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Retraction

Retracted: Idempotent Elements of the Endomorphism Semiring of a Finite Chain

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This article has been retracted as it is essentially identical in title and technical content with the published article "Idempotent Elements of the Endomorphism Semiring of a Finite Chain," by Ivan Trendafilov and Dimitrinka Vladeva, published in Comptes Rendus de L'Académie Bulgare des Sciences 66 (2013), no. 5, 621–628 (see [1]).

References

[1] I. Trendafilov and D. Vladeva, "Idempotent elements of the endomorphism semiring of a finite chain," *ISRN Algebra*, vol. 2013, Article ID 120231, 9 pages, 2013.

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Research Article

Idempotent Elements of the Endomorphism Semiring of a Finite Chain

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Idempotents yield much insight in the structure of finite semigroups and semirings. In this article, we obtain some results on (multiplicatively) idempotents of the endomorphism semiring of a finite chain. We prove that the set of all idempotents with certain fixed points is a semiring and find its order. We further show that this semiring is an ideal in a well-known semiring. The construction of an equivalence relation such that any equivalence class contains just one idempotent is proposed. In our main result we prove that such an equivalence class is a semiring and find its order. We prove that the set of all idempotents with certain jump points is a semiring.

1. Introduction

The idempotents play an essential role in the theory of finite semigroups and semirings. It is well known that in a finite semigroup some power of each element is an idempotent, so the idempotents can be taken to be like a generating system of the semigroup or the semiring. For deep results, using idempotents in the representation theory of finite semigroups, we refer the reader to [1, 2].

Let us briefly survey the contents of our paper. After the preliminaries, in Section 3 we show some facts about the fixed points of idempotent endomorphisms. The central result here is Theorem 9 where we prove that the set of all idempotents with s fixed points $k_1, \ldots, k_s$, $1 \le s \le n-1$, is a semiring of order $\prod_{m=1}^{s-1}(k_{m+1}-k_m)$. Moreover, this semiring is an ideal of the semiring of all endomorphisms having at least $k_1, \ldots, k_s$ as fixed points. In the next section we consider an equivalence relation on some finite semigroup S such that for any $x, y \in S$ follows $x \sim y$ if and only if $x^k = y^m = e$, where $k, m \in \mathbf{N}$ and e is an idempotent of S. Then we consider the equivalence classes of semigroup $S = (\widehat{\mathscr{E}}_{\mathscr{C}_n}, \cdot)$ which is, see [3], one subsemigroup of $\mathscr{PT}_n$. Here we investigate the so-called jump points of the endomorphism and prove that between any two fixed points k_i and k_{i+1} of an endomorphism, where $k_{i+1} \ne k_i$, there is a unique jump point.

The main result of the paper is Theorem 19 where we prove that such an equivalence class is a semiring of order

$$C_{k_{1,1}}\left(\prod_{i=1}^{\ell-1} C_{t_i} C_{s_i}\right) C_{n-1-k_{\ell,m_\ell}}, \tag{1}$$

where C_p is the pth Catalan number.

In the last section of the paper we consider idempotent endomorphisms with arbitrary fixed points but with certain jump points. Here we prove that the set of idempotent endomorphisms with identical jump points is a semiring.

2. Preliminaries

We consider some basic definitions and facts concerning finite semigroups and that can be found in any of [1, 2, 4, 5]. As the terminology for semirings is not completely standardized, we say what our conventions are.

An algebra $R = (R, +, \cdot)$ with two binary operations $+$ and $\cdot$ on R is called a *semiring* if

(i) $(R, +)$ is a commutative semigroup,

(ii) $(R, \cdot)$ is a semigroup,

(iii) both distributive laws hold $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for any $x, y, z \in R$.

Let $R = (R, +, \cdot)$ be a semiring. If a neutral element 0 of semigroup $(R, +)$ exists and it satisfies $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$, then it is called *zero*. If a neutral element 1 of semigroup $(R, \cdot)$ exists, it is called *identity element*.

For a semilattice $\mathscr{M}$ the set $\mathscr{E}_{\mathscr{M}}$ of the endomorphisms of $\mathscr{M}$ is a semiring with respect to the addition and multiplication defined by

(i) $h = f + g$ when $h(x) = f(x) \vee g(x)$ for all $x \in \mathscr{M}$,

(ii) $h = f \cdot g$ when $h(x) = f(g(x))$ for all $x \in \mathscr{M}$.

This semiring is called the *endomorphism semiring* of $\mathscr{M}$. In this paper all semilattices are finite chains. Following [6, 7], we fix a finite chain $\mathscr{C}_n = (\{0, 1, \ldots, n-1\}, \vee)$ and denote the endomorphism semiring of this chain with $\widehat{\mathscr{E}}_{\mathscr{C}_n}$. We do not assume that $\alpha(0) = 0$ for arbitrary $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$. So, there is not a zero in endomorphism semiring $\widehat{\mathscr{E}}_{\mathscr{C}_n}$. Subsemiring $\mathscr{E}_{\mathscr{C}_n} = \mathscr{E}^0_{\mathscr{C}_n}$ of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ consisting of all endomorphisms α with property $\alpha(0) = 0$ has zero and is considered in [6–8].

If $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ such that $f(k) = i_k$ for any $k \in \mathscr{C}_n$ we denote α as an ordered n-tuple $\wr i_0, i_1, i_2, \ldots, i_{n-1} \wr$. Note that mappings will be composed accordingly, although we will usually give preference to writing mappings on the right, so that $\alpha \cdot \beta$ means "first α, then β." The identity $\mathbf{i} = \wr 0, 1, \ldots, n-1 \wr$ and all constant endomorphisms $\kappa_i = \wr i, \ldots, i \wr$ are obviously (multiplicatively) idempotents.

An element $a \in \mathscr{C}_n$ satisfying $\alpha(a) = a$ is called a fixed point of the endomorphism α. Any $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ has at most n fixed points, and only identity $\mathbf{i}$ has just n fixed points.

For other properties of the endomorphism semiring we refer the reader to [3, 6–8].

In the following sections we use some terms from [4] having in mind that in [3] we show that some subsemigroups of the partial transformation semigroup are indeed endomorphism semirings.

3. Idempotent Endomorphisms and Their Fixed Points

The set of all idempotents (or idempotent elements) of semiring $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ is not a semiring namely; endomorphisms $\alpha = \wr 0, 0, 2 \wr$ and $\beta = \wr 0, 1, 1 \wr$ are the idempotents of semiring $\widehat{\mathscr{E}}_{\mathscr{C}_3}$ but $\alpha\beta = \wr 0, 0, 1 \wr$ is not an idempotent. The following several facts are well known or are consequences of well-known facts.

Proposition 1. *The endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ is an idempotent if and only if for any $k \in \mathscr{C}_n$, which is not a fixed point of α, image $\alpha(k)$ is a fixed point of α.*

Note that in [9] maps α_k are considered such that $\alpha_k(x) = k$ for all $x \in \mathscr{M}$, where $\mathscr{M}$ is not necessarily a finite semilattice. These maps are called constant endomorphisms.

For any $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ with one fixed point the cardinality of set $\mathrm{Im}(\alpha)$ is an arbitrary number from 2 to $n-1$. Indeed, for $\alpha = \wr n-2, n-1, \ldots, n-1 \wr$ with unique fixed point $n-1$ we have $|\mathrm{Im}(\alpha)| = 2$ and for $\beta = \wr 1, 2, \ldots, n-2, n-1, n-1 \wr$ with the same unique fixed point, $|\mathrm{Im}(\beta)| = n-1$. So, the following consequence of Proposition 1 is important.

Corollary 2. *An endomorphism with only one fixed point is an idempotent if and only if it is a constant.*

More generally follows Corollary 3.

Corollary 3. *An endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ with s fixed points $k_1, \ldots, k_s$, $1 \le s \le n-1$, is an idempotent if and only if* $\mathrm{Im}(\alpha) = \{k_1, \ldots, k_s\}$.

Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ have just $n-1$ fixed points. Then $\alpha \neq \mathbf{i}$ and $n-1 \le |\mathrm{Im}(\alpha)| < n$. So, $|\mathrm{Im}(\alpha)| = n-1$ and from Corollary 3 follows.

Corollary 4. *Every endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ with $n-1$ fixed points is an idempotent.*

Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ have just $n-2$ fixed points. Let $\alpha(k) \neq k$ and $\alpha(\ell) \neq \ell$, where $k, \ell \in \mathscr{C}_n$. If we assume that k and ℓ are not consecutive, then $\alpha(k) = k-1$ or $\alpha(k) = k+1$. Similarly, $\alpha(\ell) = \ell - 1$ or $\alpha(\ell) = \ell + 1$. Since $k-1$, $k+1$, $\ell-1$, and $\ell+1$ are fixed points, from Proposition 1 it follows that α is an idempotent. Let us assume that $\ell = k+1$. If

(1) $\alpha(k) = k-1$, $\alpha(k+1) = k-1$,

(2) $\alpha(k) = k-1$, $\alpha(k+1) = k+2$,

(3) $\alpha(k) = k+2$, $\alpha(k+1) = k+2$,

then it easily follows that α is an idempotent. Let us assume that $\alpha(k) = k+1$ and $\alpha(k+1) = k+2$. Then from Proposition 1 it follows that α is not an idempotent endomorphism. But it is clear that $\alpha^2(k) = k+2 = \alpha^3(k)$ and $\alpha^2(k+1) = k+2 = \alpha^3(k+1)$.

Thus we prove the following.

Corollary 5. *Every endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ with $n-2$ fixed points is either an idempotent or $\alpha^3 = \alpha^2$.*

By similar reasonings, we may prove the following more general fact.

Corollary 6. *Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ have s fixed points $k_1, \ldots, k_s$, where $[(n+1)/2] \le s \le n-1$ and all other two points k and ℓ, $k < \ell$, are not consecutive; that is $k \neq k_i$, $\ell \neq k_i$, where $i = 1, \ldots, s$, and $\ell \neq k+1$. Then α is an idempotent.*

Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ have just two fixed points k and ℓ, where $k < \ell$. From Corollary 3 it follows that

$$\alpha = \wr k, \ldots, k, i_{k+1}, \ldots, i_{\ell-1}, \ell, \ldots, \ell \wr, \tag{2}$$

where images i_s, $s = k+1, \ldots, \ell-1$, are equal either to k or to ℓ.

Thus the endomorphisms of such type are

$$\begin{aligned}
\alpha_0 &= \wr k, \ldots, \underset{\substack{\uparrow\\ k}}{k}, \ldots, k, \underset{\substack{\uparrow\\ \ell}}{\ell}, \ldots, \ell \wr \\
&\ldots \\
\alpha_{\ell-k-2} &= \wr k, \ldots, \underset{\substack{\uparrow\\ k}}{k}, k, \ell \ldots, \underset{\substack{\uparrow\\ \ell}}{\ell}, \ldots \ell \wr \\
\alpha_{\ell-k-1} &= \wr k, \ldots, \underset{\substack{\uparrow\\ k}}{k}, \ell \ldots, \underset{\substack{\uparrow\\ \ell}}{\ell}, \ldots, \ell \wr .
\end{aligned} \tag{3}$$

For these maps we have

$$\alpha_0 < \cdots < \alpha_{l-k-2} < \alpha_{l-k-1}. \tag{4}$$

Hence $\alpha_i + \alpha_j = \alpha_j + \alpha_i = \alpha_j$, where $j \geq i$. It is easy to see that $\alpha_i \cdot \alpha_j = \alpha_i$ for all $i, j \in \{0, \ldots l-k-2\}$. So, we prove the following.

Proposition 7. *The set of idempotent endomorphisms with two fixed points k and ℓ, $k < \ell$, is a semiring of order $l - k$.*

Example 8. The idempotent endomorphisms of semiring $\widehat{\mathscr{E}}_{\mathscr{C}_7}$ with fixed points 1 and 5 are

$$\begin{aligned}
\varphi_1 &= \wr 1\ 1\ 1\ 1\ 1\ 5\ 5 \wr, \\
\varphi_2 &= \wr 1\ 1\ 1\ 1\ 5\ 5\ 5 \wr, \\
\varphi_3 &= \wr 1\ 1\ 1\ 5\ 5\ 5\ 5 \wr, \\
\varphi_4 &= \wr 1\ 1\ 5\ 5\ 5\ 5\ 5 \wr .
\end{aligned} \tag{5}$$

The semiring consisting of these maps has the following addition and multiplication tables:

$+$	φ_1	φ_2	φ_3	φ_4
φ_1	φ_1	φ_2	φ_3	φ_4
φ_2	φ_2	φ_2	φ_3	φ_4
φ_3	φ_3	φ_3	φ_3	φ_4
φ_4	φ_4	φ_4	φ_4	φ_4

$\cdot$	φ_1	φ_2	φ_3	φ_4
φ_1	φ_1	φ_1	φ_1	φ_1
φ_2	φ_2	φ_2	φ_2	φ_2
φ_3	φ_3	φ_3	φ_3	φ_3
φ_4	φ_4	φ_4	φ_4	φ_4

(6)

Theorem 9. *The subset of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$, $n \geq 3$, of all idempotent endomorphisms with s fixed points $k_1, \ldots, k_s$, $1 \leq s \leq n-1$, is a semiring of order $\prod_{m=1}^{s-1}(k_{m+1} - k_m)$.*

Proof. Let k_1 be the least fixed point of idempotent endomorphism α. If $k < k_1$, then $\alpha(k) \leq k_1$. We assume that $\alpha(k) = j < k_1$. Then $\alpha(j) = j$ is a contradiction to the minimal choice of k_1. Hence $\alpha(k) = k_1$, that is in the first $k_1 + 1$ positions of the ordered n-tuple which represent endomorphism α occurs only k_1.

Let k_s be the biggest fixed point of idempotent endomorphism α. If $k > k_s$, then $\alpha(k) \geq k_s$. We assume that $\alpha(k) = j > k_s$.

Now follows $\alpha(j) = j$ which is a contradiction to the maximal choice of k_s. Hence $\alpha(k) = k_s$, that is the last $n - k_s$ positions of the ordered n-tuple which represent endomorphism α occurs only k_s.

We denote by p_{k_m}, where $1 \leq m \leq s$, the number of coordinates in the ordered n-tuple which represents endomorphism α equal to k_m. For p_{k_m} it follows

$$\begin{gathered}
k_1 + 1 \leq p_{k_1} \leq k_2 \\
1 \leq p_{k_2} \leq k_3 - k_2 \\
\vdots \\
1 \leq p_{k_m} \leq k_{m+1} - k_m \\
\vdots \\
n - k_s \leq p_{k_s} \leq n - k_{s-1} - 1.
\end{gathered} \tag{7}$$

So every idempotent endomorphism with s fixed points $k_1, \ldots, k_s$ has the following type:

$$\alpha = \wr \underbrace{k_1, \ldots, k_1}_{p_{k_1}}, \underbrace{k_2, \ldots, k_2}_{p_{k_2}}, \ldots, \underbrace{k_s, \ldots, k_s}_{p_{k_s}} \wr . \tag{8}$$

Let another endomorphism β have the same type:

$$\beta = \wr \underbrace{k_1, \ldots, k_1}_{p'_{k_1}}, \underbrace{k_2, \ldots, k_2}_{p'_{k_2}}, \ldots, \underbrace{k_s, \ldots, k_s}_{p'_{k_s}} \wr . \tag{9}$$

Since $(\alpha + \beta)(k_i) = \alpha(k_i) + \beta(k_i) = k_i + k_i = k_i$, where $i = 1, \ldots, s$, it follows that $\alpha + \beta$ has the type of α and β.

For these α and β it follows that $\alpha \cdot \beta = \alpha$. (So, the multiplication of idempotent endomorphisms with fixed points $k_1, \ldots, k_s$ is like the multiplication of constant endomorphisms—every idempotent is a right identity).

Thus we prove that the set of all idempotent endomorphisms with s fixed points $k_1, \ldots, k_s$, $1 \leq s \leq n-1$, is a semiring.

From the inequalities of numbers p_{k_m}, where $1 \leq m \leq s$, it follows that all possibilities to have k_m in n-tuple are $k_{m+1} - k_m$.

The order of this semiring is equal to the product of all possibilities for the fixed points k_m, where $1 \leq m \leq s$. This number is equal to product $\prod_{m=1}^{s-1}(k_{m+1} - k_m)$, and the proof is completed. □

The semiring of the idempotent endomorphisms of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ with s fixed points $k_1, \ldots, k_s$ is denoted by $\mathscr{ID}(k_1, \ldots, k_s)$.

Since the semiring of all the endomorphisms with fixed points $k_1, \ldots, k_s$ is $\bigcap_{r=1}^{s} \mathscr{E}_{\mathscr{C}_n}^{(k_r)}$, it follows that $\mathscr{ID}(k_1, \ldots, k_s)$ is a subsemiring of $\bigcap_{r=1}^{s} \mathscr{E}_{\mathscr{C}_n}^{(k_r)}$.

Remark 10. (a) From Theorem 9 semiring $\mathscr{ID}(0, \ldots, k-1, k+1, \ldots, n-1)$ has two elements: idempotent endomorphisms $k^+ = \wr 0, \ldots, k-1, k+1, k+1, \ldots n-1 \wr$ and $k^- = \wr 0, \ldots, k-1, k-1, k+1, \ldots n-1 \wr$. This semiring has the following addition and multiplication tables:

$$\begin{array}{c|cc} + & k^- & k^+ \\ \hline k^- & k^- & k^+ \\ k^+ & k^+ & k^+ \end{array} \qquad \begin{array}{c|cc} \cdot & k^- & k^+ \\ \hline k^- & k^- & k^- \\ k^+ & k^+ & k^+ \end{array} \tag{10}$$

(b) The product of endomorphisms from different semirings $\mathscr{ID}(k_1, \ldots, k_s)$ is not, in general, an idempotent. Indeed, in semiring $\widehat{\mathscr{E}}_{\mathscr{C}_8}$ for

$$\begin{aligned} \alpha &= \wr 1\ 1\ 5\ 5\ 5\ 5\ 5\ 5 \wr \in \mathscr{ID}(1,5), \\ g &= \wr 2\ 2\ 2\ 5\ 5\ 5\ 5\ 5 \wr \in \mathscr{ID}(2,5) \end{aligned} \tag{11}$$

it follows that $\alpha \cdot \beta = \wr 2\ 2\ 5\ 5\ 5\ 5\ 5\ 5 \wr$, but this endomorphism is not an idempotent.

(c) The sum of two idempotent endomorphisms such that the fixed points of the first one are part of the fixed points of another endomorphism is not surely an idempotent. For instance, $\wr 0\ 0\ 3\ 3 \wr + \wr 0\ 2\ 2\ 3 \wr = \wr 0\ 2\ 3\ 3 \wr$.

Corollary 11. *Let $n \geq 3$ and $k_1, \ldots, k_s \in \mathscr{C}_n$, where $s = 1, \ldots, n-2$. Semiring $\mathscr{ID}(k_1, \ldots, k_s)$ is an ideal of $\bigcap_{r=1}^{s} \mathscr{E}_{\mathscr{C}_n}^{(k_r)}$.*

Proof. Using Theorem 9, it will be enough to show that $\mathscr{ID}(k_1, \ldots, k_s)$ is closed under the left and right multiplications by elements of $\bigcap_{r=1}^{s} \mathscr{E}_{\mathscr{C}_n}^{(k_r)}$.

Let

$$\begin{aligned} \alpha &= \wr \underbrace{k_1, \ldots, k_1}_{p_{k_1}}, \underbrace{k_2, \ldots, k_2}_{p_{k_2}}, \ldots, \\ &\quad \underbrace{k_s, \ldots, k_s}_{p_{k_s}} \wr \in \mathscr{ID}(k_1, \ldots, k_s), \\ \beta &= \wr i_0, i_1, \ldots, i_{k_1-1}, k_1, i_{k_1+1}, \ldots, \\ &\quad i_{k_2-1}, k_2, \ldots, k_s \wr \in \bigcap_{r=1}^{s} \mathscr{E}_{\mathscr{C}_n}^{(k_r)}. \end{aligned} \tag{12}$$

Hence $\alpha \cdot \beta = \alpha$.

Now we calculate

$$\begin{aligned} \beta \cdot \alpha &= \wr i_0, i_1, \ldots, i_{k_1-1}, k_1, \\ &\quad i_{k_1+1}, \ldots, i_{k_2-1}, k_2, \ldots, k_s \wr \\ &\quad \cdot \wr k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_s, \ldots, k_s \wr . \end{aligned} \tag{13}$$

Since $i_s \leq k_1$, where $s = 0, \ldots, k_1 - 1$, then $\alpha(i_s) \leq \alpha(k_1) = k_1$, and from the proof of Theorem 9 it follows that $\alpha(i_s) = k_1$.

Since $k_1 \leq i_{k_1+1} \leq \cdots \leq i_{k_2-1} \leq k_2$, it follows that

$$\begin{aligned} k_1 = \alpha(k_1) &\leq \alpha(i_{k_1+1}) \leq \cdots \\ &\leq \alpha(i_{k_2-1}) \leq \alpha(k_2) = k_2. \end{aligned} \tag{14}$$

But endomorphism α maps all the elements between k_1 and k_2 either in k_1, or in k_2. Then $\alpha(i_{k_1+1}), \ldots, \alpha(i_{k_2-1})$ are either k_1, or k_2. We use such arguments for the next elements of $\mathrm{Im}(\beta)$ which are between the other fixed points. Thus

$$\begin{aligned} \beta \cdot \alpha = \gamma = \wr &\underbrace{k_1, \ldots, k_1}_{p'_{k_1}}, \underbrace{k_2, \ldots, k_2}_{p'_{k_2}}, \ldots, \\ &\underbrace{k_s, \ldots, k_s}_{p'_{k_s}} \wr \in \mathscr{ID}(k_1, \ldots, k_s). \end{aligned} \tag{15}$$

□

4. Roots of Idempotent Endomorphisms

Let $(S, \cdot)$ be a finite semigroup. It is well known, see [10], that for any $x \in S$ there is a positive integer $k = k(x)$ such that a^k is an idempotent element of S.

Now we consider the following relation: for any $x, y \in S$ we define

$$\begin{aligned} x \sim y &\Longleftrightarrow \exists\, k, \quad m \in \mathbf{N}, \\ x^k &= y^m = e, \end{aligned} \tag{16}$$

where e is an idempotent element of S.

Obviously, the relation $\sim$ is reflexive and symmetric.

Let $x \sim y$ and $y \sim z$. Then $x^k = y^m = e_1$ and $y^r = z^s = e_2$ where e_1 and e_2 are idempotents. Now it follows that $(y^m)^r = e_1^r = e_1$ and $(y^r)^m = e_2^m = e_2$; thus $e_1 = e_2$. Hence $x^k = z^s = e_1$; that is, $x \sim z$. So, we prove that $\sim$ is an equivalence relation on S.

Note that two different idempotents belong to different equivalence classes modulo $\sim$. If e is an idempotent, the elements of the equivalence class containing e are called *roots of idempotent e*.

The following natural question arises: *are there any finite noncommutative semigroups such that the equivalence classes (modulo the previous relation) are semigroups?*

Of course, we must avoid trivial examples of commutative semigroups, nilpotent semigroups, and others.

To answer the previous question, we consider the semigroup $(\widehat{\mathscr{E}}_{\mathscr{C}_n}, \cdot)$ and the already defined equivalence relation.

Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$. Element $\mathbf{j} \in \mathscr{C}_n$ is called *a jump point* of α if $\mathbf{j} \neq 0$, and one of the following conditions holds:

(1) $\alpha(\mathbf{j} - 1) \leq \mathbf{j} - 1$ and $\alpha(\mathbf{j}) > \mathbf{j}$,

(2) $\alpha(\mathbf{j} - 1) < \mathbf{j} - 1$ and $\alpha(\mathbf{j}) \geq \mathbf{j}$.

There are endomorphisms without jump points, namely, identity $\mathbf{i}$ and constant endomorphisms $\kappa_k = \wr k\ k\ \cdots\ k\wr$, $k = 0, 1, \ldots, n-1$. For $k, \ell \in \mathscr{C}_n$, $k < \ell$ endomorphism

$$\alpha_j(i) = \begin{cases} k, & i \le j-1, \\ \ell, & i \ge j \end{cases} \tag{17}$$

has no jump points if $j > \ell$. The endomorphisms k^+ and k^-, see Remark 10, have jump points: k and $k+1$ are jump points of k^+ and k^-, respectively.

Theorem 12. *Let $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$, $n \ge 3$, be an endomorphism with s fixed points $k_1, \ldots, k_s$, $1 \le s \le n-2$. Let for some i, $i = 1, \ldots, s-1$, fixed points k_i and k_{i+1} be non consecutive; that is, $k_{i+1} \ne k_i + 1$. Then there is a unique jump point $\mathbf{j}_i$ of α such that $k_i + 1 \le \mathbf{j}_i \le k_{i+1}$.*

Proof. For $k_i + 1$ there are two possibilities:

(1) $\alpha(k_i+1) > k_i + 1$, then $\mathbf{j}_i = k_i + 1$ is the searched jump point of α, or

(2) $\alpha(k_i+1) < k_i + 1$.

Let $\alpha(k_i + m) < k_i + m$ for any $m = 1, 2, \ldots, \ell - 1$. If $\alpha(k_i + \ell) \ge k_i + \ell$, then $\mathbf{j}_i = k_i + \ell$ is the searched jump point of α.

We assume that $\alpha(k_i + m) < k_i + m$ for all $m = 1, 2, \ldots, k_{i+1} - k_i - 1$. Now $\alpha(k_{i+1} - 1) < k_{i+1} - 1$. But $\alpha(k_{i+1}) = k_{i+1}$, so, $\mathbf{j}_i = k_{i+1}$ is the searched jump point of α.

Suppose that $\mathbf{j}$ and $\mathbf{j}^\circ$ are two jump points of α such that $k_i + 1 \le \mathbf{j} \le k_{i+1}$ and $k_i + 1 \le \mathbf{j}^\circ \le k_{i+1}$. Let $\mathbf{j} < \mathbf{j}^\circ$. Now $\alpha(\mathbf{j}) > \mathbf{j}$. Let $m \in \mathscr{C}_n$, and $\mathbf{j} \le m < \mathbf{j}^\circ$ is the maximal element such that $\alpha(m) > m$. Then $\alpha(m+1) < m+1$, and it follows that $\alpha(m) \ge m + 1 > \alpha(m+1)$ which is a contradiction. By the same arguments we show that $\mathbf{j}^\circ < \mathbf{j}$ is impossible. So $\mathbf{j} = \mathbf{j}^\circ$ is the unique jump point of α such that $k_i + 1 \le \mathbf{j} \le k_{i+1}$. □

Let k_1 be the least fixed point of endomorphism α and $i \in \mathscr{C}_n$, $i < k_1$. If we suppose that $\alpha(i) < i$, using $\alpha(0) > 0$, it follows that for some maximal element m, where $0 \le m < i$, we have $\alpha(m) > m$ and then $\alpha(m+1) < m+1$, which is impossible. So, for every $i < k_1$ it follows that $\alpha(i) > i$. Similarly for every $i > k_s$, where k_s is the biggest fixed point of α, it follows that $\alpha(i) < i$. So, we prove the following.

Corollary 13. *Every endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$, $n \ge 3$, with just ℓ fixed points $k_1, \ldots, k_\ell$, which are not consecutive, that is, $k_{i+1} \ne k_i + 1$ for $i = 1, \ldots, \ell - 1$, has just $\ell - 1$ jump points $\mathbf{j}_i$ such that $k_i + 1 \le \mathbf{j}_i \le k_{i+1}$.*

Remark 14. From Theorem 9 it follows that the number of idempotents with fixed points $k_1, \ldots, k_s$ is a semiring of order $\prod_{i=1}^{s-1}(k_{i+1} - k_i)$. But if two fixed points k_i and k_{i+1} are consecutive, the difference $k_{i+1} - k_i$ is equal to 1, and they do not appear in the product. So, the order of the semiring is equal to $\prod_{i=1}^{\ell-1}(k_{i+1} - k_i)$, where $k_1, \ldots, k_\ell$ are (after suitable renumbering) non consecutive fixed points. But for any i the difference $k_{i+1} - k_i$ is just the number of possible jump points $\mathbf{j}_{i,t} = k_i + t$, where $t = 1, \ldots, k_{i+1} - k_i$. So, for given fixed points $k_1, \ldots, k_s$ the order of semiring of idempotents is equal to the number of all $\ell - 1$-tuples $(\mathbf{j}_{1,t_1}, \mathbf{j}_{2,t_2}, \ldots, \mathbf{j}_{\ell-1,t_{\ell-1}})$, where $\mathbf{j}_{i,t_i} = k_i + t_i$ for $t_i = 1, \ldots, k_{i+1} - k_i$ and $i = 1, \ldots, \ell - 1$. Hence, any $\ell - 1$-tuple $(\mathbf{j}_{1,t_1}, \ldots, \mathbf{j}_{\ell-1,t_{\ell-1}})$ defines just one idempotent from the semiring considered.

To describe precisely all the fixed points of an arbitrary endomorphism $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ we will use new indices. Let first fixed point of α be $k_{1,1}$ and let some fixed points after $k_{1,1}$ be consecutive; that is, $k_{1,2}, \ldots, k_{1,m_1}$ are fixed points such that $k_{1,i+1} - k_{1,i} = 1$ where $i = 1, \ldots, m_1 - 1$. Let the next fixed point be $k_{2,1}$ such that $k_{2,1} - k_{1,m_1} > 1$. So we construct the first pair of two fixed points which are not consecutive. Let the following fixed points be $k_{2,2}, \ldots, k_{2,m_2}$ such that $k_{2,i+1} - k_{2,i} = 1$ where $i = 1, \ldots, m_2 - 1$. The next fixed point is $k_{3,1}$ and $k_{3,1} - k_{2,m_2} > 1$. Let the last pair of two non consecutive fixed points be $k_{\ell-1,m_{\ell-1}}$ and $k_{\ell,1}$. Then the last fixed points are $k_{\ell,2}, \ldots k_{\ell,m_\ell}$ such that $k_{\ell,i+1} - k_{\ell,i} = 1$ where $i = 1, \ldots, m_\ell - 1$. So, we construct a partition of a set of fixed points of α such that we can distinguish the fixed points which are not consecutive.

Let $\mathbf{j}_{i,t_i}$ be the jump points of α such that $\mathbf{j}_{i,t_i} = k_{i,m_i} + t_i$, where $t_i = 1, \ldots, k_{i+1,1} - k_{i,m_i}$ and $i = 1, \ldots, \ell - 1$.

An endomorphism α with fixed points $k_{1,1}, \ldots, k_{\ell,m_\ell}$ and jump points $\mathbf{j}_{i,t_i}$ from the previous definitions is called *a endomorphism of type*

$$\begin{aligned} &[k_{1,1}, \ldots, k_{1,m_1}, \mathbf{j}_{1,t_1}, k_{2,1}, \ldots, k_{\ell-1,m_{\ell-1}}, \\ &\quad \mathbf{j}_{\ell-1,t_{\ell-1}}, k_{\ell,1}, \ldots, k_{\ell,m_\ell}]. \end{aligned} \tag{18}$$

Let us consider endomorphism ε of this type such that

(i) $\varepsilon(x) = k_{1,1}$ for any $0 \le x \le k_{1,1}$,

(ii) $\varepsilon(x) = k_{i,m_i}$ for any $k_{i,m_i} \le x \le \mathbf{j}_{i,t_i} - 1$, where $i = 1, \ldots, \ell - 1$,

(iii) $\varepsilon(x) = k_{i+1,1}$ for any $\mathbf{j}_{i,t_i} \le x \le k_{i+1,1}$, where $i = 1, \ldots, \ell - 1$,

(iv) $\varepsilon(x) = k_{\ell,m_\ell}$ for any $k_{\ell,m_\ell} \le x \le n - 1$.

Now it is easy to show that this endomorphism is an idempotent.

Let $\overline{\varepsilon}$ be another idempotent of the same type. Then from the reasonings just before Corollary 13 it follows that

(i) $\overline{\varepsilon}(x) = k_{1,1}$ for any $0 \le x \le k_{1,1}$,

(ii) $\overline{\varepsilon}(x) = k_{\ell,m_\ell}$ for any $k_{\ell,m_\ell} \le x \le n - 1$.

Since $\overline{\varepsilon}$ is an idempotent, we conclude that for some x, where $k_{i,m_i} \le x \le k_{i+1,1}$, it follows either $\overline{\varepsilon}(x) = k_{i,m_i}$, or $\overline{\varepsilon}(x) = k_{i+1,1}$. By using that $\overline{\varepsilon}$ is of type (18) it follows that $\overline{\varepsilon}(x) = \varepsilon(x)$ for all x, where $k_{i,m_i} \le x \le k_{i+1,1}$ and $i = 1, \ldots, \ell - 1$.

Hence, there is only one idempotent of the given type (18). This endomorphism is

$$\varepsilon = \wr k_{1,1}, \ldots, \underset{\substack{\uparrow \\ k_{1,1}}}{k_{1,1}}, \ldots, \underset{\substack{\uparrow \\ k_{1,m_1}}}{k_{1,m_1}}, \ldots, k_{1,m_1}, \underset{\substack{\uparrow \\ \mathbf{j}_{1,t_1}}}{k_{2,1}}, \ldots, \tag{19}$$

$$k_{\ell-1,m_{\ell-1}}, \underset{\substack{\uparrow \\ \mathbf{j}_{\ell-1,t_{\ell-1}}}}{k_{\ell,1}}, \ldots, \underset{\substack{\uparrow \\ k_{\ell,m_\ell}}}{k_{\ell,m_\ell}}, \ldots, k_{\ell,m_\ell} \wr .$$

Lemma 15. *Let $\varepsilon \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ be an idempotent endomorphism. If $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ is a root of ε, then both endomorphisms α and ε have the same fixed points.*

Proof. Let k be a fixed point of ε. If $\alpha(k) < k$, then $\alpha^2(k) \leq \alpha(k) < k$, and by the same arguments $\alpha^m(k) < k$ for arbitrary natural number m, but this is a contradiction to $\alpha^m = \varepsilon$ for some $m \in \mathbf{N}$. Analogously, assuming that $\alpha(k) > k$, we obtain a contradiction to equality $\alpha^m = \varepsilon$. Hence $\alpha(k) = k$.

Let k be a fixed point of α. Since it is a fixed point of α^m for arbitrary natural m, it follows that it is a fixed point of ε. □

Lemma 16. *Let $\varepsilon \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ be an idempotent endomorphism. If $\alpha \in \widehat{\mathscr{E}}_{\mathscr{C}_n}$ is a root of ε, then both endomorphisms α and ε have the same jump points.*

Proof. Let $\mathbf{j}$ be a jump point of ε and $\varepsilon(\mathbf{j}-1) \leq \mathbf{j}-1$, $\varepsilon(\mathbf{j}) > \mathbf{j}$. If we assume that $\alpha(\mathbf{j}-1) > \mathbf{j}-1$, then $\alpha^m(\mathbf{j}-1) > \mathbf{j}-1$ for arbitrary natural m which is a contradiction to $\alpha^m = \varepsilon$ for some $m \in \mathbf{N}$.

Analogously, if we assume that $\alpha(\mathbf{j}) \leq \mathbf{j}$, it follows that $\alpha^m(\mathbf{j}) \leq \mathbf{j}$ for arbitrary natural m which is also a contradiction. Hence $\mathbf{j}$ is a jump point of α such that $\alpha(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\alpha(\mathbf{j}) > \mathbf{j}$. From similar reasonings, it follows that if $\mathbf{j}$ is a jump point of ε such that $\varepsilon(\mathbf{j}-1) < \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) \geq \mathbf{j}$ for every endomorphism α, which is a root of ε, we have that $\mathbf{j}$ is a jump point of α, $\alpha(\mathbf{j}-1) < \mathbf{j}-1$, and $\alpha(\mathbf{j}) \geq \mathbf{j}$.

Let $\mathbf{j}$ be a jump point of α such that $\alpha(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\alpha(\mathbf{j}) > \mathbf{j}$. Then for arbitrary natural m it follows $\alpha^m(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\alpha^m(\mathbf{j}) > \mathbf{j}$. So, $\mathbf{j}$ is a jump point of ε, $\varepsilon(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) > \mathbf{j}$. Analogously, $\alpha(\mathbf{j}-1) < \mathbf{j}-1$ and $\alpha(\mathbf{j}) \geq \mathbf{j}$ imply $\alpha^m(\mathbf{j}-1) < \mathbf{j}-1$ and $\alpha^m(\mathbf{j}) \geq \mathbf{j}$ for every natural number m, so $\mathbf{j}$ is a jump point of ε, $\varepsilon(\mathbf{j}-1) < \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) \geq \mathbf{j}$. □

Immediately from Lemmas 15 and 16 follows Proposition 17.

Proposition 17. *All the endomorphisms of one equivalence class modulo $\sim$ are of the same type.*

Let the idempotent endomorphism ε from (19) be an element of equivalence class E. Then E is called an equivalence class of type (18).

Lemma 18. *Let E be an equivalence class of type* (18). *Then any $\alpha \in E$ satisfies the following conditions:*

(A) *$\alpha(x) > x$, where $0 \leq x < k_{1,1}$, $\mathbf{j}_{1,t_1} \leq x < k_{2,1}$, $\mathbf{j}_{2,t_2} \leq x < k_{3,1}, \ldots$, or $\mathbf{j}_{\ell-1,t_{\ell-1}} \leq x < k_{\ell,1}$;*

(B) *$\alpha(x) < x$, where $k_{1,m_1} < x < \mathbf{j}_{1,t_1}$, $k_{2,m_2} < x < \mathbf{j}_{2,t_2}, \ldots$, $k_{\ell-1,m_{\ell-1}} < x < \mathbf{j}_{\ell-1,t_{\ell-1}}$, or $k_{\ell,m_\ell} < x \leq n-1$.*

Proof. We assume that for some x such that

$$\begin{aligned} &0 \leq x < k_{1,1}, \quad \text{or} \quad \mathbf{j}_{1,t_1} \leq x < k_{2,1}, \\ &\text{or} \quad \mathbf{j}_{2,t_2} \leq x < k_{3,1}, \ldots, \\ &\text{or} \quad \mathbf{j}_{\ell-1,t_{\ell-1}} \leq x < k_{\ell,1} \end{aligned} \tag{20}$$

it follows that $\alpha(x) \leq x$. Then $\alpha^2(x) \leq \alpha(x) \leq x$ and by induction $\alpha^m(x) \leq x$ for arbitrary natural m. It is a contradiction to the assumption that $\alpha^m = \varepsilon$ for some natural m, where ε has the form (19). Thus we prove that $\alpha(x) > x$.

Analogously, we assume that for some x such that

$$\begin{aligned} &k_{1,m_1} < x < \mathbf{j}_{1,t_1}, \\ &\text{or} \quad k_{2,m_2} < x < \mathbf{j}_{2,t_2}, \ldots, \\ &\text{or} \quad k_{\ell-1,m_{\ell-1}} < x < \mathbf{j}_{\ell-1,t_{\ell-1}}, \\ &\text{or} \quad k_{\ell,m_\ell} < x \leq n-1 \end{aligned} \tag{21}$$

it follows $\alpha(x) \geq x$. Then by induction $\alpha^m(x) \geq x$ for arbitrary natural m. This contradicts the assumption that $\alpha^m = \varepsilon$ for some natural m, where ε has the form (19). Thus we prove that $\alpha(x) < x$. □

Note that if two endomorphisms have the same fixed points but different jump points, their product can posses a new fixed point. For instance, endomorphisms $\alpha = \wr 2,2,2,4,5,5,5 \wr \sim \varepsilon_1 = \wr 2,2,2,5,5,5,5 \wr$ and $\beta = \wr 2,2,2,2,3,5,5 \wr \sim \varepsilon_2 = \wr 2,2,2,2,2,5,5 \wr$ have fixed points 2 and 5 but $\alpha \cdot \beta = \wr 2,2,2,3,5,5,5 \wr$ and $\beta \cdot \alpha = \wr 2,2,2,2,4,5,5 \wr$ have new fixed points 3 and 4, respectively.

Let us remind that the jump points, which we use in (18), are defined by equality $\mathbf{j}_{i,t_i} = k_{i,m_i} + t_i$. So it follows that $\mathbf{j}_{i,t_i} - k_{i,m_i} = t_i$. Now we denote $k_{i+1,1} - \mathbf{j}_{i,t_i} = s_i$. The indices t_i and s_i will be used in the next theorem.

Theorem 19. *Every equivalence class modulo $\sim$ of type* (18) *is a subsemiring of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$, $n \geq 2$. The order of this semiring is*

$$C_{k_{1,1}} \left(\prod_{i=1}^{\ell-1} C_{t_i} C_{s_i} \right) C_{n-1-k_{\ell,m_\ell}}, \tag{22}$$

where C_p is the pth Catalan number.

Proof. Let E be an equivalence class of type (18) and $\alpha, \beta \in E$. Using Lemma 18 it follows that

(A) $\alpha(x) > x$ and $\beta(x) > x$, where $0 \leq x < k_{1,1}$, $\mathbf{j}_{1,t_1} \leq x < k_{2,1}$, $\mathbf{j}_{2,t_2} \leq x < k_{3,1}, \ldots$, or $\mathbf{j}_{\ell-1,t_{\ell-1}} \leq x < k_{\ell,1}$. Then $(\alpha+\beta)(x) = \alpha(x) \vee \beta(x) > x \vee x = x$ and $(\alpha \cdot \beta)(x) = \beta(\alpha(x)) \geq \beta(x) > x$.

(B) $\alpha(x) < x$ and $\beta(x) < x$, where $k_{1,m_1} < x < \mathbf{j}_{1,t_1}$, $k_{2,m_2} < x < \mathbf{j}_{2,t_2}, \ldots$, or $k_{\ell-1,m_{\ell-1}} < x < \mathbf{j}_{\ell-1,t_{\ell-1}}$, or $k_{\ell,m_\ell} < x \leq n-1$. Then $(\alpha+\beta)(x) = \alpha(x) \vee \beta(x) < x \vee x = x$ and $(\alpha \cdot \beta)(x) = \beta(\alpha(x)) \leq \beta(x) < x$.

Thus we prove that equivalence class E is a subsemiring of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$.

For the second part of the proof of the theorem we needed two combinatorial lemmas.

Lemma 20. *The number of the ordered p-tuples $(i_0, \ldots, i_{p-1})$, where*

(1) $i_r \in \{0, \ldots, p-1\}$ *for* $r = 0, \ldots, p-1$,

(2) $i_r \leq i_{r+1}$ *for* $r = 0, \ldots, p-1$,

(3) $i_r < r$ *for* $r = 1, \ldots, p-1$,

is the pth Catalan number $C_p = (1/p)\binom{2p-1}{p-1}$.

The proof of this lemma is a part of the proof of Proposition 3.4 of [7]. This result (using the so-called Dyck paths) and many other applications of Catalan numbers can be found in [11].

Lemma 21. *The number of the ordered p-tuples $(i_0, \ldots, i_{p-1})$, where*

(1) $i_r \in \{1, \ldots, p\}$ *for* $r = 0, \ldots, p-1$, $i_p = p$,

(2) $i_r \leq i_{r+1}$ *for* $r = 0, \ldots, p-1$,

(3) $i_r > r$ *for* $r = 0, \ldots, p-1$

is the pth Catalan number $C_p = (1/p)\binom{2p-1}{p-1}$.

Proof of Lemma 21. Let $(i_0, \ldots, i_{p-1})$ be the ordered p-tuple. Since $i_r > r$ for any $r = 0, \ldots, p-1$, follows that $p = i_p \geq i_{p-1} > p-1$; that is, $i_{p-1} = p$. Instead of the p-tuple $(i_0, \ldots, i_{p-1})$ we consider the p-tuple $(j_0, \ldots, j_{p-1})$, where $j_m = p - i_{p-m-1}$, $m = 0, \ldots, p-1$. Then $j_m < m$ and $j_m \leq j_{m+1}$. From Lemma 20 it follows that the number of all k-tuples of this kind is C_k.

Now we continue the proof of the theorem.

For all the intervals considered in part (A) (from the beginning of this proof) we apply Lemma 21. Then, for all the intervals considered in part (B) we apply Lemma 20. Hence we find that the number of the endomorphisms in the equivalence class modulo $\sim$ of type (1) is $C_{k_{1,1}}(\prod_{i=1}^{\ell-1} C_{t_i} C_{s_i}) C_{n-1-k_{\ell,m_\ell}}$. □

Remark 22. The equivalence relation $\sim$ is not a congruence on $\widehat{\mathscr{E}}_{\mathscr{C}_n}$. For $n = 8$ we consider the endomorphisms

$$\begin{aligned}
\alpha &= \wr 2,2,2,2,2,6,7,7\wr \sim \varepsilon_1 = \wr 2,2,2,2,2,7,7,7\wr,\\
\beta &= \wr 2,2,2,2,3,7,7,7\wr \sim \varepsilon_1 = \wr 2,2,2,2,2,7,7,7\wr,\\
\gamma &= \wr 1,1,1,3,3,4,5,6\wr \sim \varepsilon_2 = \wr 1,1,1,3,3,3,3,3\wr .
\end{aligned} \tag{23}$$

Now we compute $\alpha \cdot \gamma = \wr 1,1,1,1,1,5,6,6\wr$ and $\beta \cdot \gamma = \wr 1,1,1,1,3,6,6,6\wr$. So, it follows that $\alpha \sim \beta$, but $\alpha \cdot \gamma \nsim \beta \cdot \gamma$.

5. The Crucial Role of Jump Points

Here we consider the idempotent endomorphisms with arbitrary fixed points but we assume that each endomorphism has $\mathbf{j}_1, \ldots, \mathbf{j}_r$ for jump points.

Two jump points $\mathbf{j}_s$ and $\mathbf{j}_{s+1}$ of the idempotent ε are called *consecutive* if $\mathbf{j}_{s+1} = \mathbf{j}_s + 1$. First, let us answer the question: are there any consecutive jump points of the idempotent endomorphism?

Yes, for instance, $\varepsilon = \wr 1,1,1,3,5,5\wr \in \widehat{\mathscr{E}}_{\mathscr{C}_6}$ is an idempotent and jump points 3 and 4 are consecutive. Note that 3 is also a fixed point of ε. So, we modify the question: If the first jump point is not a fixed point, is it possible fort the next point to be a jump point?

The answer is negative. Indeed, if ε is an idempotent and $\varepsilon(\mathbf{j}) = k > j$, then from Proposition 1 it follows that $\varepsilon(k) = k$, and, since $\mathbf{j} + 1 \leq k$, we have $\varepsilon(\mathbf{j}+1) \leq k$. So, $\mathbf{j}+1$ is not a jump point of ε.

Lemma 23. *Let ε be an idempotent and $\mathbf{j}_s$ and let $\mathbf{j}_{s+1}$ be non consecutive jump points of ε. Then in interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$ one of the following holds:*

(1) *ε is a constant endomorphism.*

(2) *ε is an identity.*

(3) *ε is an identity in interval $[\mathbf{j}_s, k]$ and a constant endomorphism in interval $[k, \mathbf{j}_{s+1}-1]$.*

(4) *ε is a constant endomorphism in interval $[\mathbf{j}_s, k]$, and an identity in interval $[k, \mathbf{j}_{s+1}-1]$.*

(5) *ε is a constant endomorphism in interval $[\mathbf{j}_s, k]$, an identity in interval $[k, \ell]$ and a constant endomorphism in interval $[\ell, \mathbf{j}_{s+1}-1]$.*

Proof. Let $\varepsilon(\mathbf{j}_s) = \mathbf{j}_s$. Since $\mathbf{j}_s + 1$ is not a jump point, there are two possibilities: $\varepsilon(\mathbf{j}_s+1) = \mathbf{j}_s$ or $\varepsilon(\mathbf{j}_s+1) = \mathbf{j}_s + 1$. In the first case the equality $\varepsilon(\mathbf{j}_s+2) = \mathbf{j}_s + 1$ is impossible; see Proposition 1. So, either $\varepsilon(\mathbf{j}_s+2) \geq \mathbf{j}_s + 2$ and $\mathbf{j}_s + 2$ is the next jump point or $\varepsilon(\mathbf{j}_s+2) = \mathbf{j}_s$. From these reasonings, it follows that ε is a constant endomorphism in interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$. When $\varepsilon(\mathbf{j}_s+1) = \mathbf{j}_s + 1$ for point $\mathbf{j}_s + 2$, there are three possibilities. If $\varepsilon(\mathbf{j}_s+2) > \mathbf{j}_s + 2$, then $\mathbf{j}_s + 2$ is the next jump point and ε is an identity in the interval between the considered jump points. If $\varepsilon(\mathbf{j}_s+2) = \mathbf{j}_s + 1$, by the previous reasonings, we find that ε is an identity in interval $[\mathbf{j}_s, \mathbf{j}_s+1]$ and a constant endomorphism in the second interval. Let $\varepsilon(\mathbf{j}_s+2) = \mathbf{j}_s + 2$. Now if for any $k \leq \mathbf{j}_{s+1}-1$, it follows that $\varepsilon(\mathbf{j}_s+k) = \mathbf{j}_s + k$. Then ε is an identity in the whole interval. If for some k we have $\varepsilon(\mathbf{j}_s+k+1) = \varepsilon(\mathbf{j}_s+k) = \mathbf{j}_s + k$, then ε is an identity in interval $[\mathbf{j}_s, k]$ and a constant endomorphism in interval $[k, \mathbf{j}_{s+1}-1]$.

Let $\varepsilon(\mathbf{j}_s) = k > \mathbf{j}_s$. Since ε is an idempotent, it follows that $\varepsilon(k) = k$; that is ε is a constant endomorphism in interval $[\mathbf{j}_s, k]$. If $\varepsilon(k+1) = k$ then ε is a constant endomorphism in the whole interval. Let $\varepsilon(k+1) = k+1$, and for any ℓ, where $\ell = k+1, \ldots, \mathbf{j}_{s+1}-1$, we have $\varepsilon(\ell) = \ell$. So, ε is a constant endomorphism in interval $[\mathbf{j}_s, k]$ and an identity in interval $[k, \mathbf{j}_{s+1}-1]$. In the end, if there is some ℓ, $\ell = k+1, \ldots, \mathbf{j}_{s+1}-1$, such that $\varepsilon(\ell+1) = \varepsilon(\ell) = \ell$, then ε is a constant

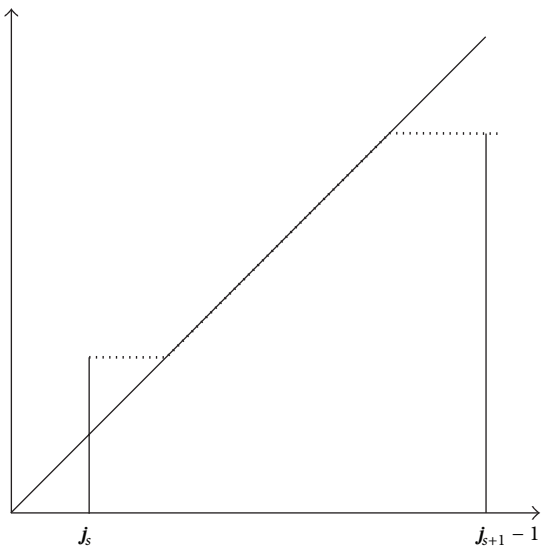

Figure 1

endomorphism in interval $[\mathbf{j}_s, k]$, an identity in interval $[k, \ell]$, and a constant endomorphism in interval $[\ell, \mathbf{j}_{s+1}-1]$. □

It is straightforward to show that if one of the conditions (1)–(5) of Lemma 23 holds for endomorphism ε, then ε is an idempotent.

From Lemma 23 it follows that the graph of the arbitrary idempotent endomorphism has the shape displayed on Figure 1. Note that each of the three segments of this graph can have length zero.

Let $\mathbf{j}_s$ and $\mathbf{j}_{s+1}$ be consecutive jump points of the idempotent ε. Then the graph of ε is a particular case of those displayed in Figure 1 containing one point $(\mathbf{j}_s, \varepsilon(\mathbf{j}_s))$, where $\varepsilon(\mathbf{j}_s) = \mathbf{j}_s$.

Lemma 24. *Let ε and $\overline{\varepsilon}$ be idempotent endomorphisms of $\widehat{\mathscr{C}}_{\mathscr{C}_n}$ with the same jump points $\mathbf{j}_1, \ldots, \mathbf{j}_r$. Then $\varepsilon + \overline{\varepsilon}$ is also an idempotent endomorphism with the same jump points.*

Proof. First we will prove that $\mathbf{j}_1, \ldots, \mathbf{j}_r$ are jump points of endomorphism $\varepsilon + \overline{\varepsilon}$. Let $\mathbf{j}$ be any common jump point of ε and $\overline{\varepsilon}$. If $\mathbf{j}$ is also a fixed point of ε and $\overline{\varepsilon}$, that is, both idempotents satisfied the inequalities $\varepsilon(\mathbf{j}-1) < \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) \geq \mathbf{j}$, $\overline{\varepsilon}(\mathbf{j}-1) < \mathbf{j}-1$ and $\overline{\varepsilon}(\mathbf{j}) \geq \mathbf{j}$, then $(\varepsilon+\overline{\varepsilon})(\mathbf{j}-1) < \mathbf{j}-1$ and $(\varepsilon+\overline{\varepsilon})(\mathbf{j}) \geq \mathbf{j}$. When for ε it follows that $\varepsilon(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) > \mathbf{j}$ or for $\overline{\varepsilon}$ it follows that $\overline{\varepsilon}(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\overline{\varepsilon}(\mathbf{j}) > \mathbf{j}$, then $(\varepsilon+\overline{\varepsilon})(\mathbf{j}-1) \leq \mathbf{j}-1$ and $(\varepsilon+\overline{\varepsilon})(\mathbf{j}) > \mathbf{j}$.

Now we will prove that sum $\varepsilon + \overline{\varepsilon}$ is an idempotent endomorphism. Let us, like in Lemma 23, fix two nonconsecutive jump points $\mathbf{j}_s$ and $\mathbf{j}_{s+1}$ of idempotents ε and $\overline{\varepsilon}$ and the interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$.

When the idempotent ε satisfies some the of conditions (1)–(5) of Lemma 23, we denote this by $\varepsilon_{(1)}, \ldots, \varepsilon_{(5)}$.

When ε satisfies (1) and $\overline{\varepsilon}$ satisfies (5), we denote this by $\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(5)}$. Let, in this case, ε be equal to constant C in interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$, and $\overline{\varepsilon}$ is consecutively equal to constant k, identity $\mathbf{i}$, and constant ℓ. If $C \leq k$ or $C \geq \ell$, then $\varepsilon + \overline{\varepsilon} = C$. If $k < C < \ell$, then $\varepsilon + \overline{\varepsilon}$ satisfy condition (5). This reasoning we denote by $\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(5)} \Rightarrow (\varepsilon+\overline{\varepsilon})_{(1)}$ or $(\varepsilon+\overline{\varepsilon})_{(5)}$.

The verification of all fifteen possibilities is easy:

$$\begin{gathered}
\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(1)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(1)}, \\
\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(2)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(1)} \\
\text{or} \quad (\varepsilon+\overline{\varepsilon})_{(2)} \quad \text{or} \quad (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(3)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(1)} \\
\text{or} \ (\varepsilon+\overline{\varepsilon})_{(3)} \quad \text{or} \quad (\varepsilon+\overline{\varepsilon})_{(5)}, \\
\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(4)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(1)} \\
\text{or} \quad (\varepsilon+\overline{\varepsilon})_{(4)} \quad \text{or} \quad (\varepsilon+\overline{\varepsilon})_{(5)}, \\
\varepsilon_{(1)} \,\&\, \overline{\varepsilon}_{(5)} \Rightarrow (\varepsilon+\overline{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon+\overline{\varepsilon})_{(5)}, \\
\varepsilon_{(2)} \,\&\, \overline{\varepsilon}_{(2)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(2)}, \\
\varepsilon_{(2)} \,\&\, \overline{\varepsilon}_{(3)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(2)}, \\
\varepsilon_{(2)} \,\&\, \overline{\varepsilon}_{(4)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(2)} \,\&\, \overline{\varepsilon}_{(5)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(3)} \,\&\, \overline{\varepsilon}_{(3)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(3)}, \\
\varepsilon_{(3)} \,\&\, \overline{\varepsilon}_{(4)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(3)} \,\&\, \overline{\varepsilon}_{(5)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(5)}, \\
\varepsilon_{(4)} \,\&\, \overline{\varepsilon}_{(4)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(4)} \,\&\, \overline{\varepsilon}_{(5)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(4)}, \\
\varepsilon_{(5)} \,\&\, \overline{\varepsilon}_{(5)} \Longrightarrow (\varepsilon+\overline{\varepsilon})_{(5)}.
\end{gathered} \tag{24}$$

Thus we prove that the sum of two idempotent endomorphisms in interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$ is also an idempotent in this interval. So, it follows that $\varepsilon + \overline{\varepsilon}$ is an idempotent endomorphism in interval $[\mathbf{j}_1, \mathbf{j}_r]$. For $0 \leq x < \mathbf{j}_1$ each of idempotents ε and $\overline{\varepsilon}$ satisfies one of conditions (1)–(5) of Lemma 23. So, from the previous reasonings it follows that sum $\varepsilon + \overline{\varepsilon}$ is an idempotent endomorphism in $[0, \mathbf{j}_1 - 1]$. In a similar way we show that $\varepsilon + \overline{\varepsilon}$ is an idempotent endomorphism in $[\mathbf{j}_r, n-1]$. Hence, $\varepsilon + \overline{\varepsilon}$ is an idempotent endomorphism in the whole chain $\mathscr{C}_n$. □

Lemma 25. *Let ε and $\overline{\varepsilon}$ be idempotent endomorphisms of $\widehat{\mathscr{C}}_{\mathscr{C}_n}$ with the same jump points $\mathbf{j}_1, \ldots, \mathbf{j}_r$. Then $\varepsilon \cdot \overline{\varepsilon}$ is also an idempotent endomorphism with the same jump points.*

Proof. First we will prove that $\mathbf{j}_1, \ldots, \mathbf{j}_r$ are the jump points of endomorphism $\varepsilon \cdot \overline{\varepsilon}$. Let $\mathbf{j}$ be any common jump point of ε and $\overline{\varepsilon}$. Let $\mathbf{j}$ be also a fixed point of ε or of $\overline{\varepsilon}$; that is, $\varepsilon(\mathbf{j}-1) < \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) \geq \mathbf{j}$ or $\overline{\varepsilon}(\mathbf{j}-1) < \mathbf{j}-1$ and $\overline{\varepsilon}(\mathbf{j}) \geq \mathbf{j}$. Then it follows that in all cases $(\varepsilon \cdot \overline{\varepsilon})(\mathbf{j}-1) = \overline{\varepsilon}(\varepsilon(\mathbf{j}-1)) < \mathbf{j}-1$ and $(\varepsilon \cdot \overline{\varepsilon})(\mathbf{j}) = \overline{\varepsilon}(\varepsilon(\mathbf{j})) \geq \mathbf{j}$. Let $\mathbf{j}$ be not a fixed point of both idempotents; that is, $\varepsilon(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\varepsilon(\mathbf{j}) > \mathbf{j}$ and also $\overline{\varepsilon}(\mathbf{j}-1) \leq \mathbf{j}-1$ and $\overline{\varepsilon}(\mathbf{j}) > \mathbf{j}$. At last,

we obtain that $(\varepsilon \cdot \bar{\varepsilon})(\mathbf{j}-1) = \bar{\varepsilon}(\varepsilon(\mathbf{j}-1)) \leq \bar{\varepsilon}(\mathbf{j}-1) \leq \varepsilon(\mathbf{j}-1)$ and $(\varepsilon \cdot \bar{\varepsilon})(\mathbf{j}) = \bar{\varepsilon}(\varepsilon(\mathbf{j}-1)) > \bar{\varepsilon}(\mathbf{j}-1) > \varepsilon(\mathbf{j}-1)$.

Now we will prove that $\varepsilon \cdot \bar{\varepsilon}$ is an idempotent endomorphism. Like in the proof of the previous lemma, we fix two nonconsecutive jump points $\mathbf{j}_s$ and $\mathbf{j}_{s+1}$ of idempotents ε and $\bar{\varepsilon}$ and interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$.

Using the notations from the proof of Lemma 24 we easily verify all twenty-five possibilities:

$$\begin{gathered}
\varepsilon_{(1)} \cdot \bar{\varepsilon}_{(1)} = \varepsilon_{(1)}, \\
\varepsilon_{(1)} \cdot \bar{\varepsilon}_{(2)} = \varepsilon_{(1)}, \\
\varepsilon_{(1)} \,\&\, \bar{\varepsilon}_{(3)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)}, \\
\varepsilon_{(1)} \,\&\, \bar{\varepsilon}_{(4)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)}, \\
\varepsilon_{(1)} \,\&\, \bar{\varepsilon}_{(5)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)}, \\
\varepsilon_{(2)} \cdot \bar{\varepsilon}_{(1)} = \bar{\varepsilon}_{(1)}, \\
\varepsilon_{(2)} \cdot \bar{\varepsilon}_{(2)} = \bar{\varepsilon}_{(2)}, \\
\varepsilon_{(2)} \cdot \bar{\varepsilon}_{(3)} = \bar{\varepsilon}_{(3)}, \\
\varepsilon_{(2)} \cdot \bar{\varepsilon}_{(4)} = \bar{\varepsilon}_{(4)}, \\
\varepsilon_{(2)} \cdot \bar{\varepsilon}_{(5)} = \bar{\varepsilon}_{(5)}, \\
\varepsilon_{(3)} \cdot \bar{\varepsilon}_{(1)} = \bar{\varepsilon}_{(1)}, \\
\varepsilon_{(3)} \cdot \bar{\varepsilon}_{(2)} = \varepsilon_{(3)}, \\
\varepsilon_{(3)} \,\&\, \bar{\varepsilon}_{(3)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(3)}, \qquad (25) \\
\varepsilon_{(3)} \,\&\, \bar{\varepsilon}_{(4)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(3)} \,\&\, \bar{\varepsilon}_{(5)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(4)} \cdot \bar{\varepsilon}_{(1)} = \bar{\varepsilon}_{(1)}, \\
\varepsilon_{(4)} \cdot \bar{\varepsilon}_{(2)} = \varepsilon_{(4)}, \\
\varepsilon_{(4)} \,\&\, \bar{\varepsilon}_{(3)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(4)} \,\&\, \bar{\varepsilon}_{(4)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(4)}, \\
\varepsilon_{(4)} \,\&\, \bar{\varepsilon}_{(5)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(5)} \cdot \bar{\varepsilon}_{(1)} = \bar{\varepsilon}_{(1)}, \\
\varepsilon_{(5)} \cdot \bar{\varepsilon}_{(2)} = \varepsilon_{(5)}, \\
\varepsilon_{(5)} \,\&\, \bar{\varepsilon}_{(3)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(5)} \,\&\, \bar{\varepsilon}_{(4)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}, \\
\varepsilon_{(5)} \,\&\, \bar{\varepsilon}_{(5)} \Longrightarrow (\varepsilon \cdot \bar{\varepsilon})_{(1)} \quad \text{or} \quad (\varepsilon \cdot \bar{\varepsilon})_{(5)}.
\end{gathered}$$

Thus we prove that the product of both idempotent endomorphisms in interval $[\mathbf{j}_s, \mathbf{j}_{s+1}-1]$ is also an idempotent in this interval. It easily implies that $\varepsilon \cdot \bar{\varepsilon}$ is an idempotent endomorphism in the whole interval $[\mathbf{j}_1, \mathbf{j}_r]$. In interval $[0, \mathbf{j}_1 - 1]$ each one of the idempotents ε and $\bar{\varepsilon}$ satisfies one of the conditions (1)–(5), of Lemma 23. So, from the previous reasonings it follows that the product $\varepsilon \cdot \bar{\varepsilon}$ is an idempotent endomorphism in $[0, \mathbf{j}_1 - 1]$. In a similar way we show that $\varepsilon \cdot \bar{\varepsilon}$ is an idempotent endomorphism in $[\mathbf{j}_r, n-1]$. Hence, $\varepsilon \cdot \bar{\varepsilon}$ is an idempotent endomorphism in the whole $\mathscr{C}_n$. □

Immediately from Lemmas 24 and 25 follows Theorem 26.

Theorem 26. *The set of the idempotent endomorphisms of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ with the same jump points is a subsemiring of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$.*

From Lemmas 24 and 25, and Figure 1 it is easy to prove the following.

Proposition 27. *The set of the idempotent endomorphisms of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ without jump points is a subsemiring of $\widehat{\mathscr{E}}_{\mathscr{C}_n}$ of order $\binom{n+1}{2}$.*

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