyclic Graphs of the Generalized Quaternion Groups

The quaternion group Q_8 is also an important example of finite nonabelian groups; it is given by

$$Q_8 = \left\langle -1, i, j, k : (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \right\rangle. \tag{6}$$

As a generalization of Q_8, the generalized quaternion group Q_{4n} is defined as

$$Q_{4n} = \left\langle a, b : b^2 = a^n,\ a^{2n} = e,\ bab^{-1} = a^{-1} \right\rangle, \tag{7}$$

where e is the identity element and $n \geq 2$ (if $n = 2$, then $Q_{4n} = Q_8$). Clearly, Q_{4n} has order $4n$ as a list

$$Q_{4n} = \left\{ a, a^2, \ldots, a^{2n-1}, e, b, ab, \ldots, a^{2n-1}b \right\}. \tag{8}$$

Moreover, $Z(Q_{4n}) = \{e, a^n\}$ and $|a^ib| = 4$, where $1 \leq i \leq 2n$.

Table 1

Cyclic graph	Isomorphic graph	Vertex degree sequences	Clique number	Geodetic number	Planarity
$\Gamma_{Z_2\times Z_2}$	$K_{1,3}$	3, 1, 1, 1	2	3	Planar
Γ_{S_3}		5, 2, 2, 1, 1, 1	3	5	Planar
$\Gamma_{Z_2\times Z_2\times Z_2}$	$K_{1,7}$	7, 1, 1, 1, 1, 1, 1, 1	2	7	Planar
$\Gamma_{Z_2\times Z_4}$		7, 5, 3, 3, 3, 3, 1, 1	4	6	Planar
Γ_{D_8}		7, 3, 3, 3, 1, 1, 1, 1	4	7	Planar
Γ_{Q_8}		7, 7, 3, 3, 3, 3, 3, 3	4	6	Planar
$\Gamma_{Z_3\times Z_3}$		8, 2, 2, 2, 2, 2, 2, 2, 2	3	8	Planar
$\Gamma_{D_{10}}$		9, 4, 4, 4, 4, 1, 1, 1, 1, 1	5	9	Nonplanar
$\Gamma_{Z_2\times Z_6}$		11, 9, 9, 5, 5, 5, 3, 3, 3, 3, 1, 1	6	10	Nonplanar
$\Gamma_{D_{12}}$		11, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1	6	11	Nonplanar
$\Gamma_{Q_{4\times 3}}$		11, 11, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3	6	10	Nonplanar
Γ_{A_4}		11, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1	3	11	Planar
$\Gamma_{D_{14}}$		13, 6, 6, 6, 6, 6, 6, 1, 1, 1, 1, 1, 1, 1	7	13	Nonplanar

Lemma 33. *$Cyc(Q_{4n}) = \{e, a^n\}$.*

Proof. Since $(a^ib)^2 = b^2 = a^n$ and $|a^ib| = 4$ for all i, $\langle a^ib, a^n\rangle = \langle a^ib\rangle$; that is, $a^n \in \mathrm{Cyc}_{Q_{4n}}(a^ib)$ for all i. On the other hand, it is obvious that $\langle a^j, a^n\rangle$ is a cyclic subgroup of Q_{4n} for all j as $\langle a^j, a^n\rangle \le \langle a\rangle$, where $1 \le j \le 2n$. Consequently $a^n \in Cyc_{Q_{4n}}(a^j)$ for all j, namely, $a^n \in \mathrm{Cyc}(Q_{4n})$. However, $\mathrm{Cyc}(Q_{4n}) \subseteq Z(Q_{4n})$, so $\mathrm{Cyc}(Q_{4n}) = Z(Q_{4n}) = \{e, a^n\}$. □

Proposition 34. *Let $\Gamma_{Q_{4n}}$ be the cyclic graph of Q_{4n}. Then*

(1) $\deg_{Q_{4n}}(a^ib) = 3$ *for all* $1 \le i \le 2n$;

(2) $\deg_{Q_{4n}}(a^j) = 2n - 1$ *for all* $1 \le j < n$ *and* $n < j < 2n$;

(3) $\deg_{Q_{4n}}(e) = \deg_{Q_{4n}}(a^n) = 4n - 1$.

Proof. (1) Since $|a^ib| = 4$ for all $1 \le i \le 2n$, $\deg_{Q_{4n}}(a^ib) \ge 3$. Obviously, $\{e, a^n, (a^ib)^{-1}\} \subseteq N_{\Gamma_{Q_{4n}}}(a^ib)$. If a^ib and a^jb are joined by an edge, then $\langle a^ib, a^jb\rangle$ is a cyclic subgroup of order 4; Note that $\langle a^ib\rangle$ is a cyclic subgroup of order 4, then $a^ib = a^jb$ or $a^ib = (a^jb)^{-1}$. On the other hand, it is easy to see that $\langle a^ib, a^j\rangle$ cannot be cyclic, where $1 \le j < n$ and $n < j < 2n$. Consequently, we have $\{e, a^n, (a^ib)^{-1}\} = N_{\Gamma_{Q_{4n}}}(a^ib)$, and so $\deg_{Q_{4n}}(a^ib) = 3$.

(2) By the proof of (1), we see that $\langle a^ib, a^j\rangle$ is not cyclic for all $1 \le j < n$ and $n < j < 2n$, so $\deg_{Q_{4n}}(a^j) = 2n - 1$.

(3) Obviously by Lemma 33. □

Corollary 35. *Let $n \ge 2$. Then $|E(\Gamma_{D_{4n}})| = 2n^2 + 4n$.*

Corollary 36. *Let $n \ge 2$. Then $\omega(\Gamma_{D_{4n}}) = \chi(\Gamma_{D_{4n}}) = 2n$.*

Theorem 37. *Let $\Gamma_{Q_{4n}}$ be the cyclic graph of Q_{4n}. Then $\Gamma_{Q_{4n}}$ is planar if and only if $n = 2$.*

Proof. Suppose $n = 2$. It is easy to see that Γ_{Q_8} is planar. Now assume that $\Gamma_{Q_{4n}}$ is a planar graph. Then $\omega(\Gamma_{D_{2n}}) < 5$ since K_5 is nonplanar. By Theorem 19, there exists no the element g of Q_{4n} such that $|g| \ge 5$. However $|a| = 2n$, and hence $n = 2$. □

Theorem 38. *Let $\Gamma_{Q_{4n}}$ be the cyclic graph of Q_{4n} and $n \ge 2$. Then*

(1) $\Gamma_{Q_{4n}}$ *is not Eulerian;*

(2) $\Gamma_{Q_{4n}}$ *is not Hamiltonian.*

Proof. (1) It is similar to the proof of (3) in Theorem 28.

(2) Let $k(\Gamma_{Q_{4n}})$ denote the number of components in the graph $\Gamma_{Q_{4n}}$. By Theorem 6.5 of [5] on page 145, if $\Gamma_{Q_{4n}}$ is Hamiltonian, then for every nonempty proper set S of vertices of $\Gamma_{Q_{4n}}$, we have $k(\Gamma_{Q_{4n}} - S) \le |S|$. Now suppose $S = \{e, a^n\}$. Then the number of components of the resulting graph $\Gamma_{Q_{4n}} - S$ is equal to $n + 1$. However, $n + 1 > |S|$, a contradiction. □

6. The Cyclic Graphs of Noncyclic Groups of Order up to 14

It is significant to obtain detailed information on the cyclic graphs of some noncyclic groups of lower order. In this section, we present a table on the cyclic graphs of noncyclic groups of order up to 14, as shown in Table 1.

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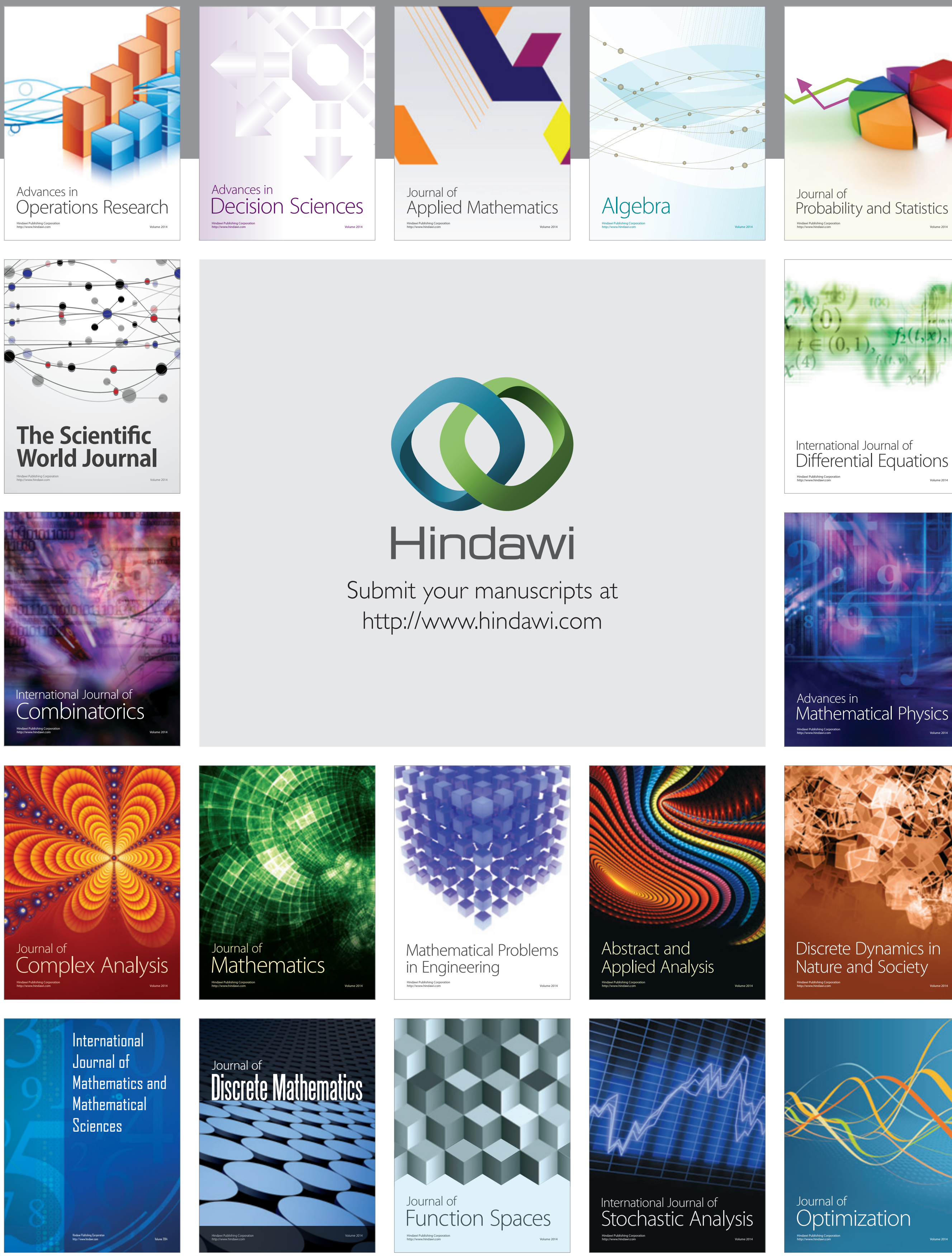
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