Research Article

Finite 1-Regular Cayley Graphs of Valency 5

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Let $\Gamma = \text{Cay}(G, S)$ and $G \leq X \leq \text{Aut}\Gamma$. We say Γ is $(X, 1)$ -regular Cayley graph if X acts regularly on its arcs. Γ is said to be core-free if G is core-free in some $X \leq \text{Aut}(\text{Cay}(G, S))$. In this paper, we prove that if an $(X, 1)$ -regular Cayley graph of valency 5 is not normal or binormal, then it is the normal cover of one of two core-free ones up to isomorphism. In particular, there are no core-free 1-regular Cayley graphs of valency 5.

1. Introduction

We assume that all graphs in this paper are finite, simple, and undirected.

Let Γ be a graph. Denote the vertex set, arc set, and full automorphism group of Γ by $V\Gamma$, $A\Gamma$, and $\text{Aut}\Gamma$, respectively. A graph Γ is called X -vertex-transitive or X -arc-transitive if X acts transitively on $V\Gamma$ or $A\Gamma$, where $X \leq \text{Aut}\Gamma$. Γ is simply called vertex-transitive, arc-transitive for the case where $X = \text{Aut}\Gamma$. In particular, Γ is called $(X, 1)$ -regular if $X \leq \text{Aut}\Gamma$ acts regularly on its arcs and then 1-regular when $X = \text{Aut}\Gamma$.

Let G be a finite group with identity element 1. For a subset S of G with $1 \notin S = S^{-1} := \{x^{-1} \mid x \in S\}$, the Cayley graph $\text{Cay}(G, S)$ of G (with respect to S) is defined as the graph with vertex set G such that $x, y \in G$ are adjacent if and only if $yx^{-1} \in S$. It is easy to see that a Cayley graph $\text{Cay}(G, S)$ has valency $|S|$, and it is connected if and only if $\langle S \rangle = G$.

Li proved in [1] that there are only finite number of core-free s -transitive Cayley graphs of valency k for $s \in \{2, 3, 4, 5, 7\}$ and $k \geq 3$ and that, with the exceptions $s = 2$ and $(s, k) = (3, 7)$, every s -transitive Cayley graph is a normal cover of a core-free one. It was proved in [2] that there are 15 core-free s -transitive cubic Cayley graphs up to isomorphism, and there are no core-free 1-regular cubic Cayley graphs. A natural problem arises. Characterize 1-transitive Cayley graphs, in particular, which graphs are 1-regular? Until now, the result about 1-regular graphs mainly

focused constructing examples. For example, Frucht gave the first example of cubic 1-regular graph in [3]. After then, Conder and Praeger constructed two infinite families of cubic 1-regular graphs in [4]. Marušič [5] and Malnič et al. [6] constructed two infinite families of tetravalent 1-regular graphs. Classifying such graphs has aroused great interest. Motivated by above results and problem, we consider 1-regular Cayley graphs of valency 5 in this paper.

A graph Γ can be viewed as a Cayley graph of a group G if and only if $\text{Aut}\Gamma$ contains a subgroup that is isomorphic to G and acts regularly on the vertex set. For convenience, we denote this regular subgroup still by G . If $X \leq \text{Aut}\Gamma$ contains a normal subgroup that is regular and isomorphic to G , then Γ is called an X -normal Cayley graph of G ; if G is not normal in X but has a subgroup which is normal in X and semiregular on $V\Gamma$ with exactly two orbits, then Γ is called an X -bi-normal Cayley graph; furthermore if $X = \text{Aut}\Gamma$, Γ is called normal or bi-normal. Some characterization of normal and bi-normal Cayley graphs has given in [1, 2].

For a Cayley graph $\Gamma = \text{Cay}(G, S)$, Γ is said to be core-free (with respect to G) if G is core-free in some $X \leq \text{Aut}\Gamma$; that is, $\text{Core}_X(G) = \bigcap_{x \in X} G^x = 1$.

The main result of this paper is the following assertion.

Theorem 1. Let $\Gamma = \text{Cay}(G, S)$ be an $(X, 1)$ -regular Cayley graph of valency 5, where $G \leq X \leq \text{Aut}\Gamma$. Let $n(G)$ be the number of nonisomorphic core-free $(X, 1)$ -regular Cayley

TABLE 1

Number	X	G	$n(G)$	1-regular	Remark
1	A_5	A_4	1	No	Icosahedron
2	S_5	S_4	1	No	

graph of valency 5 with the regular subgroup equal to G . Then either

- (i) Γ is an X -normal or X -bi-normal Cayley graph or
- (ii) Γ is a nontrivial normal cover of one line of Table 1.

In particular, there are no core-free 1-regular Cayley graphs of valency 5.

By Theorem 1, we can get the following remark immediately.

Remark 2. Let $\Gamma = \text{Cay}(G, S)$ be an 1-regular Cayley graph of valency 5. Then Γ is normal or bi-normal.

2. Examples

In this section we give some examples of graphs appearing in Theorem 1.

Example 3. Let $M = \langle a \rangle \cong \mathbb{Z}_{11}$ be a cyclic group. Assume that $\tau \in \text{Aut}(M)$ is of order 10 and $X = M : \langle \tau \rangle \cong \mathbb{Z}_{11} : \mathbb{Z}_{10}$. Let

$$G = M : \langle \tau^5 \rangle \cong D_{22}. \quad (1)$$

Suppose that

$$S = g^{\langle \tau^2 \rangle} = \{g, g^{\tau^2}, g^{\tau^4}, g^{\tau^6}, g^{\tau^8}\}, \quad (2)$$

where $g \in G$ is an involution such that $g \neq \tau^5$. Let $\Gamma = \text{Cay}(G, S)$ be the Cayley graph of the dihedral group G with respect to S . Then Γ is a connected $(X, 1)$ -regular Cayley graph of valency 5. In particular, Γ is X -normal.

Proof. Let

$$G = \langle a \rangle : \langle b \rangle = \{1, a, a^2, \dots, a^{10}, b, ab, a^2b, \dots, a^{10}b\} \cong D_{22}, \quad (3)$$

where $b = \tau^5$.

Noting $o(a) = 11$, we may assume that $a^\tau = a^9$. Since the involution $g \in G$ is not equal to b , we may let $g = a^i b$ for some $1 \leq i < 11$ such that $(9, i) = 1$. Then $g^{\tau^2} = (a^i b)^{\tau^2} = a^{81i} b = a^{4i} b$, and so $g^{\tau^2} g^{-1} = a^{3i} \in \langle g^{\langle \tau^2 \rangle} \rangle = \langle S \rangle$. Thus the element $g^{\tau^2} g^{-1}$ is of order 11 as $(3i, 11) = 1$. So $\langle S \rangle = \langle g^{\langle \tau^2 \rangle} \rangle = G$; that is, $\Gamma = \text{Cay}(G, S)$ is connected.

Obviously, $G \triangleleft X \leq \text{Aut}\Gamma$ and $X_1 = \langle \tau^2 \rangle$. However, $|X| = 55 = |\text{Aut}\Gamma|$; then Γ is an $(X, 1)$ -regular normal Cayley graph of G of valency 5. $\square$

Example 4. Let $G = \langle a, b \mid a^5 = b^2 = 1, a^b = a^{-1} \rangle \cong D_{10}$. Set $S = \{b, ab, a^2b, a^3b, a^4b\}$ and $\Gamma = \text{Cay}(G, S)$. Then $\Gamma \cong K_{5,5}$ and $\text{Aut}\Gamma = S_5 \wr S_2$. Let $X = (\mathbb{Z}_5 \times \mathbb{Z}_5) : \mathbb{Z}_2 \not\cong D_{10} \times \mathbb{Z}_5$ such that $G \leq X \leq \text{Aut}\Gamma$. It follows that $\text{Core}_X(G) \cong \mathbb{Z}_5$. Then $X_\alpha \cong \mathbb{Z}_5$ for $\alpha \in V\Gamma$, and furthermore Γ is $(X, 1)$ -regular. Obviously G is not normal in X . However, $\text{Core}_X(G) \triangleleft X$ is semiregular and has exactly two orbits on $V\Gamma$; then Γ is an $(X, 1)$ -regular Cayley graph of valency 5. In particular, Γ is X -bi-normal.

3. The Proof of Main Results

In this section, we will prove our main results. We first present some properties about normal Cayley graphs.

For a Cayley graph $\Gamma = \text{Cay}(G, S)$, we have a subgroup of $\text{Aut}(G)$:

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}. \quad (4)$$

Clearly it is a subgroup of the stabilizer in $\text{Aut}\Gamma$ of the vertex corresponding to the identity 1 of G . Since Γ is connected, $\text{Aut}(G, S)$ acts faithfully on S . By Godsil [7, Lemma 2.1], the normalizer $N_{\text{Aut}\Gamma}(G) = G : \text{Aut}(G, S)$. So $\Gamma = \text{Cay}(G, S)$ is a normal Cayley graph if and only if $\text{Aut}(G, S) = (\text{Aut}\Gamma)_1$.

Let $\Gamma = \text{Cay}(G, S)$ be an $(X, 1)$ -regular Cayley graph of valency 5 such that $G \leq X \leq \text{Aut}\Gamma$. Then S contains at least one involution. Let $K = \text{Core}_X(G)$, which is the core of G in X .

Lemma 5. Assume that $K = 1$. Then $(X, G) = (A_5, A_4)$ or (S_5, S_4) .

Proof. Let H be the stabilizer in X of the vertex corresponding to the identity of G . Then $H \cong \mathbb{Z}_5$, $H \cap G = 1$, and $X = GH$. Let $[X : G]$ be the set of right cosets of G in X . Consider the action of X on $[X : G]$ by the right multiplication. Then we get that X is a primitive permutation group of degree 5 and G is a stabilizer of X . Since Γ has valency 5, $|G| = |V\Gamma| \geq 6$, and so $|X| = |G||H| \geq 30$. Then we can show $X \cong A_5$ or S_5 , and then $G = A_4$ or S_4 , respectively. $\square$

Lemma 6. Suppose that $G = A_4$ and $X = A_5$. Then Γ is the icosahedron graph. Moreover, $\text{Aut}\Gamma = A_5 \times \mathbb{Z}_2$ and Γ is not 1-regular.

Proof. Note that $X = GH$, where $X \cong A_5$, $G \cong A_4$, and $H \cong \mathbb{Z}_5$. Since X has no nontrivial normal subgroup, Γ is not bipartite. So Γ is the icosahedron graph. Further by Magma [8], $\text{Aut}\Gamma = A_5 \times \mathbb{Z}_2$, so Γ is not 1-regular. $\square$

Lemma 7. Suppose that $G = S_4$ and $X = S_5$. Then the graph Γ is not 1-regular and there is only one isomorphism class of these graphs.

Proof. Note that $G = S_4$, $X = S_5$, and $X = GH$. Let $H = \langle \sigma \rangle$, where $\sigma = (1\ 2\ 3\ 4\ 5)$. By considering the right multiplication action of X on the right cosets of G in X , G can be viewed as a stabilizer of X acting on $\{1, 2, 3, 4, 5\}$. Without lost generality, we may assume that 1 is fixed by G . Take an involution $\tau \in S$. Then, by [2], $\tau \in S_5 \setminus N_{S_5}(H)$ and we can identify S with $H\tau H \cap G$. Note that $\tau \in G \leq S_4$

and $N_S(H) = H : \text{Aut}(H) = \langle (1\ 2\ 3\ 4\ 5) \rangle : \langle (2\ 3\ 5\ 4) \rangle \cong \mathbb{Z}_5 : \mathbb{Z}_4$; then τ is one of the following: $(2\ 5)$, $(3\ 5)$, $(2\ 3)$, $(3\ 4)$, $(4\ 5)$, $(2\ 4)$, $(2\ 3)(4\ 5)$, and $(2\ 4)(3\ 5)$. Note $H = \langle (1\ 2\ 3\ 4\ 5) \rangle$. Assume that $\tau = (2\ 5)$; by calculation, we have $(2\ 5) := h_1$, $\tau \cdot (1\ 2\ 3\ 4\ 5) = (1\ 2)(3\ 4\ 5) := h_2$, $\tau \cdot (1\ 3\ 5\ 2\ 4) = (1\ 3\ 5\ 4) := h_3$, $\tau \cdot (1\ 4\ 2\ 5\ 3) = (1\ 4\ 2\ 3) := h_4$, and $\tau \cdot (1\ 5\ 4\ 3\ 2) = (1\ 5)(2\ 4\ 3) := h_5$. Then $H(2\ 5)H = \{Hh_1, Hh_2, Hh_3, Hh_4, Hh_5\} = \{(2\ 5), (1\ 5)(2\ 3\ 4), (1\ 4\ 5\ 3), (1\ 2)(3\ 5\ 4), (1\ 3\ 2\ 4), (1\ 5)(2\ 4\ 3), (1\ 4\ 2\ 3), (1\ 3\ 5\ 4), (1\ 2)(3\ 4\ 5), (2\ 5\ 3\ 4), (1\ 5\ 2\ 4), (1\ 4\ 5)(2\ 3), (1\ 3\ 5), (1\ 3\ 5\ 2), (1\ 2\ 5)(3\ 4), (2\ 4), (1\ 5\ 4)(2\ 3), (1\ 2\ 3)(4\ 5), (3\ 5), (1\ 5\ 2)(3\ 4), (1\ 4\ 2\ 5), (1\ 4), (1\ 3\ 2)(4\ 5), (1\ 2\ 5\ 3), (2\ 4\ 3\ 5)\}$. Thus the corresponding S is $\{(2\ 5), (2\ 5\ 3\ 4), (2\ 4), (3\ 5), (2\ 4\ 3\ 5)\}$ since $1^s = 1$ for each $s \in H(2\ 5)H$. By similar argument, for every τ , we can work out S explicitly, which is one of the following four cases: $S_1 = \{(2\ 5), (2\ 5\ 3\ 4), (2\ 4), (3\ 5), (2\ 4\ 3\ 5)\}$, $S_2 = \{(2\ 3), (3\ 4), (4\ 5), (2\ 3\ 4\ 5), (2\ 5\ 4\ 3)\}$, $S_3 = \{(2\ 3)(4\ 5), (2\ 3\ 5), (2\ 5\ 3), (2\ 4\ 5), (2\ 5\ 4)\}$, and $S_4 = \{(2\ 4)(3\ 5), (2\ 4\ 3), (3\ 5\ 4), (2\ 3\ 4), (3\ 4\ 5)\}$.

Now let $A = \text{Aut}\Gamma$. We declare that $X \neq A$. Assume that $X = A$. Note that $G = N_A(G) = G\text{Aut}(G, S)$; then $\text{Aut}(G, S) = 1$. Let $\sigma = (2\ 5)(3\ 4)$. Since $\sigma = (2\ 5\ 3\ 4)^{(2\ 5)} \cdot (3\ 5) = (3\ 4)^{(2\ 3)} \cdot (2\ 3\ 4\ 5) = (2\ 3)(4\ 5) \cdot (2\ 5\ 3) \cdot (2\ 5\ 4) = (2\ 4)(3\ 5) \cdot (2\ 4\ 3) \cdot (3\ 5\ 4)$, $\sigma \in G$ and $S^\sigma = S$ for any possible S . Therefore $\sigma \in \text{Aut}(G, S)$, which leads to a contradiction. So the assertion is right; that is, Γ is not 1-regular.

Let $G_i = \langle S_i \rangle$ and $\Gamma_i = \text{Cay}(G_i, S_i)$ for $i \in \{1, 2, 3, 4\}$. Set $\gamma = (2\ 4\ 5\ 3)$, then $S_1^\gamma = S_2$ and $S_3^\gamma = S_4$. It follows that $G_1^\gamma = G_2$ and $G_3^\gamma = G_4$, namely, $\Gamma_1 \cong \Gamma_2$ and $\Gamma_3 \cong \Gamma_4$. Now we consider $G_1 = \langle S_1 \rangle$. Note that $(2\ 4) = (2\ 5)^{(2\ 4\ 3\ 5)}$ and $(3\ 5) = (2\ 5\ 3\ 4)^{(2\ 5)} \cdot (2\ 5\ 3\ 4)^2$, then $G_1 = \langle (2\ 5), (2\ 5\ 3\ 4) \rangle$. Since $(2\ 5\ 4) = (2\ 3)(4\ 5) \cdot (2\ 3\ 5)$, $G_3 = \langle S_3 \rangle = \langle (2\ 3)(4\ 5), (2\ 3\ 5) \rangle$. On the other hand, $(2\ 5\ 3\ 4)^4 = (2\ 5)^2 = ((2\ 5) \cdot (2\ 5\ 3\ 4))^3 = 1$ and $(2\ 3\ 5)^3 = ((2\ 3)(4\ 5))^2 = ((2\ 3)(4\ 5) \cdot (2\ 3\ 5))^3 = 1$, then $G_1 \cong S_4$ and $G_3 \cong A_4$. By the assumption, Γ_3 is not the graph satisfying conditions. So far we get the result that there is only one isomorphism class of graphs when $G = S_4$. $\square$

To finish our proof, we need to introduce some definitions and properties. Assume that Γ is an X -vertex transitive graph with X being a subgroup of $\text{Aut}\Gamma$. Let N be a normal subgroup of X . Denote the set of N -orbits in $V\Gamma$ by V_N . The normal quotient Γ_N of Γ induced by N is defined as the graph with vertex set V_N , and two vertices $B, C \in V_N$ are adjacent if there exist $u \in B$ and $v \in C$ such that they are adjacent in Γ . It is easy to show that X/N acts transitively on the vertex set of Γ_N . Assume further that Γ is X -edge-transitive. Then X/N acts transitively on the edge set of Γ_N , and the valency $\text{val}(\Gamma) = m\text{val}(\Gamma_N)$ for some positive integer m . If $m = 1$, then Γ is called a normal cover of Γ_N .

Proof of Theorem 1. Let $\Gamma = \text{Cay}(G, S)$ be an $(X, 1)$ -regular Cayley graph of valency 5, where $G \leq X \leq \text{Aut}\Gamma$. Then it is trivial to see that Γ is connected. Let $N = \text{Core}_X(G)$ be the core of G in X . Assume that N is not trivial. Then either $G = N$ or $|G : N| \geq 2$. The former implies $G \trianglelefteq X$; that is, Γ

is an X -normal Cayley graph with respect to G . For the case where $|G : N| = 2$, it is easy to verify Γ is an X -bi-normal Cayley graph. Suppose that $|G : N| > 2$; namely, N has at least three orbits on $V\Gamma$. Since $\text{val}(\Gamma) = 5$ is a prime and Γ is $(X, 1)$ -regular, Γ is a cover of Γ_N and $G/N \leq X/N \leq \text{Aut}\Gamma_N$. We have that Γ_N is a Cayley graph of G/N and Γ_N is core-free with respect to G/N . Now suppose that N is trivial, then Γ is a core-free one. According to Lemmas 5, 6, and 7, there are two core-free $(X, 1)$ -regular Cayley graphs of valency 5 (up to isomorphism) as in Table 1. As far, Theorem 1 holds.

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